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THE EXPECTED INDEPENDENT DOMINATION NUMBER OF RANDOM DIRECTED ROOTED TREES

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ABSTRACT. We derive a formula for the expected value $\mu(n)$ of the independent domination number of a random directed rooted tree with n labeled vertices and determine the asymptotic behavior of $\mu(n)$ as n goes to infinity.

1. Introduction

Let D be a digraph. A subset S of vertices of D is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that (u, v) is an arc of D. The *domination number* of D is the number $\alpha(D)$ of vertices in any smallest dominating subset of vertices in D. A subset I of vertices of D is an *independent set* of D if no two vertices of I are joined by an arc in D. The *independence number* of D is the number $\beta(D)$ of vertices in any largest independent subset of vertices in D. An *independent dominating set* of D is an independent and dominating set of D. The *independent domination number* of D is the number $\alpha'(D)$ of vertices in any smallest independent dominating subset of vertices in D. For definitions not given here see, for example, [2] and [7].

There are $\binom{2n}{n}/(n+1)$ binary trees T with $2n+1$ vertices. Let $\mu(2n+1)$ denote the expected value of the independent domination number $\alpha'(T)$ over the set of such binary trees. Lee showed in [5] that

$$\mu(2n+1) = \sum (k+1)2^k \frac{\langle n \rangle_k}{\langle 2n \rangle_k}$$

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for $n = 0, 1, 2, \ldots$, where the inner sum is over all even integers k with $1 \leq k \leq n$ and $\langle n \rangle_k$ denotes the falling factorial $\langle n \rangle_k = n(n-1)\cdots(n-k+1)$, and that

$$\frac{\mu(2n+1)}{2n+1} \longrightarrow \frac{1}{2}$$

as $n \to \infty$. We want to do similar work for the expected independent domination number of directed rooted trees.

A *directed rooted tree* is an oriented rooted tree in which every direction is led away from the root. There are n^{n-1} directed rooted trees T with n labeled vertices. Let $\mu(n)$ denote the expected value of the independent domination number $\alpha'(T)$ over the set of such trees.

Our goal is to show that

$$\mu(n) = \sum \binom{n-1}{k} n^{-k}(k+1)!,$$

where the sum is over all even integers k with $0 \leq k \leq n-1$, and that

$$\frac{\mu(n)}{n} \longrightarrow \frac{1}{2}$$

as $n \to \infty$.

2. Some lemmas

An *oriented tree* is a tree in which each edge is assigned a unique direction. A digraph might have no independent dominating sets as we can see in 3-cycles. However, every oriented tree has a unique independent dominating set [5]. Therefore, we have following lemma.

LEMMA 1. *Every directed rooted tree has a unique independent dominating set.*

Let y_n denote the number of directed rooted trees T with n labeled vertices. Clearly, $y_1 = 1$. If we remove the root r of T, along with all arcs incident from r, we obtain a (possibly empty) ordered collection of disjoint directed rooted trees, or 1-*branches*, whose roots were originally joined from r. If we classify these trees according to the number of 1-branches attached from the root, count the number of ways of selecting the root vertex and forming 1-branches on the remaining vertices, and bear in mind that the ordering of the 1-branches is immaterial, we find that

$$y_n = n \sum_{j=1}^{n-1} \frac{1}{j!} \sum \binom{n-1}{a_1, \ldots, a_j} y_{a_1} \cdots y_{a_j} \tag{2.1}$$

for $n \geq 2$, where the inner sum is over all solutions in positive integers to the equation $a_1 + \cdots + a_j = n - 1$. If we let

$$y = y(x) = \sum_{n=1}^{\infty} y_n \frac{x^n}{n!}$$

be the exponential generating function for directed rooted trees, then it follows from equation (2.1) that

$$y = x + x \sum_{n=1}^{\infty} \frac{y^n}{n!} = xe^y. \tag{2.2}$$

This implies that

$$y = \sum_{n=1}^{\infty} n^{n-1} \frac{x^n}{n!} \tag{2.3}$$

by Lagrange's inversion formula. This, of course, is a well-known argument (see [1] or [p. 26, 7]).

On the other hand, we can find the exponential generating function y for directed rooted trees using a slightly different approach. Let T be a directed rooted tree with n labeled vertices and root r, and let T_r denote the directed rooted subtree of T induced by $N^+[r]$, where $N^+[r]$ consists of the root r and its outneighbors. Suppose the outdegree of r is k. If we remove T_r from T, along with all arcs incident with vertices in T_r, we obtain an ordered k-tuple of disjoint (possibly empty) collections of directed rooted trees, or 2-*branches*.

If we classify these trees according to the outdegree k of the root r, then the generating function for ordered k-tuples of disjoint collections of 2-branches is $(e^y)^k$ and the generating function for T_r is $(k+1)(x^{k+1}/(k+1)!)$. Note that different orderings of collections of 2-branches yield different directed rooted trees. Thus the generating function for directed rooted trees is

$$y = \sum_{k=0}^{\infty} (e^y)^k \cdot (k+1) \frac{x^{k+1}}{(k+1)!} = xe^{xe^y}, \tag{2.4}$$

which is equivalent to $y = xe^y$.

The following lemma is a straightforward consequence of the definition of 2-branch and Lemma 1.

LEMMA 2. *Let T be a directed rooted tree. Then the independent domination number of T is one more than the sum of the independent domination numbers of all 2-branches of T.*

For $1 \le k \le n$, let $y_{n,k}$ denote the number of directed rooted trees with n labeled vertices whose independent domination number is exactly k. Let

$$Y = Y(x, z) = \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} y_{n,k} z^k\Big) \frac{x^n}{n!}.$$

It follows by a slight extension of the argument used to establish equation (2.4) that

$$Y = zxe^{xe^Y}. \tag{2.5}$$

The factor z is present in equation (2.5) because of Lemma 2. Notice that $y = Y(x, 1)$.

LEMMA 3. *Let $\mu(n)$ denote the expected value of the independent domination numbers of the y_n directed rooted trees with n labeled vertices and define*

$$M(x) = \sum_{n=1}^{\infty} \mu(n) y_n \frac{x^n}{n!} = \sum_{n=1}^{\infty} \mu(n) n^{n-1} \frac{x^n}{n!}. \tag{2.6}$$

Then we have

$$M(x) = \frac{y}{1-y^2} \tag{2.7}$$

$$= \sum_{k=0}^{\infty} y^{2k+1}. \tag{2.8}$$

Proof. It is easy to see that

$$M(x) = \sum_{n=1}^{\infty} \mu(n) y_n \frac{x^n}{n!} = Y_z(x, 1).$$

If we differentiate both sides of equation (2.5) with respect to z, set $z = 1$, use the fact that equations (2.2) and (2.4) are equivalent, and solve for $Y_z(x, 1)$, we find the required result. □

3. A formula for $\mu(n)$

We know that $M(x)$ is the exponential generating function for the total sums of the independent domination numbers of directed rooted trees. Therefore, using power series expansion of $M(x)$ in x, we could find directly the expected value $\mu(n)$ of the independent domination numbers of directed rooted trees for small n. Actually, it follows from (2.8), (2.3), and routine use of *Mathematica* that

$$\begin{aligned}[*]\cdots M(x) \quad = \quad & x + 2\frac{x^2}{2!} + 15\frac{x^3}{3!} + 136\frac{x^4}{4!} + 1645\frac{x^5}{5!} + 24336\frac{x^6}{6!} \\ & +426979\frac{x^7}{7!} + 8658560\frac{x^8}{8!} + 199234809\frac{x^9}{9!} \\ & +5128019200\frac{x^{10}}{10!} + 145969492471\frac{x^{11}}{11!} \\ & +4552809182208\frac{x^{12}}{12!} + 154404454932325\frac{x^{13}}{13!} \\ & +5656950010320896\frac{x^{14}}{14!} + 222655633595044875\frac{x^{15}}{15!} \\ & + \cdots .\end{aligned}$$

Here is a table for $\mu(n)$ and $\mu(n)/n$. The entries for $n \le 5$ were verified by drawing all of the diagrams for directed rooted trees with up to 5 vertices.

n	y_n	$\mu(n)y_n$	$\mu(n)$	$\mu(n)/n$
1	1	1	1.00	1.0000
2	2	2	1.00	0.5000
3	9	15	1.66	0.5555
4	64	136	2.12	0.5312
5	625	1645	2.63	0.5264
6	7776	24336	3.12	0.5216
7	117649	426979	3.62	0.5184
8	2097152	8658560	4.12	0.5160
9	43046721	199234809	4.62	0.5142
10	1000000000	5128019200	5.12	0.5128
11	see [9]	see [*]	5.62	0.5114
12	see [9]	see [*]	6.12	0.5100
13	see [9]	see [*]	6.62	0.5083
14	see [9]	see [*]	7.12	0.5061
15	see [9]	see [*]	7.62	0.5035

Furthermore, we can derive a reasonably explicit formula for $\mu(n)$ as follows.

THEOREM 4. *The expected value of the independent domination numbers of directed rooted trees with n labeled vertices is*

$$\mu(n) = \sum \binom{n-1}{k} n^{-k}(k+1)!,$$

where the sum is over all even integers k with $0 \le k \le n-1$.

Proof. Let

$$\phi(\lambda) = e^{\lambda} \quad \text{and} \tag{3.1}$$

$$f(\lambda) = \frac{\lambda}{1-\lambda^2} = \sum_{i=0}^{\infty} \lambda^{2i+1}. \tag{3.2}$$

It follows from (2.2) and (3.1) that $y = x\phi(y)$ and from Lagrange's inversion formula (see [p. 17, 3]) that

$$f(y) = f(0) + \sum_{n=1}^{\infty} \left[\left(\frac{d}{d\lambda}\right)^{n-1}\Big(f'(\lambda)\phi^n(\lambda)\Big)\right]_{\lambda=0} \frac{x^n}{n!}. \tag{3.3}$$

It follows from (2.7), (3.2), and (3.3) that

$$\begin{aligned}
M(x) &= f(y) \\
&= \sum_{n=1}^{\infty} \left[\left(\frac{d}{d\lambda}\right)^{n-1}\Big(f'(\lambda)\phi^n(\lambda)\Big)\right]_{\lambda=0} \frac{x^n}{n!} \\
&= \sum_{n=1}^{\infty} \left[\left(\frac{d}{d\lambda}\right)^{n-1}\Big(\sum_{i=0}^{\infty}(2i+1)\lambda^{2i}\cdot e^{n\lambda}\Big)\right]_{\lambda=0} \frac{x^n}{n!} \\
&= \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1}\binom{n-1}{k}\Big(\sum_{i}(2i+1)_{k+1}\lambda^{2i-k}\Big)n^{n-1-k}e^{n\lambda}\right]_{\lambda=0} \frac{x^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum \binom{n-1}{k} n^{n-1-k}(k+1)!\frac{x^n}{n!},
\end{aligned}$$

where the inner sum is over all even integers k with $0 \le k \le n-1$. Thus it follows from (2.6) and (2.3) that

$$M(x) = \sum_{n=1}^{\infty} \mu(n) n^{n-1}\frac{x^n}{n!} \tag{3.4}$$

$$= \sum_{n=1}^{\infty}\sum \binom{n-1}{k} n^{n-1-k}(k+1)!\frac{x^n}{n!}, \tag{3.5}$$

where the inner sum is over all even integers k with $0 \leq k \leq n-1$. Comparing the coefficients of $x^n/n!$ in (3.4) and (3.5), we obtain the required result. $\square$

4. The asymptotic behavior of $\mu(n)$

To determine the asymptotic behavior of $\mu(n)/n$, we need the following lemma [5].

LEMMA 5. *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be power series with radii of convergence $\rho_1 \geq \rho_2$, respectively. Suppose that $A(x)$ converges absolutely at $x = \rho_1$. Suppose that $b_n > 0$ for all n and that b_{n-1}/b_n approaches a limit b as $n \to \infty$. If $\sum_{n=0}^{\infty} c_n x^n = A(x)B(x)$, then $c_n \sim A(b)b_n$.*

Now we can state the main result of this paper.

THEOREM 6. *The expected value of the independent domination numbers of directed rooted trees with n labeled vertices is*

$$\mu(n) \sim \frac{1}{2}n.$$

Proof. Let

$$A(x) = \frac{y}{1+y} = \sum_{n=1}^{\infty} a_n x^n \quad \text{and} \tag{4.1}$$

$$B(x) = \frac{1}{1-y} = \sum_{n=0}^{\infty} b_n x^n \tag{4.2}$$

where y is the exponential generating function for directed rooted trees, that is, $y = xe^y = \sum_{n=1}^{\infty} n^{n-1}\frac{x^n}{n!}$. Using the same argument as in Lemma 4, we obtain

$$a_n = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}n^{n-1-k}(k+1)! \tag{4.3}$$

$$= \frac{n^{n-1}}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}n^{-k}(k+1)!, \tag{4.4}$$

$$b_n = \frac{n^n}{n!}, \tag{4.5}$$

where we adopt the convention that $0^0 = 1$. It is easy to see that (2.3) is convergent for $|x| \leq e^{-1}$ and from (2.2) that $y(e^{-1}) = 1$. Thus (4.1) and (4.2) are valid for $|x| < e^{-1}$ (see [pp. 112–114, 4]).

First, it follows from (4.5) that

$$\frac{b_{n-1}}{b_n} \to \frac{1}{e}$$

as $n \to \infty$.

Secondly, we want to show that (4.1) converges absolutely at $x = e^{-1}$ and that $A(e^{-1}) = 1/2$. Let

$$c_k = (-1)^k \binom{n-1}{k} n^{-k}(k+1)!$$

for $k = 0, \ldots, n-1$ so that

$$A(e^{-1}) = \sum_{n=1}^{\infty} a_n e^{-n} = \sum_{n=1}^{\infty} n^{n-1} \frac{e^{-n}}{n!} \sum_{k=0}^{n-1} c_k. \tag{4.6}$$

Now we can show by an elementary but tedious argument that $\sum_{k=0}^{n-1} c_k$ is bounded; we omit the details of this argument. Moreover, we already know that

$$\sum_{k=1}^{\infty} n^{n-1} \frac{e^{-n}}{n!}$$

is absolutely convergent. Therefore, (4.6) is absolutely convergent (see [p. 73, 4]), that is, (4.1) converges absolutely at $x = e^{-1}$. Now, it follows from $y(e^{-1}) = 1$ and (4.1) that $A(e^{-1}) = 1/2$.

Finally, it follows from Lemma 5 that

$$\mu(n) \frac{n^{n-1}}{n!} \sim A(e^{-1}) b_n = \frac{1}{2} \frac{n^n}{n!}$$

since

$$M = \sum_{n=1}^{\infty} \mu(n) n^{n-1} \frac{x^n}{n!} = \frac{y}{1-y^2} = A(x)B(x).$$

Hence, we obtain

$$\mu(n) \sim \frac{1}{2} n$$

as we required. □

We know [6] that the expected independence number $\nu'(n)$ of directed rooted trees with n labeled vertices is

$$\nu'(n) \sim (.5671 \cdots) n.$$

It is easy to see that $\alpha'(T) \leq \beta(T)$ for any directed rooted tree T with labeled vertices. Our result

$$\mu(n) \sim .5n$$

is consistent with these two facts.

5. A new proof for an equality

The following equality appears in [8]:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k}(k+1)! = n^n,$$

which was proved by using inverse pairs

$$a_n = \sum_{k=0}^{n} \binom{n}{k} k^{n-k} b_k, \qquad b_n = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} kn^{n-1-k} a_k$$

and

$$n! = \sum_{k=0}^{n} (-1)^{k+n} \binom{n}{k} k^n.$$

In this section, we will give another proof for this equality with the aid of $M(x)$ in section 3.

Since $y_{n,k}$ is actually the number of directed rooted trees with n labeled vertices in which the number of vertices on even levels is k, $M(x)$ is the exponential generating function for the vertices on even levels of directed rooted trees with labeled vertices. Now, let $w_{n,k}$ be the number of directed rooted trees with n labeled vertices in which the number of vertices on odd levels is k and let

$$W(x, z) = \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} w_{n,k} z^k \Big) \frac{x^n}{n!}.$$

Then, it follows that

$$W(x, z) = xe^Y$$

since the number of vertices on odd levels of a directed rooted tree T is the sum of the vertices on even levels of all 1-branches of T. Let $\nu(n)$ denote the expected value of the number of vertices on odd levels of the n^{n-1} directed rooted trees with n labeled vertices and define

$$N(x) = \sum_{n=1}^{\infty} \nu(n) n^{n-1} \frac{x^n}{n!}.$$

Then, it follows from the same argument as in establishing the formula for $M(x)$ that

$$\begin{aligned} N(x) &= W_z(x,1) = xe^y M(x) = y\frac{y}{1-y^2} \\ &= \sum_{n=1}^{\infty}\sum \binom{n-1}{k} n^{n-1-k}(k+1)!\frac{x^n}{n!}, \end{aligned}$$

where the inner sum is over all odd integers k with $0 \le k \le n-1$. Therefore, we obtain

$$M(x) + N(x) = \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k}(k+1)!\frac{x^n}{n!}. \tag{5.1}$$

On the other hand, $M(x)+N(x)$ is the exponential generating function for the total sums of the numbers of vertices in all directed rooted trees. Therefore, we obtain

$$M(x) + N(x) = \sum_{n=1}^{\infty} n^n \frac{x^n}{n!}. \tag{5.2}$$

Comparing the coefficients of $x^n/n!$ in (5.1) and (5.2), we obtain the equality.

We can observe that the expected value $\mu(n)$ of the number of vertices on even levels of the n^{n-1} directed rooted trees with n labeled vertices is

$$\begin{aligned} \mu(n) &= \sum \binom{n-1}{k} n^{-k}(k+1)! \\ &\sim \frac{1}{2}n, \end{aligned}$$

where the sum is over all even integers k with $0 \le k \le n-1$ and that the expected value $\nu(n)$ of the number of vertices on odd levels of the n^{n-1} directed rooted trees with n labeled vertices is

$$\begin{aligned} \nu(n) &= \sum \binom{n-1}{k} n^{-k}(k+1)! \\ &\sim \frac{1}{2}n, \end{aligned}$$

where the sum is over all odd integers k with $0 \le k \le n-1$.

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