

THE SPECTRAL GEOMETRY OF EINSTEIN MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let (M, g) be a compact m dimensional Einstein manifold with smooth boundary. Let $\Delta_{p, \mathcal{B}}$ be the realization of the p form valued Laplacian with a suitable boundary condition \mathcal{B} . Let $\text{Spec}(\Delta_{p, \mathcal{B}})$ be the spectrum where each eigenvalue is repeated according to multiplicity. We show that certain geometric properties of the boundary may be spectrally characterized in terms of this data where we fix the Einstein constant.

1. Introduction

Let Δ_p be the Laplace-Beltrami operator acting on the space of smooth p forms over a compact m dimensional Riemannian manifold M with smooth boundary. If the boundary is non-empty, then we impose boundary conditions defined by a suitably chosen operator \mathcal{B} to define the realization $\Delta_{p, \mathcal{B}}$. Let ∇ be the Levi-Civita connection of M and let e_m be the inward unit normal vector field on the boundary. Then, for example, Dirichlet and Neumann boundary conditions are defined by the corresponding Dirichlet and Neumann boundary operators:

$$\mathcal{B}_D\phi := \phi|_{\partial M} \quad \text{and} \quad \mathcal{B}_N\phi := \nabla_{e_m}\phi|_{\partial M} \quad \text{for} \quad \phi \in C^\infty(\Lambda^p M).$$

In addition to the boundary conditions defined by these operators, there are also boundary conditions arising from index theory. Near the boundary, we decompose a differential form

$$\phi = \sum_I \phi_I dy^I + \sum_J \psi_J dx^m \wedge dy^J$$

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into tangential and normal components. *Absolute boundary conditions* are then defined by the operator

$$\mathcal{B}_a\phi := \{\sum_I \partial_m^x \phi_I dy^I\}|_{\partial M} \oplus \{\sum_J \psi_J dy^J\}|_{\partial M}.$$

Dually, we may use the Hodge \star operator to define the *relative boundary operator* by setting:

$$\mathcal{B}_r\phi := \mathcal{B}_a \star \phi.$$

In previous work [4], we extended a result of Patodi [3] from the context of closed Riemannian manifolds to the context of compact Riemannian manifolds with boundary. As we were interested in determining if the Einstein condition was spectrally determined, we worked in the context of manifolds of constant scalar curvature to show:

THEOREM 1.1. *Let (M_i, g_i) be compact Riemannian manifolds with smooth boundaries and constant scalar curvatures τ_i for $i = 1, 2$. Let \mathcal{B} define either Dirichlet or Neumann boundary conditions. Assume $\text{Spec}(\Delta_{p,\mathcal{B}})(M_1) = \text{Spec}(\Delta_{p,\mathcal{B}})(M_2)$ for $p = 0, 1, 2$. Then if (M_1, g_1) is Einstein, then (M_2, g_2) is Einstein and $\tau_1 = \tau_2$.*

In this paper, instead of studying the geometry of the interior, we turn our attention to the geometry of the boundary. Motivated by Theorem 1.1, we shall assume henceforth that the manifolds under consideration are Einstein and we shall fix the Einstein constant, or, equivalently, the scalar curvature τ .

We recall some basic definitions. Let indices i, j range from 1 to m and index a local orthonormal frame $\{e_i\}$ for the tangent bundle of M . Near the boundary, we further normalize the frame and assume that e_m is the inward unit geodesic normal vector field. Let indices a, b range from 1 to $m - 1$ and index the induced local orthonormal frame $\{e_a\}$ for the tangent bundle of the boundary.

We adopt the Einstein convention and sum over repeated indices. Let L be the second fundamental form and let R_{ijkl} be the Riemann curvature tensor. The *normalized mean curvature* κ , the *Ricci tensor* ρ , and the *scalar curvature* τ are then given by:

$$\kappa := L_{aa}, \quad \rho_{ij} := R_{ikkj}, \quad \text{and} \quad \tau := \rho_{ii}.$$

Since M is Einstein, $\rho = \lambda g$ where λ is the *Einstein constant*. This implies that $\tau = m\lambda$. Thus fixing the Einstein constant is equivalent to fixing the scalar curvature.

DEFINITION 1.2. We say that the boundary of (M, g) is:

1. *totally geodesic* if the second fundamental form vanishes identically. Equivalently, this means that if a geodesic in M is tangent to the boundary at a single point, then the geodesic stays in ∂M .
2. *minimal* if the normalized mean curvature vanishes identically. Equivalently, this means that the volume of the boundary is infinitesimally stationary.
3. *totally umbillic* if at each point of the boundary, the second fundamental form has only one eigenvalue; the eigenvalue in question is allowed to vary with the point of the boundary.
4. *strongly totally umbillic* if the eigenvalue in (3) is independent of the boundary point chosen.

We can now state the main results of this paper. We first consider both Dirichlet and Neumann boundary conditions:

THEOREM 1.3. For $i = 1, 2$, let (M_i, g_i) be compact Einstein manifolds with smooth boundaries. Assume that $\tau_1 = \tau_2$ and that

$$\begin{aligned} \text{Spec}(\Delta_{0, \mathcal{B}_D})(M_1) &= \text{Spec}(\Delta_{0, \mathcal{B}_D})(M_2), \quad \text{and} \\ \text{Spec}(\Delta_{0, \mathcal{B}_N})(M_1) &= \text{Spec}(\Delta_{0, \mathcal{B}_N})(M_2) \end{aligned}$$

where \mathcal{B}_D and \mathcal{B}_N define Dirichlet and Neumann boundary conditions, respectively. Then:

1. If ∂M_1 is totally geodesic, then ∂M_2 is totally geodesic.
2. If ∂M_1 is minimal, then ∂M_2 is minimal.
3. If ∂M_1 is totally umbillic, then ∂M_2 is totally umbillic.
4. If ∂M_1 is strongly totally umbillic, then ∂M_2 is strongly totally umbillic.

In the previous Theorem, we studied two different boundary conditions for the operator Δ_0 . In the next Theorem, we study two different operators, Δ_0 and Δ_1 , and impose either relative or absolute boundary conditions.

THEOREM 1.4. For $i = 1, 2$, let (M_i, g_i) be compact Einstein manifolds with smooth boundaries. Assume that $\tau_1 = \tau_2$ and that

$$\begin{aligned} \text{Spec}(\Delta_{0, \mathcal{B}})(M_1) &= \text{Spec}(\Delta_{0, \mathcal{B}})(M_2), \quad \text{and} \\ \text{Spec}(\Delta_{1, \mathcal{B}})(M_1) &= \text{Spec}(\Delta_{1, \mathcal{B}})(M_2) \end{aligned}$$

where \mathcal{B} denotes either relative or absolute boundary conditions. Then:

1. If ∂M_1 is totally geodesic, then ∂M_2 is totally geodesic.
2. If ∂M_1 is minimal, then ∂M_2 is minimal.

3. If ∂M_1 is totally umbilic, then ∂M_2 is totally umbilic.
4. If ∂M_1 is strongly totally umbilic, then ∂M_2 is strongly totally umbilic.

Here is a brief outline to the remainder of this paper. In Section 2, we review some facts concerning boundary geometry which we shall need. In Section 3, we recall some previous results concerning the heat trace asymptotics. In Section 4, we use these results to complete the proof of Theorems 1.3 and 1.4.

2. The geometry of the boundary

Central to our proof of Theorems 1.3 and 1.4 is the following integral characterization of certain geometric properties. Let dy denote the Riemannian measure on the boundary and let dx denote the Riemannian measure on the interior. To simplify the notation, let

$$f[M] = \int_M f(x)dx \quad \text{and} \quad f[\partial M] = \int_{\partial M} f(y)dy$$

where f is a scalar function.

THEOREM 2.1. *Let M be a compact m dimensional Riemannian manifold with smooth boundary ∂M .*

1. ∂M is totally geodesic if and only if $L_{ab}L_{ab}[\partial M] = 0$.
2. ∂M is minimal if and only if $L_{aa}L_{bb}[\partial M] = 0$.
3. ∂M is totally umbilic if and only if

$$\{(m-1)L_{ab}L_{ab} - L_{aa}L_{bb}\}[\partial M] = 0.$$

4. ∂M is strongly totally umbilic if and only if there exists a constant μ so that $\{L_{ab}L_{ab} - 2\mu L_{aa} + \mu^2(m-1)\}[\partial M] = 0$.

Proof. The first two assertions are immediate. To prove Assertion (3), we let $\{\kappa_1(y), \dots, \kappa_{m-1}(y)\}$ be the eigenvalues of the second fundamental form at a point y of the boundary. Then the second fundamental form is umbilic at y if and only if $\kappa_1(y) = \dots = \kappa_{m-1}(y)$ or equivalently if

$$0 = \sum_{i < j} (\kappa_i - \kappa_j)^2.$$

Assertion (3) now follows since we have that

$$\begin{aligned} \sum_i \kappa_i(y)^2 &= L_{ab}L_{ab}(y), \\ \sum_{i,j} \kappa_i(y)\kappa_j(y) &= L_{aa}L_{bb}(y), \quad \text{and} \\ 0 &\leq \sum_{i < j} (\kappa_i(y) - \kappa_j(y))^2 = (m-1)L_{ab}L_{ab}(y) - L_{aa}L_{bb}(y). \end{aligned}$$

Finally, to prove assertion (4), we note that the second fundamental form is μ times the identity at a point y of the boundary if and only if

$$0 = |L - \mu \text{id}|^2 = L_{ab}(y)L_{ab}(y) - 2\mu L_{aa}(y) + (m - 1)\mu^2.$$

Since $|L - \mu \text{id}|^2$ is non-negative, Assertion (4) now holds. □

3. Heat trace asymptotics

To deal with Dirichlet, Neumann, and absolute boundary conditions in a common framework, it is useful to introduce the more general notion of *mixed boundary conditions*. Let χ be a self-adjoint endomorphism of $\Lambda^p(M)|_{\partial M}$ so that $\chi^2 = \text{id}$. Let Π_{\pm} be orthonormal projection on the ± 1 eigenspaces of χ . Let S be an auxiliary endomorphism of range Π_+ . The mixed boundary operator $\mathcal{B}_{\chi,S}$ is then defined by

$$\mathcal{B}_{\chi,S}\phi := \{\Pi_+(\phi_{;m} + S\phi)\}|_{\partial M} \oplus \{\Pi_-\phi\}|_{\partial M}.$$

EXAMPLE 3.1. Let $\mathcal{B} = \mathcal{B}_{\chi,S}$.

1. If we take $\chi = -\text{id}$, then \mathcal{B} defines Dirichlet boundary conditions.
2. If we take $\chi = \text{id}$, then \mathcal{B} defines Neumann boundary conditions.
3. Let $\text{ext}(e_i)$ denote left exterior multiplication by the covector e_i and let $\text{int}(e_i)$ be the dual operation, left interior multiplication by the covector e_i . Let Π_+ be projection on $\Lambda(\partial M)$, let Π_- be projection on $\Lambda(\partial M)^\perp$, and let

$$S = -\Pi_+ \text{ext}(e_a) \text{int}(e_b) L_{ab} \Pi_+.$$

Then $\mathcal{B}_{\chi,S}$ defines absolute boundary conditions, see, for example, the discussion in [2]. We note for future reference that

$$\chi_{;a} = 2L_{ab} \{ \text{ext}(e_b) \text{int}(e_m) + \text{ext}(e_m) \text{int}(e_b) \}.$$

Let $\mathcal{B} = \mathcal{B}_{\chi,S}$ and let $e^{-t\Delta_{p,\mathcal{B}}}$ be the fundamental solution of the heat equation. The pseudo-differential calculus established by Seeley [5, 6] shows operator is of trace class and as $t \downarrow 0$ there is a complete asymptotic expansion with locally computable coefficients in the form:

$$\text{Tr}_{L^2} e^{-t\Delta_{p,\mathcal{B}}} \sim \sum_{n \geq 0} t^{(n-m)/2} a_n(\Delta_{p,\mathcal{B}}).$$

Let ‘;’ denote multiple covariant differentiation. The Weitzenböck formula permits us to express

$$\Delta_p \omega = -(\omega_{;kk} + E_p \omega)$$

where E_p is a suitably chosen expression in the curvature tensor. For example, we have that

$$(3.1) \quad E_0 = 0 \quad \text{and} \quad E_1(e_i) = -\rho_{ij}e_j.$$

The following result is a special case of a more general result established by Branson and Gilkey [1].

THEOREM 3.2. *Let M be a compact Riemannian manifold which has a smooth boundary ∂M . Let $\mathcal{B} = \mathcal{B}_{\chi,S}$ define mixed boundary conditions on $\Lambda^p(M)$.*

1. $a_0(\Delta_{p,\mathcal{B}}) = (4\pi)^{-m/2} \text{Tr}\{\text{id}\}[M].$
2. $a_1(\Delta_{p,\mathcal{B}}) = (4\pi)^{-(m-1)/2} \frac{1}{4} \text{Tr}\{\chi\}[\partial M].$
3. $a_2(\Delta_{p,\mathcal{B}}) = (4\pi)^{-m/2} \frac{1}{6} \{ \text{Tr}\{6E_p + \tau\}[M] + \text{Tr}\{2L_{aa} + 12S\}[\partial M] \}.$
4. $a_3(\Delta_{p,\mathcal{B}}) = (4\pi)^{-(m-1)/2} \frac{1}{384} \text{Tr}\{96\chi E_p + 16\chi\tau - 8\chi\rho_{mm}$
 $+ [13\Pi_+ - 7\Pi_-]L_{aa}L_{bb} + [2\Pi_+ + 10\Pi_-]L_{ab}L_{ab} + 96SL_{aa}$
 $+ 192S^2 - 12\chi_{;a}\chi_{;a}\}[\partial M].$

4. Proof of Theorems 1.3 and 1.4

Let \mathcal{B} denote Dirichlet, Neumann, absolute or relative boundary conditions. The heat trace asymptotics $a_n(\Delta_{p,\mathcal{B}})$ are spectral invariants. Consequently by Theorem 3.2,

$$\{\text{vol}(M), \text{vol}(\partial M)\}$$

are spectral invariants. We have fixed the Einstein constant and set the scalar curvature $\tau = c$. Thus

$$\tau[M], \quad \tau[\partial M], \quad \text{and} \quad \rho_{mm}[\partial M]$$

are spectral invariants as well. The formula for a_2 then shows that

$$L_{aa}[\partial M]$$

is spectrally determined. In light of Theorem 2.1, to complete the proof of Theorems 1.3 and 1.4 it suffices to show

$$(4.1) \quad L_{aa}L_{bb}[\partial M] \quad \text{and} \quad L_{ab}L_{ab}[\partial M]$$

are spectrally determined by $\{\Delta_{0,\mathcal{B}_D}, \Delta_{0,\mathcal{B}_N}\}$, by $\{\Delta_{0,\mathcal{B}_a}, \Delta_{1,\mathcal{B}_a}\}$, or by $\{\Delta_{0,\mathcal{B}_r}, \Delta_{1,\mathcal{B}_r}\}$.

We shall suppress the coefficients of certain invariants in what follows since they define invariants which are already known to be spectrally determined; we denote such coefficients by a generic symbol \star . We

use the discussion in Example 3.1, the formulae in Equation 3.1, and Theorem 3.2 to compute:

$$\begin{aligned}
 a_3(\Delta_0, \mathcal{B}_D) &= (4\pi)^{(1-m)/2} \frac{1}{384} \{ \star\tau + \star\rho_{mm} - 7L_{aa}L_{bb} \\
 &\quad + 10L_{ab}L_{ab} \} [\partial M], \\
 a_3(\Delta_0, \mathcal{B}_N) &= (4\pi)^{(1-m)/2} \frac{1}{384} \{ \star\tau + \star\rho_{mm} + 13L_{aa}L_{bb} \\
 &\quad + 2L_{ab}L_{ab} \} [\partial M].
 \end{aligned}$$

Since the coefficient matrix

$$\begin{pmatrix} -7 & 10 \\ 13 & 2 \end{pmatrix}$$

is non-singular, the invariants given in Equation (4.1) are spectral invariants as desired; Theorem 1.3 now follows.

To establish Theorem 1.4, we must perform similar computations for absolute and for relative boundary conditions.

Absolute boundary conditions are pure Neumann boundary conditions on 0 forms. By Example 3.1,

$$\begin{aligned}
 \text{Tr}_{\Lambda^1 M} \{ [13\Pi_+ - 7\Pi_-] L_{aa}L_{bb} \} &= (13m - 20)L_{aa}L_{bb}, \\
 \text{Tr}_{\Lambda^1 M} \{ [2\Pi_+ + 10\Pi_-] L_{ab}L_{ab} \} &= (2m + 8)L_{ab}L_{ab}, \\
 \text{Tr}_{\Lambda^1 M} \{ 96SL_{aa} \} &= -96L_{aa}L_{bb}, \\
 \text{Tr}_{\Lambda^1 M} \{ 192S^2 \} &= 192L_{ab}L_{ab}, \\
 \text{Tr}_{\Lambda^1 M} \{ -12\chi_{:a}\chi_{:a} \} &= -96L_{ab}L_{ab}.
 \end{aligned}$$

It is now an easy matter to use Theorem 3.2 to see that

$$\begin{aligned}
 a_3(\Delta_0, \mathcal{B}_a) &= (4\pi)^{(1-m)/2} \frac{1}{384} \{ \star\tau + \star\rho_{mm} + 13L_{aa}L_{bb} \\
 &\quad + 2L_{ab}L_{ab} \} [\partial M], \\
 a_3(\Delta_1, \mathcal{B}_a) &= (4\pi)^{(1-m)/2} \frac{1}{384} \{ \star\tau + \star\rho_{mm} + (13m - 116)L_{aa}L_{bb} \\
 &\quad + (2m + 104)L_{ab}L_{ab} \} [\partial M].
 \end{aligned}$$

The desired result for absolute boundary conditions now follows as the determinant of the coefficient matrix

$$\begin{pmatrix} 13 & 2 \\ 13m - 116 & 2m + 104 \end{pmatrix}$$

is 1584 which is different from zero.

Relative boundary conditions are Dirichlet boundary conditions on 0 forms. We use duality to see relative boundary conditions on 1 forms have the same spectral asymptotics as absolute boundary conditions on

$m - 1$ forms. On $m - 1$ forms, we compute:

$$\begin{aligned}\mathrm{Tr}_{\Lambda^{m-1}M}\{[13\Pi_+ - 7\Pi_-]L_{aa}L_{bb}\} &= (-7m + 20)L_{aa}L_{bb}, \\ \mathrm{Tr}_{\Lambda^{m-1}M}\{[2\Pi_+ + 10\Pi_-]L_{ab}L_{ab}\} &= (10m - 8)L_{ab}L_{ab}, \\ \mathrm{Tr}_{\Lambda^{m-1}M}\{96S L_{aa}\} &= -96L_{aa}L_{bb}, \\ \mathrm{Tr}_{\Lambda^{m-1}M}\{192S^2\} &= 192L_{aa}L_{bb}, \\ \mathrm{Tr}_{\Lambda^{m-1}M}\{-12\chi_{:a}\chi_{:a}\} &= -96L_{ab}L_{ab}.\end{aligned}$$

It is now an easy matter to use Theorem 3.2 to see that

$$\begin{aligned}a_3(\Delta_{0,\mathcal{B}_r}) &= (4\pi)^{(1-m)/2} \frac{1}{384} \{ \star\tau + \star\rho_{mm} - 7L_{aa}L_{bb} \\ &\quad + 10L_{ab}L_{ab} \} [\partial M], \\ a_3(\Delta_{1,\mathcal{B}_r}) &= (4\pi)^{(1-m)/2} \frac{1}{384} \{ \star\tau + \star\rho_{mm} + (-7m + 116)L_{aa}L_{bb} \\ &\quad + (10m + 104)L_{ab}L_{ab} \} [\partial M].\end{aligned}$$

The coefficient matrix

$$\begin{pmatrix} & -7 & 10 \\ -7m + 116 & 10m - 104 & \end{pmatrix}$$

has determinant -432 which again is different from zero. \square

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