

ON SOME GRONWALL TYPE INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT. The aim of the present paper is to establish some non-linear integral inequalities in two independent variables which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in the qualitative theory of certain partial differential equations.

1. Introduction

The integral inequalities involving functions of one and more than one independent variables which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential equations.

Let $u : [\alpha, \alpha + h] \rightarrow R$ be a continuous real-valued function satisfying the inequality

$$0 \leq u(t) \leq \int_{\alpha}^t [a + bu(s)] ds \quad \text{for } t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \leq ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by T. H. Gronwall [8] in the year 1919, and is the prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several publications on this subject, the paper of Bellman

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[3] is very well known: Let $x(t)$ and $k(t)$ be real valued nonnegative continuous functions for $t \geq \alpha$. If a is a constant, $a \geq 0$, and

$$x(t) \leq a + \int_{\alpha}^t k(s)x(s) ds, \quad t \geq \alpha,$$

then

$$x(t) \leq a \exp\left(\int_{\alpha}^t k(s) ds\right), \quad \text{for } t \geq \alpha.$$

It is clear that Bellman's result contains that of Gronwall. This is the reason why inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of Gronwall type". The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [8] and Guiliano [9]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Some applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [13], Bihari [4], and Langenhop [10]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

In [14], Pachpatte proved the following interesting integral inequality: Let $u(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$ be nonnegative continuous functions defined for $x, y \in R_+$, assume that $a(x, y)$ is nondecreasing in $x \in R_+$. If

$$(1.1) \quad u(x, y) \leq a(x, y) + \int_0^x b(s, y)u(s, y) ds + \int_0^x \int_y^{\infty} c(s, t)u(s, t) dt ds,$$

for $x, y \in R_+$, then

$$u(x, y) \leq p(x, y) \left[a(x, y) + A(x, y) \exp\left(\int_0^x \int_0^y v(r, s) dr ds\right) \right]$$

for $x, y \in R_+$, where

$$p(x, y) = \exp\left(\int_0^x b(s, y) ds\right),$$

$$A(x, y) = \int_0^x \int_y^\infty c(s, t)p(s, t)a(s, t) dt ds.$$

In this paper we obtain bounds in the inequality (1.1) for function of two independent variables when the function $u(x, y)$ in the right-hand side of the inequality (1.1) is replaced by the function $u^p(x, y)$ for $p \geq 0, p \neq 1$. We also provide some integral inequalities and some applications of these integral inequalities for finding the boundedness of the solutions to hyperbolic partial differential equations.

2. Integral inequalities

In this section we state and prove some new nonlinear integral inequalities in two independent variables. Throughout the paper, all the functions which appear in the inequalities are assumed to be realvalued and all the integrals are involved in existence on the domains of their definitions. We shall introduce some notation: R denotes the set of real numbers and $R_+ = [0, \infty)$ is the given subset of R . The first order partial derivatives of a functions $z(x, y)$ defined for $x, y \in R$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$ respectively.

THEOREM 2.1. *Let $u(x, y), a(x, y), b(x, y), c(x, y)$ be nonnegative continuous functions for $x, y \in R_+$, and let $a(x, y)$ be nondecreasing in each of the variables for $x, y \in R_+$. Suppose that*

(2.1)

$$u(x, y) \leq a(x, y) + \int_0^x b(s, y)u^p(s, y) ds + \int_0^x \int_y^\infty c(s, t)u^p(s, t) dt ds$$

for $x, y \in R_+$, where $p \geq 0, p \neq 1$ be a constant,

$$\int_0^x \int_y^\infty c(s, t) dt ds < \infty$$

and $\int_0^x b(s, y)u^p(s, y) ds$ is nonincreasing in $y \in R$. Then

(2.2)

$$u(x, y) \leq \left[a^q(x, y) + q \int_0^x b(s, y) ds + q \int_0^x \int_y^\infty c(s, t) dt ds \right]^{1/q}$$

for $x \in [0, X), y \in [0, Y)$, where $q = 1-p, X$ and Y are chosen so that the expression between [...] is positive in the subintervals $[0, X)$ and $[0, Y)$.

Proof. Let $X > 0$ and $Y > 0$ be fixed. Then for $0 \leq x \leq X, 0 \leq y \leq Y$ we have

(2.3)

$$u(x, y) \leq a(X, Y) + \int_0^x b(s, y)u^p(s, y) ds + \int_0^x \int_y^\infty c(s, t)u^p(s, t) dt ds.$$

Define a function $v(x, y)$ by the right-hand side of (2.3). Then the function $v(x, y)$ is nonincreasing in y , $u(x, y) \leq v(x, y)$, $v(0, y) = a(X, Y)$ and

$$\begin{aligned} \frac{\partial v(x, y)}{\partial x} &= b(x, y)u^p(x, y) + \int_y^\infty c(x, t)u^p(x, t) dt \\ &\leq b(x, y)v^p(x, y) + \int_y^\infty c(x, t)v^p(x, t) dt \\ (2.4) \quad &\leq \left(b(x, y) + \int_y^\infty c(x, t) dt \right) v^p(x, y), \end{aligned}$$

since $u(x, t) \leq v(x, t) \leq v(x, y)$. Define a function $z(x, y)$ by $z(x, y) = v^q(x, y)/q$. Then, from (2.4) we have

$$\begin{aligned} \frac{\partial z}{\partial x}(x, y) &= v^{q-1}(x, y) \frac{\partial v}{\partial x}(x, y) \\ &\leq v^{q-1}(x, y) \left(b(x, y) + \int_y^\infty c(x, t) dt \right) v^p(x, y) \\ (2.5) \quad &= b(x, y) + \int_y^\infty c(x, t) dt, \end{aligned}$$

where $q = 1 - p$. Integrating (2.5) over s from 0 to x , and the change of variable yields

$$z(x, y) \leq \frac{1}{q}v^q(0, y) + \int_0^x b(s, y) ds + \int_0^x \int_y^\infty c(s, t) dt ds,$$

or

$$v^q(x, y) \begin{cases} \leq \\ \geq \end{cases} a^q(X, Y) + q \int_0^x b(s, y) ds + q \int_0^x \int_y^\infty c(s, t) dt ds,$$

where \leq (respectively, \geq) holds for $q > 0$ (respectively, $q < 0$). In both cases this estimate implies

$$v(x, y) \leq \left[a^q(X, Y) + q \int_0^x b(s, y) ds + q \int_0^x \int_y^\infty k(s, t) dt ds \right]^{1/q}$$

for $0 \leq x \leq X, 0 \leq y \leq Y$. Setting $x = X$ and $y = Y$ and changing notation we arrive at (2.2). \square

THEOREM 2.2. *Let $u(x, y), a(x, y), b(x, y), c(x, y)$ be nonnegative continuous functions for $x \geq 0, y \geq 0$, and let $a(x, y)$ be nondecreasing in each of the variables for $x \geq 0, y \geq 0$. Suppose that*

$$u(x, y) \leq a(x, y) + \int_0^x b(s, y)u^p(s, y) ds + \int_0^x \int_0^y c(s, t)u^p(s, t) dt ds$$

for $x \geq 0, y \geq 0$, where $p \geq 0, p \neq 1$ be a constant and $\int_0^x b(s, y)u^p(s, y) ds$ be nondecreasing in $y \geq 0$. Then

$$u(x, y) \leq \left[a^q(x, y) + q \left(\int_0^x b(s, y) ds + \int_0^x \int_0^y c(s, t) dt ds \right) \right]^{1/q}$$

for $x \in [0, X), y \in [0, Y)$, where $q = 1-p, X$ and Y are chosen so that the expression between [...] is positive in the subintervals $[0, X)$ and $[0, Y)$.

Proof. The proof of Theorem 2.2 follows by an argument similar to that given for the proof of Theorem 2.1 with some minor changes. \square

By a reasoning similar to the proof of Theorem 2.1 we also can prove the following assertion.

THEOREM 2.3. *Let $u(x, y), a(x, y), b(x, y), c(x, y)$ be nonnegative continuous functions in R_+^2 , and let $a(x, y)$ be nonincreasing in each of the variables in $x \geq 0, y \geq 0$. Suppose that*

$$u(x, y) \leq a(x, y) + \int_x^\infty b(s, y)u^p(s, y) ds + \int_x^\infty \int_y^\infty c(s, t)u^p(s, t) dt ds$$

for $x \geq 0, y \geq 0, p \geq 0, p \neq 1$ be a constant,

$$\int_x^\infty b(s, y) ds < \infty, \quad \int_x^\infty \int_y^\infty k(s, t) dt ds < \infty,$$

and $\int_x^\infty b(s, y)u^p(s, y) ds$ be nonincreasing in y . Then

$$u(x, y) \leq \left[a^q(x, y) + q \left(\int_x^\infty b(s, y) ds + \int_x^\infty \int_y^\infty c(s, t) dt ds \right) \right]^{1/q}$$

for $x \in [0, X), y \in [0, Y)$, where $q = 1-p, X$ and Y are chosen so that the expression between [...] is positive in the subintervals $[0, X)$ and $[0, Y)$.

3. Further integral inequalities

In this section we consider further nonlinear integral inequalities for functions of two independent variables. In what follows, $J_1 = [0, X)$ and $J_2 = [0, Y)$ are given subsets of real numbers R , and denote by $\Delta = J_1 \times J_2$. The first order partial derivatives of $z(x, y)$ defined for $x, y \in R$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$, respectively.

LEMMA 3.1. Let $a, b, u \in C(\Delta, R_+)$, and $k \geq 1$ be constant, and let

$$(3.1) \quad u(x, y) \leq k + \int_0^x a(s, y)u(s, y) ds + \int_0^x \int_0^y b(s, t)u(s, t) dt ds$$

for $(x, y) \in \Delta$, where $\int_0^x a(s, y)u(s, y) ds$ be nondecreasing in y , then

$$(3.2) \quad u(x, y) \leq k \exp\left(\int_0^x a(s, y) ds + \int_0^x \int_0^y b(s, t) dt ds\right)$$

for $(x, y) \in \Delta$.

Proof. Let $k \geq 1$, and define a function $z(x, y)$ by the right-hand side of (3.1). Then $z(x, y) \geq 1$, $z(0, y) = k$, $u(x, y) \leq z(x, y)$, and

$$\begin{aligned} z_x(x, y) &= a(x, y)u(x, y) + \int_0^y b(x, t)u(x, t) dt \\ &\leq z(x, y) \left(a(x, y) + \int_0^y b(x, t) dt \right). \end{aligned}$$

The last estimate reduces to the inequality

$$(3.3) \quad \frac{z_x(x, y)}{z(x, y)} \leq a(x, y) + \int_0^y b(x, t) dt.$$

Keeping y fixed in (3.3), setting $x = \sigma$, and integrating it with respect to σ from 0 to x , $x \in J_1$, and making the change of variable yields

$$(3.4) \quad z(x, y) \leq k \exp\left(\int_0^x a(s, t) ds + \int_0^x \int_0^y b(s, t) dt ds\right).$$

Using (3.4) in $u(x, y) \leq z(x, y)$, we get the inequality in (3.2). \square

THEOREM 3.2. *Let $a, b, c \in C(\Delta, R_+ - \{0\})$, $u \in C(\Delta, R_+)$ and $0 < p \leq 1$ be a constant. If*

$$u(x, y) \leq a(x, y) + \int_0^x b(s, y)u^p(s, y) ds + \int_0^x \int_0^y c(s, t)u^p(s, t) dt ds$$

for $x \geq 0, y \geq 0$, and $\int_0^x b(s, y)a^p(s, y) ds$ be nondecreasing in y , then

$$(3.5) \quad u(x, y) \leq a(x, y) + f_1(x, y) \exp(A_1(x, y) + B_1(x, y))$$

for $(x, y) \in \Delta$, where

(3.6)

$$f_1(x, y) = \int_0^x b(s, y)a^p(s, y) ds + \int_0^x \int_0^y c(s, t)a^p(s, t) dt ds,$$

$$A_1(x, y) = p \int_0^x b(s, t)a^{p-1}(s, t) ds,$$

$$B_1(x, y) = p \int_0^x \int_0^y c(s, t)a^{p-1}(s, t) dt ds$$

for $(x, y) \in \Delta$.

Proof. We deduce from the hypothesis on $u(x, y)$ that $u(x, y) \leq a(x, y) + z(x, y)$, where the function $z(x, y)$ is defined by

$$z(x, y) = \int_0^x b(s, y)u^p(s, y) dt ds + \int_0^x \int_0^y c(s, t)u^p(s, t) dt ds.$$

By applying some generalizations of Bernoulli's inequality $(1 + x)^a \leq 1 + ax$, where $0 < a \leq 1$ and $-1 < x$, it is easy to observe that

$$(3.7) \quad \begin{aligned} (a(x, y) + z(x, y))^p &\leq a^p(x, y) \left(1 + \frac{z(x, y)}{a(x, y)}\right)^p \\ &\leq a^p(x, y) + pa^{p-1}(x, y)z(x, y) \end{aligned}$$

for $0 < p \leq 1, a : \Delta \rightarrow R_+ - \{0\}$. From the definition of $z(x, y)$ and (3.7) we get

$$\begin{aligned} z(x, y) &\leq \int_0^x b(s, y)(a(s, y) + z(s, y))^p dt ds \\ &\quad + \int_0^x \int_0^y c(s, t)(a(s, t) + z(s, t))^p dt ds. \\ &\leq f_1(x, y) + p \int_0^x b(s, y)a^{p-1}(s, y)z(s, y) ds \\ &\quad + p \int_0^x \int_0^y c(s, t)a^{p-1}(s, t)z(s, t) dt ds, \end{aligned}$$

where the function $f_1(x, y)$ is defined by (3.6). First, we assume that $f(x, y) > 0$ for $(x, y) \in \Delta$. We get that

$$(3.8) \quad \frac{z(x, y)}{f_1(x, y)} \leq 1 + p \int_0^x b(s, y) a^{p-1}(s, y) \frac{z(s, y)}{f_1(s, y)} ds \\ + p \int_0^x \int_0^y c(s, t) a^{p-1}(s, t) \frac{z(s, t)}{f_1(s, t)} dt ds.$$

From the Lemma 3.1, the previous inequality (3.8) yields

$$(3.9) \quad \frac{z(x, y)}{f_1(x, y)} \\ \leq \exp \left(p \int_0^x b(s, y) a^{p-1}(s, y) ds + p \int_0^x \int_0^y c(s, t) a^{p-1}(s, t) dt ds \right).$$

Using inequality (3.9) in $u(x, y) \leq a(x, y) + z(x, y)$, we get the required inequality in (3.5). \square

LEMMA 3.3. Let $a, b, u \in C(\Delta, R_+)$, and $k \geq 1$ is a constant, and let

$$u(x, y) \leq k + \int_0^x a(s, y) u(s, y) ds + \int_0^x \int_y^\infty b(s, t) u(s, t) dt ds$$

for $(x, y) \in \Delta$, where $\int_0^x a(s, y) u(s, y) ds$ be nondecreasing in y and

$$\int_0^x \int_y^\infty b(s, t) dt ds < \infty,$$

then

$$u(x, y) \leq k \exp \left(\int_0^x a(s, y) ds + \int_0^x \int_y^\infty b(s, t) dt ds \right)$$

for $(x, y) \in \Delta$.

Proof. The proof of Lemma 3.3 follows by an argument similar to that given for the proof of Lemma 3.1 with some minor changes. \square

THEOREM 3.4. Let $a, b, c \in C(\Delta, R_+ - \{0\})$, $u \in C(\Delta, R_+)$ and $0 < p \leq 1$ is a constant. If

$$u(x, y) \leq a(x, y) + \int_0^x b(s, y) u^p(s, y) ds + \int_0^x \int_y^\infty c(s, t) u^p(s, t) dt ds$$

for $x, y \geq 0$, where $\int_0^x b(s, y)a^p(s, y) ds$ be nondecreasing in y and

$$\int_0^x \int_y^\infty c(s, t) dt ds < \infty,$$

then

$$u(x, y) \leq a(x, y) + f_2(x, y) \exp(A_1(x, y) + B_2(x, y))$$

for $(x, y) \in \Delta$, where

$$f_2(x, y) = \int_0^x b(s, y)a^p(s, y) ds + \int_0^x \int_y^\infty c(s, t)a^p(s, t) dt ds,$$

$$B_2(x, y) = p \int_0^x \int_y^\infty c(s, t)a^{p-1}(s, t) dt ds$$

for $(x, y) \in \Delta$ and $A_1(x, y)$ is defined by Theorem 3.2.

Proof. The proof of Theorem 3.4 follows by an argument similar to that given for the proof of Theorem 3.2 with some minor changes. □

By a reasoning similar to the proof of Lemma 3.1 we also can prove the following assertion.

LEMMA 3.5. Let $a, b \in C(\Delta, R_+)$, $u \in C(\Delta, R_+)$ and $k \geq 1$ is a constant, and let

$$u(x, y) \leq k + \int_x^\infty a(s, y)u(s, y) ds + \int_y^\infty \int_y^\infty b(s, t)u(s, t) dt ds$$

for $(x, y) \in \Delta$, where $\int_0^x a(s, y)u(s, y) ds$ be nonincreasing in y ,

$$\int_x^\infty a(s, y) ds < \infty \quad \text{and} \quad \int_y^\infty \int_y^\infty b(s, t) dt ds < \infty,$$

then

$$u(x, y) \leq k \exp\left(\int_x^\infty a(s, y) ds + \int_x^\infty \int_y^\infty b(s, t) dt ds\right)$$

for $(x, y) \in \Delta$.

By a reasoning similar to the proof of Theorem 3.2 we also can prove the following assertion.

THEOREM 3.6. Let $a, b, c \in C(\Delta, R_+ - \{0\})$, $u \in C(\Delta, R_+)$ and $0 < p \leq 1$ is a constant, and let

$$u(x, y) \leq a(x, y) + \int_x^\infty b(s, y)u^p(s, y) ds + \int_x^\infty \int_y^\infty c(s, t)u^p(s, t) dt ds$$

for $x \geq 0, y \geq 0$, $\int_0^x b(s, y)a^p(s, y) ds$ be nonincreasing in y ,

$$\int_x^\infty a(s, y) ds < \infty \quad \text{and} \quad \int_y^\infty \int_y^\infty b(s, t) dt ds < \infty,$$

then

$$u(x, y) \leq a(x, y) + f_3(x, y) \exp(A_2(x, y) + B_3(x, y))$$

for $(x, y) \in \Delta$, where

$$f_3(x, y) = \int_x^\infty b(s, y)a^p(s, y) ds + \int_x^\infty \int_y^\infty c(s, t)a^p(s, t) dt ds,$$

$$A_2(x, y) = p \int_x^\infty b(s, t)a^{p-1}(s, t) ds,$$

$$B_3(x, y) = p \int_x^\infty \int_y^\infty c(s, t)a^{p-1}(s, t) dt ds$$

for $(x, y) \in \Delta$.

4. Applications

In this section we present some immediate applications of Theorem 3.6 to study certain properties of solutions of the following terminal value problem for the hyperbolic partial differential equation

$$(4.1) \quad u_{xy}(x, y) = h(x, y, u(x, y)) + r(x, y),$$

$$(4.2) \quad u(x, \infty) = \sigma_\infty(x), u(\infty, y) = \tau_\infty(y), u(\infty, \infty) = k,$$

where $h : R_+^2 \times R \rightarrow R, r : R_+^2 \rightarrow R, \sigma_\infty, \tau_\infty(y) : R_+ \rightarrow R$ are continuous functions and k is a real constant. Our first aim is to derive the bound on the solution of the problem (4.1)-(4.2).

EXAMPLE 1. Suppose that the function h in (4.1) satisfies the condition

$$(4.3) \quad |h(x, y, u)| \leq c(x, y) |u|,$$

$$(4.4) \quad \left| \sigma_\infty(x) + \tau_\infty(y) - k + \int_x^\infty \int_y^\infty r(s, t) dt ds \right| \leq a(x, y) + \int_x^\infty b(s, y)u(s, y) ds,$$

where $a(x, y)$, $b(x, y)$ and $c(x, y)$ are as defined in Theorem 3.6. If $u(x, y)$ be a solution of (4.1) with the conditions (4.2), then it can be written as (see [1, p. 80])

$$(4.5) \quad u(x, y) = \sigma_\infty(x) + \tau_\infty(y) - k + \int_x^\infty \int_y^\infty (h(s, t, u(s, t)) + r(s, t)) dt ds$$

for $x, y \in R$. From (4.3), (4.4) and (4.5) we get

$$(4.6) \quad |u(x, y)| \leq a(x, y) + \int_x^\infty b(s, y)|u| ds + \int_x^\infty \int_y^\infty c(s, t)|u| dt ds.$$

Now, a suitable application of Theorem 3.6 to (4.6) yields the required estimate following

$$(4.7) \quad u(x, y) \leq a(x, y) + f(x, y) \exp\left(\int_x^\infty b(s, t) ds + \int_x^\infty \int_y^\infty c(s, t) dt ds\right)$$

for $(x, y) \in \Delta$, where

$$f(x, y) = \int_x^\infty b(s, y)a(s, y) ds + \int_x^\infty \int_y^\infty c(s, t)a(s, t) dt ds$$

for $(x, y) \in \Delta$. The right-hand side of (4.7) gives us the bound on the solution $u(x, y)$ of (4.1)-(4.2) in terms of the known functions. Thus, if the right-hand side of (4.7) is bounded, then we assert that the solution of (4.1)-(4.2) is bounded for $(x, y) \in \Delta$.

In the next we derive from Theorem 3.2 the boundedness of the solutions of the initial boundary value problem for partial differential equations of the form

$$(4.8) \quad u_{xy}(x, y) = f((x, y, u(x, y))),$$

$$(4.9) \quad u(x, 0) = a_1(x), u(0, y) = a_2(y), a_1(0) = a_2(0),$$

where $f \in C(\Delta \times R^2, R)$, $a_1 \in C^1(J_1, R)$ and $a_2 \in C^1(J_2, R)$.

EXAMPLE 2. Assume that $f : \Delta \times R^2 \rightarrow R$ is a continuous function for which there exist continuous positive functions $a(x, y), b(x, y)$ for $(x, y) \in \Delta$ such that

$$(4.10) \quad |f(x, y, u)| \leq c(x, y) |u|,$$

$$(4.11) \quad |a_1(x) + a_2(y) - a_1(0)| \leq a(x, y) + \int_0^x b(s, y) |u(s, y)| ds,$$

for $k : \Delta \rightarrow R_+ - \{0\}$, where $a(x, y), b(x, y), c(x, y)$ and $\int_0^x b(s, y)a(s, y) ds$ are as defined in Theorem 3.2. If $u(x, y)$ be a solution of (4.8) with the conditions (4.9), then it can be written as

$$(4.12) \quad u(x, y) = a_1(x) + a_2(y) - a_1(0) + \int_0^x \int_0^y f(s, t, u(s, t)) dt ds.$$

Using (4.10) and (4.11) in (4.12), we obtain

$$(4.13) \quad |u(x, y)| \leq a(x, y) + \int_0^x b(s, y)|u| ds + \int_0^x \int_0^y c(s, t)|u| dt ds.$$

Now, a suitable application of Theorem 3.2 to (4.13) yields the required estimate following

$$(4.14) \quad u(x, y) \leq a(x, y) + f(x, y) \exp\left(\int_0^x b(s, t) ds + \int_0^x \int_0^y c(s, t) dt ds\right)$$

for $(x, y) \in \Delta$, where

$$f(x, y) = \int_0^x b(s, y)a(s, y) ds + \int_0^x \int_0^y c(s, t)a(s, t) dt ds$$

for $(x, y) \in \Delta$. The right-hand side of (4.14) gives us the bound on the solution $u(x, y)$ of (4.8)-(4.9) in terms of the known functions. Thus, if the right-hand side of (4.14) is bounded, then we assert that the solution of (4.8)-(4.9) is bounded for $(x, y) \in \Delta$.

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