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A NOTE ON THE HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC EQUATION

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ABSTRACT. In this paper we prove the Hyers-Ulam-Rassias stability by considering the cases that the approximate remainder φ is defined by $f(x * y) + f(x * y^{-1}) - 2f(x) - 2f(y) = \varphi(x, y)$, $f(x*y*z)+f(x)+f(y)+f(z)-f(x*y)-f(y*z)-f(z*x) = \varphi(x, y, z)$, where $(G, *)$ is a group, X is a real or complex Hausdorff topological vector space, and f is a function from G into X.

1. Introduction

In 1940, S. M. Ulam [31] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, D. H. Hyers [5] proved that if $f : V \to X$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in V$, where V and X are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in V$.

Th. M. Rassias [21] gave a generalization of the Hyers' result in the following way:

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THEOREM 1.1. *Let $f : V \to X$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p < 1$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$(for all $x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique linear mapping $T : V \to X$ such that

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p$$

for all $x \in V$ (for all $x \in V \setminus \{0\}$ if $p < 0$).

Th. M. Rassias [27] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Z. Gajda [3] following the same approach as in Th. M. Rassias [21], gave an affirmative solution to Rassias' question for $p > 1$.

However, it was showed that a similar result for the case $p = 1$ does not hold (see [3, 28]). Recently, P. Găvruta [4] also obtained a further generalization of the Hyers-Rassias theorem (see also [6-11, 16, 19, 20, 22-24]).

Lee and Jun [17, 18] also obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$ (see also [14]):

In 1983, the stability theorem for the quadratic functional equation

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$$

was proved F. Skof [30] for the function $f : V \to X$. In 1984, P. W. Cholewa [1] extended the Skof's result to the case where V is an Abelian group G. In 1992, S. Czerwik [2] gave a generalization of the Skof-Cholewa's result in the following way:

THEOREM 1.2. *Let $p \neq 2$, $\theta > 0$ be real numbers. Suppose that the function $f : V \to X$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

Then there exists exactly one quadratic function $g : V \to X$ such that

$$\|f(x) - g(x)\| \leq c + k\theta\|x\|^p$$

for all x in V if $p \geq 0$ and for all $x \in V \setminus \{0\}$ if $p < 0$, where: when $p < 2$, $c = \frac{\|f(0)\|}{3}$, $k = \frac{2}{4-2^p}$ and g is given by (2.4) with g instead of q. When $p > 2$, $c = 0$, $k = \frac{2}{2^p-4}$ and $g(x) = \lim_{n\to\infty} 4^n f(2^{-n}x)$ for all x in V. Also, if the mapping $t \to f(tx)$ from R to X is continuous for each fixed x in V, then $g(tx) = t^2 g(x)$ for all t in R.

Since then, the stability problem of the quadratic equation have been extensively investigated by a number of mathematician ([25, 26, 29]). In 2001, Jun and Lee [13] proved the stability of the Pexiderized quadratic inequalities:

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \leq \varphi(x,y),$$
$$\|f(x+y) + g(x-y) - 2h(x) - 2k(y)\| \leq \varphi(x,y).$$

Throughout this paper, we denote by G a group and by X a real or complex Hausdorff topological space. By $\mathbb{N}$ we denote the set of positive integers. e stands for the unit of G, while it is 0 instead of e if G is an abelian group. W. Jian [12] obtained the Hyers-Ulam-Rassias stability theory by considering the cases where the approximate remainder φ is defined by

$$f(x*y) - f(x) - f(y) = 0 \quad (\forall x, y \in G),$$
$$f(x*y) - g(x) - h(y) = 0 \quad (\forall x, y \in G),$$
$$2f((x*y)^{1/2}) - f(x) - f(y) = 0 \quad (\forall x, y \in G),$$

where f, g, h are functions from G into X. In this paper, using the direct method, we obtain some generalizations of the Hyers-Ulam-Rassias stability of the following two kinds of the functional equations.

$$f(x*y) + f(x*y^{-1}) - 2f(x) - 2f(y) = 0 \text{ for all } x, y \in G, \tag{1.1}$$

$$\begin{aligned} &f(x*y*z) + f(x) + f(y) + f(z) \\ &- f(x*y) - f(y*z) - f(z*x) = 0 \text{ for all } x, y \in G. \end{aligned} \tag{1.2}$$

A function $Q : G \to X$ is called quadratic on G if $Q(x*y) + Q(x*y^{-1}) - 2Q(x) - 2Q(y) = 0$ and a function $A : G \to X$ is called additive on G if $A(x*y) - A(x) - A(y) = 0$.

2. The stability of the functional equation (1.1).

In this section, we prove the stability of the functional equation (1.1).

THEOREM 2.1. *Let $\varphi : G\backslash\{e\}\times G\backslash\{e\} \to X$ be a mapping satisfying the conditions*

$$\text{(T.1)} \qquad \lim_{n\to\infty} \frac{\varphi(x^{2^n}, y^{2^n})}{4^n} = 0$$

for all $x, y \in G\backslash\{e\}$ and

$$\text{(T.2)} \qquad \tilde{\varphi}(x^i, x^j) := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{4^{k+1}} \varphi(x^{i\cdot 2^k}, x^{j\cdot 2^k}) \in X$$

for all $x \in G\backslash\{e\}$ and for any fixed $i, j = 0, 1, 2, 3, \cdots$. Suppose that the function $f : G \to X$ satisfies

$$\text{(2.1)} \qquad f(x * y) + f(x * y^{-1}) - 2f(x) - 2f(y) = \varphi(x, y)$$

for all $x, y \in G\backslash\{e\}$ and

$$\text{(2.2)} \qquad f((x * y)^{2^n}) = f(x^{2^n} * y^{2^n})$$

for all $x, y \in G$ and $n \in \mathbb{N}$. Then the limit $Q(x) = \lim_{n\to\infty} f(x^{2^n})/4^n$ exists for all $x \in G$, and Q is quadratic. In this case, the equation

$$f(x) - f(e)/3 = Q(x) - \tilde{\varphi}(x, x)$$

holds for all $x \in G$.

Proof. Replacing y by x in (2.1), we easily obtain

$$f(x) - f(e)/3 = \frac{f(x^2) - f(e)/3}{4} - \frac{\varphi(x, x)}{4}$$

for all $x \in G\backslash\{e\}$. Replacing x by x^{2^n} and dividing by 4^n in the above equation, we have

$$\frac{f(x^{2^n}) - f(e)/3}{4^n} = \frac{f(x^{2^{n+1}}) - f(e)/3}{4^{n+1}} - \frac{1}{4^{n+1}} \varphi(x^{2^n}, x^{2^n})$$

for all $x \in G\backslash\{e\}$. From the above equation, we obtain

$$f(x) - f(e)/3 = \frac{f(x^{2^{n+1}}) - f(e)/3}{4^{n+1}} - \sum_{k=0}^{n} \frac{1}{4^{k+1}} \varphi(x^{2^k}, x^{2^k})$$

for all $x \in G\backslash\{e\}$. From (2.1), we can take the limit in the above equation as $n \to \infty$, and $\lim_{n\to\infty} \frac{f(x^{2^{n+1}})-f(e)/3}{4^{n+1}} \in X$ exists for all $x \in G\backslash\{e\}$ for all $n \in \mathbb{N}$. In this case, the equation

$$f(x) - f(e)/3 = \lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - f(e)/3}{4^{n+1}} - \tilde{\varphi}(x, x)$$

holds for all $x \in G$. Let

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n}$$

for all $x \in G$. Replacing x by x^{2^n} and dividing by 4^n in (2.1), we have

$$\frac{f(x^{2^n} * y^{2^n})}{4^n} + \frac{f(x^{2^n} * y^{-2^n})}{4^n} - \frac{2f(x^{2^n})}{4^n} - \frac{2f(y^{2^n})}{4^n} = \frac{\varphi(x^{2^n}, y^{2^n})}{4^n},$$

for all $x, y \in G\backslash\{e\}$ and for all $n \in \mathbb{N}$. Taking the limit in the above equation as $n \to \infty$, we obtain

$$Q(x * y) + Q(x * y^{-1}) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in G\backslash\{e\}$. Since $Q(e) = 0$, we easily obtain

$$Q(x * y) + Q(x * y^{-1}) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in G$. □

THEOREM 2.2. *Let $\varphi : G \times G \to X$ be a mapping satisfying the conditions* (T.1) *and* (T.2) *for all $x, y \in G$. Suppose that the function $f : G \to X$ satisfies*

$$f(x * y) + f(x * y^{-1}) - 2f(x) - 2f(y) = \varphi(x, y) \quad (\forall x, y \in G),$$

and the condition (2.2) *in Theorem 2.1 for all $x, y \in G$.*

Then the limit $Q(x) = \lim_{n\to\infty} f(x^{2^n})/4^n$ exists for any $x \in G$ and Q is quadratic. In this case, the equation

$$f(x) = Q(x) - \tilde{\varphi}(x, x) - \frac{\varphi(e, e)}{6}$$

holds for all $x \in G$.

3. The stability of the functional equation (1.2).

In this section, we prove the stability of the functional equation (1.2).

THEOREM 3.1. *Let G be a groupoid. Let $\varphi : G \setminus \{e\} \times G \setminus \{e\} \times G \setminus \{e\} \to X$ be a mapping satisfying the conditions*

$$\lim_{n\to\infty} \frac{\varphi(x^{2^n}, y^{2^n}, z^{2^n})}{4^n} = 0 \tag{3.1}$$

for all $x, y, z \in G\setminus\{e\}$ and

$$\tilde{\varphi}(x^i, x^j, x^l) := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{4^{k+1}} \varphi(x^{i\cdot 2^k}, x^{j\cdot 2^k}, x^{l\cdot 2^k}) \in X \tag{3.2}$$

for all $x \in G\setminus\{e\}$ and for any fixed $i, j, l = 0, 1, 2, 3, \cdots$. Suppose that the function $f : G \to X$ satisfies

$$f(x*y*z)+f(x)+f(y)+f(z)-f(x*y)-f(y*z)-f(z*x) = \varphi(x,y,z) \tag{3.3}$$

for all $x, y, z \in G$ and

$$f((x*y*z)^{2^n}) = f(x^{2^n} * y^{2^n} * z^{2^n}) \tag{3.4}$$

for all $x, y, z \in G$ and $n \in \mathbb{N}$. Then the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n}$$

exists for all $x \in G$ and Q satisfies generalized quadratic equation (1.2). In this case, the equation

$$Q(x) = [f(x^2) - 2f(x)] + \tilde{\varphi}(x^2, x, x) + 2\tilde{\varphi}(x, x, x)$$

holds for all $x \in G \setminus \{e\}$.

Proof. Replacing x, y, z by x, x, x in (3.3), we have

$$f(x^3) - 3f(x^2) + 3f(x) = \varphi(x, x, x) \tag{3.5}$$

for all $x \in G \setminus \{e\}$. Replacing x, y, z by x^2, x, x in (3.3) respectively, we have

$$f(x^4) + 2f(x) - 2f(x^3) = \varphi(x^2, x, x) \tag{3.6}$$

for all $x \in G \setminus \{e\}$. From (3.5) and (3.6), we obtain

$$f(x^4) - 6f(x^2) + 8f(x) = \varphi(x^2, x, x) + 2\varphi(x, x, x) \tag{3.7}$$

for all $x \in G \setminus \{e\}$. From (3.7), we know that

$$\frac{f(x^4) - 2f(x^2)}{4} - [f(x^2) - 2f(x)] = \frac{1}{4}[\varphi(x^2, x, x) + 2\varphi(x, x, x)].$$

By above equation, we obtain

$$\begin{aligned} &\frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n} - [f(x^2) - 2f(x)] \\ &= \sum_{k=1}^{n} \frac{\varphi(x^{2^k}, x^{2^{k-1}}, x^{2^{k-1}}) + 2\varphi(x^{2^{k-1}}, x^{2^{k-1}}, x^{2^{k-1}})}{4^k} \end{aligned}$$

for all $x \in G \setminus \{e\}$. By the above equation and (3.2), we can define $Q : G \to X$ by

$$\begin{aligned} Q(x) &= \lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n} \\ &= [f(x^2) - 2f(x)] + \tilde{\varphi}(x^2, x, x) + 2\tilde{\varphi}(x, x, x) \end{aligned}$$

for all $x \in G$. From (3.3), we easily obtain

$$\begin{aligned} &\frac{f(x^{2^{n+1}} * y^{2^{n+1}} * z^{2^{n+1}}) - 2f(x^{2^n} * y^{2^n} * z^{2^n})}{4^n} + \frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n} \\ &+ \frac{f(y^{2^{n+1}}) - 2f(y^{2^n})}{4^n} + \frac{f(z^{2^{n+1}}) - 2f(z^{2^n})}{4^n} \\ &- \frac{f(x^{2^{n+1}} * y^{2^{n+1}}) - 2f(x^{2^n} * y^{2^n})}{4^n} \\ &- \frac{f(y^{2^{n+1}} * z^{2^{n+1}}) - 2f(y^{2^n} * z^{2^n})}{4^n} \\ &- \frac{f(z^{2^{n+1}} * x^{2^{n+1}}) - 2f(z^{2^n} * x^{2^n})}{4^n} \\ = &\frac{\varphi(x^{2^{n+1}}, y^{2^{n+1}}, z^{2^{n+1}})}{4^{n+1}} + 2\frac{\varphi(x^{2^n}, y^{2^n}, z^{2^n})}{4^n} \end{aligned}$$

for all $x, y, z \in G\setminus\{e\}$. Taking the limit in the above equation as $n \to \infty$, we obtain

$$Q(x * y * z) + Q(x) + Q(y) + Q(z) - Q(x * y) - Q(y * z) - Q(z * x) = 0$$

for all $x, y, z \in G \setminus \{e\}$. Since $Q(e) = 0$, Q satisfies the equation (1.2) for all $x \in G$. □

THEOREM 3.2. *Let G be a groupoid and let E be a Banach space. Let $\varphi : G \setminus \{e\} \times G \setminus \{e\} \times G \setminus \{e\} \to \mathbb{R}$ be a mapping satisfying the conditions (3.1) and (3.2) for all $x, y, z \in G$. Suppose that the function $f : G \to E$ satisfies*

(3.8)
$$\|f(x*y*z)+f(x)+f(y)+f(z)-f(x*y)-f(y*z)-f(z*x)\| = \varphi(x,y,z)$$

for all $x, y, z \in G \setminus \{e\}$ and condition (3.4) in Theorem 3.1. Then the limit $Q(x) = \lim_{n\to\infty} f(x^{2^n})/4^n$ exists for any $x \in G$ and Q satisfies the generalized quadratic equation (1.2). In this case, the inequality

$$\|Q(x) - \frac{1}{2}[f(x^2) - 2f(x)]\| \le \frac{1}{2}\tilde{\varphi}(x^2, x, x) + \tilde{\varphi}(x, x, x)$$

holds for all $x \in G \setminus \{e\}$.

Proof. Replacing x, y, z by x, x, x in (3.8), we have

$$\|f(x^3) - 3f(x^2) + 3f(x)\| = \varphi(x, x, x) \tag{3.9}$$

for all $x \in G \setminus \{e\}$. Replacing x, y, z by x^2, x, x in (3.8), we have

$$\|f(x^4) + 2f(x) - 2f(x^3)\| = \varphi(x^2, x, x)$$

for all $x \in G \setminus \{e\}$. From (3.9) and the above equation, we know that

$$\|f(x^4) - 6f(x^2) + 8f(x)\| \le \varphi(x^2, x, x) + 2\varphi(x, x, x)$$

for all $x \in G \setminus \{e\}$. From the above equation, we know that

$$\|[f(x^4) - 4f(x^2)] - 2[f(x^2) - 4f(x)]\| \le \varphi(x^2, x, x) + 2\varphi(x, x, x)$$

for all $x \in G \setminus \{e\}$. From above equation, we know that

$$\begin{aligned}&\|[f(x^{2^{n+1}}) - 4f(x^{2^n})] - 2^n[f(x^2) - 4f(x)]\| \\ &\le \sum_{k=1}^{n} 2^{n-k}[\varphi(x^{2^k}, x^{2^{k-1}}, x^{2^{k-1}}) + 2\varphi(x^{2^{k-1}}, x^{2^{k-1}}, x^{2^{k-1}})]\end{aligned}$$

for all $x \in G \setminus \{e\}$. Dividing by 4^n in the above equation, we obtain

$$\begin{aligned}&\|\frac{f(x^{2^{n+1}}) - 4f(x^{2^n})}{4^n} - \frac{f(x^2) - 4f(x)}{2^n}\| \\ &\le \sum_{k=1}^{n} \frac{\varphi(x^{2^k}, x^{2^{k-1}}, x^{2^{k-1}}) + 2\varphi(x^{2^{k-1}}, x^{2^{k-1}}, x^{2^{k-1}})}{4^k}\frac{1}{2^{n-k}}\end{aligned} \tag{3.10}$$

for all $x \in G \setminus \{e\}$. Let

$$\psi(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\varphi(x^{2^k}, x^{2^{k-1}}, x^{2^{k-1}}) + 2\varphi(x^{2^{k-1}}, x^{2^{k-1}}, x^{2^{k-1}})}{4^k} \frac{1}{2^{n-k}}$$

for all $x \in G \setminus \{e\}$. Then

$$\begin{aligned}
&\psi(x) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\varphi(x^{2^k}, x^{2^{k-1}}, x^{2^{k-1}}) + 2\varphi(x^{2^{k-1}}, x^{2^{k-1}}, x^{2^{k-1}})}{4^k} \frac{1}{2^{n-k}} \\
&= \lim_{n\to\infty} \frac{\varphi(x^{2^n}, x^{2^{n-1}}, x^{2^{n-1}}) + 2\varphi(x^{2^{n-1}}, x^{2^{n-1}}, x^{2^{n-1}})}{4^n} \\
&\quad + \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n-1} \frac{\varphi(x^{2^k}, x^{2^{k-1}}, x^{2^{k-1}}) + 2\varphi(x^{2^{k-1}}, x^{2^{k-1}}, x^{2^{k-1}})}{4^k} \frac{1}{2^{n-1-k}} \\
&= 0 + \frac{\psi(x)}{2}
\end{aligned}$$

for all $x \in G \setminus \{e\}$. Hence we know that $\psi(x) = 0$ for all $x \in G \setminus \{e\}$. Taking the limit in (3.10) as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 4f(x^{2^n})}{4^n} = 0 \tag{3.11}$$

for all $x \in G \setminus \{e\}$. By Theorem 3.1, the limit

$$\lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n} \tag{3.12}$$

exists for all $x \in G \setminus \{e\}$ and the inequality

$$\| \lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n} - [f(x^2) - 2f(x)] \| \leq \tilde{\varphi}(x^2, x, x) + 2\tilde{\varphi}(x, x, x)$$

holds for all $x \in G$. By (3.11) and (3.12), the limit

$$Q(x) = \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n}$$

exists for all $x \in G$ and the inequality

$$\begin{aligned}
&\|Q(x) - \frac{1}{2}[f(x^2) - 2f(x)]\| \\
&= \| \lim_{n\to\infty} \frac{f(x^{2^n})}{4^n} - \frac{1}{2}[f(x^2) - 2f(x)]\| \\
&\le \frac{1}{2}\| \lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 4f(x^{2^n})}{4^n}\| \\
&\quad + \frac{1}{2}\| \lim_{n\to\infty} \frac{f(x^{2^{n+1}}) - 2f(x^{2^n})}{4^n} - [f(x^2) - 2f(x)]\| \\
&\le \frac{1}{2}\tilde{\varphi}(x^2, x, x) + \tilde{\varphi}(x, x, x)
\end{aligned}$$

holds for all $x \in G$. From (3.8), we easily obtain

$$\begin{aligned}
&\|\frac{f(x^{2^n} * y^{2^n} * z^{2^n})}{4^n} + \frac{f(x^{2^n})}{4^n} + \frac{f(y^{2^n})}{4^n} \\
&+ \frac{f(z^{2^n})}{4^n} - \frac{f(x^{2^n} * y^{2^n})}{4^n} - \frac{f(y^{2^n} * z^{2^n})}{4^n} - \frac{f(z^{2^n} * x^{2^n})}{4^n}\| \\
&= \frac{\varphi(x^{2^n}, y^{2^n}, z^{2^n})}{4^n}
\end{aligned}$$

for all $x, y, z \in G\backslash\{e\}$. Taking the limit in the above equation as $n \to \infty$, we obtain

$$Q(x * y * z) + Q(x) + Q(y) + Q(z) - Q(x * y) - Q(y * z) - Q(z * x) = 0$$

for all $x, y, z \in G \setminus \{e\}$. Since $Q(e) = 0$, we easily obtain

$$Q(x * y * z) + Q(x) + Q(y) + Q(z) - Q(x * y) - Q(y * z) - Q(z * x) = 0$$

for all $x, y, z \in G$. □

THEOREM 3.3. *Let G be a group and let X be a topological vector space. Let $\varphi : G \setminus \{e\} \times G \setminus \{e\} \times G \setminus \{e\} \to X$ be a mapping satisfying the conditions* (3.1) *and* (3.2) *for all $x, y, z \in G \setminus \{e\}$. Suppose that the function $f : G \to X$ satisfies the equation* (3.3) *and the condition* (3.4) *in Theorem 3.1. Then the limit $Q(x) = \lim_{n\to\infty}[f(x^{2^n}) + f(x^{-2^n})]/(2\cdot 4^n)$ exists for all $x \in G$ and Q is quadratic. In this case, the equation*

$$Q(x) = \frac{1}{2}[f(x) + f(x^{-1})] - \frac{2}{3}f(e) + \frac{\tilde{\varphi}(x, x, x^{-1}) - \tilde{\varphi}(x, x^{-1}, x^{-1})}{2}$$

holds for all $x \in G \setminus \{e\}$.

Proof. Let $f_1(x) = \frac{1}{2}[f(x) + f(x^{-1})]$. From the equation (3.3), we easily obtain

$$
\begin{aligned}
&f_1(x*y*z) + f_1(x) + f_1(y) + f_1(z) - f_1(x*y) - f_1(y*z) \\
&- f_1(z*x) \\
&= \frac{1}{2}[\varphi(x,y,z) + \varphi(z^{-1},y^{-1},x^{-1})]
\end{aligned} \tag{3.13}
$$

for all $x, y, z \in G\setminus\{e\}$ and we know that the equations $f_1(x) = f_1(x^{-1})$, $f_1(e) = f(e)$ for all $x \in G$ from the definition of f_1. Replacing z by x^{-1} in (3.13), we have

$$
\begin{aligned}
&f_1(x*y) + f_1(x*y^{-1}) - 2f_1(x) - f_1(y) - f_1(x*y*x^{-1}) + f(e) \\
&= -\frac{1}{2}[\varphi(x,y,x^{-1}) + \varphi(x,y^{-1},x^{-1})]
\end{aligned} \tag{3.14}
$$

for all $x, y \in G\setminus\{e\}$.

Replacing y by x in (3.14), we have

$$
f_1(x^2) - 4f_1(x) + 2f(e) = -\frac{1}{2}[\varphi(x,x,x^{-1}) + \varphi(x,x^{-1},x^{-1})] \tag{3.15}
$$

for all $x \in G\setminus\{e\}$.

Dividing by 4 in (3.15), we have

$$
f_1(x) - \frac{2}{3}f(e) - \frac{1}{4}[f_1(x^2) - \frac{2}{3}f(e)] = \frac{1}{8}[\varphi(x,x,x^{-1}) + \varphi(x,x^{-1},x^{-1})] \tag{3.16}
$$

for all $x \in G\setminus\{e\}$. Replacing x by x^{2^n} and dividing by 4^n in (3.16), we have

$$
\begin{aligned}
&\frac{f_1(x^{2^n}) - \frac{2}{3}f(e)}{4^n} - \frac{f_1(x^{2^{n+1}}) - \frac{2}{3}f(e)}{4^{n+1}} \\
&= \frac{\varphi(x^{2^n},x^{2^n},x^{-2^n}) + \varphi(x^{2^n},x^{-2^n},x^{-2^n})}{2\cdot 4^{n+1}}
\end{aligned}
$$

for all $n \in N$ and $x \in G\setminus\{e\}$. Induction argument implies

$$
\begin{aligned}
&f_1(x) - \frac{2}{3}f(e) - \frac{f_1(x^{2^n}) - \frac{2}{3}f(e)}{4^n} \\
&= \sum_{n=0}^{\infty} \frac{\varphi(x^{2^n},x^{2^n},x^{-2^n}) + \varphi(x^{2^n},x^{-2^n},x^{-2^n})}{2\cdot 4^{n+1}}
\end{aligned} \tag{3.17}
$$

for all $n \in N$ and $x \in G \setminus \{e\}$. Taking the limit in (3.17) as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{f_1(x^{2^n}) - \frac{2}{3}f(e)}{4^n} = f_1(x) - \frac{2}{3}f(e) - \frac{\tilde{\varphi}(x,x,x^{-1}) + \tilde{\varphi}(x,x^{-1},x^{-1})}{2}$$

for all $x \in G \setminus \{e\}$. Therefore we can define $Q : G \to X$ by

$$Q(x) = \lim_{n\to\infty} \frac{f_1(x^{2^n})}{4^n}$$

for all $x \in G$. From (3.4), we know that

$$\begin{aligned} f_1(x^{2^n} * y^{2^n} * x^{-2^n}) &= f_1((x^{2^k})^{2^{n-k}} * (y^{2^k})^{2^{n-k}} * (x^{-2^k})^{2^{n-k}}) \\ &= f_1((x^{2^k} * y^{2^k} * x^{-2^k})^{2^{n-k}}) \\ &= f_1(x^{2^k} * y^{2^n} * x^{-2^k}) \end{aligned} \tag{3.18}$$

for all $x, y \in G$ and for all $n \geq k = 0, 1, 2, 3, \cdots$. From the above equation, we know

$$f_1(x^2 * y^{2^n} * x^{-2}) = f_1(x * y^{2^n} * x^{-1})$$

for all $x, y \in G$. Replacing y by $x^{-1} * y * x$ in the above equation, we have

$$f_1(x * y^{2^n} * x^{-1}) = f_1(y^{2^n})$$

for all $x, y \in G$. From (3.14) and the definition of Q, we easily get

$$Q(x * y) + Q(x * y^{-1}) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in G \setminus \{e\}$. Since $Q(e) = 0$, we have

$$Q(x * y) + Q(x * y^{-1}) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in G$. □

THEOREM 3.4. *Let G be a group and let X be a topological vector space. Let $\varphi : G \setminus \{e\} \times G \setminus \{e\} \times G \setminus \{e\} \to X$ be a mapping satisfying the conditions*

$$\lim_{n\to\infty} \frac{\varphi(x^{2^n}, y^{2^n}, z^{2^n})}{2^n} = 0 \tag{3.19}$$

for all $x, y, z \in G\setminus\{e\}$ and

$$\hat{\varphi}(x^i, x^j, x^l) := \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^{k+1}} \varphi(x^{i\cdot 2^k}, x^{j\cdot 2^k}, x^{l\cdot 2^k}) \in X \tag{3.20}$$

for all $x \in G\setminus\{e\}$ and for any fixed $i, j, l = 0, 1, 2, 3, \cdots$. Suppose that the function $f : G \to X$ satisfies the conditions (3.3) and (3.4) in Theorem 3.1. Then the limit $T(x) = \lim_{n\to\infty}[f(x^{2^n}) - f(x^{-2^n})]/2^{n+1}$ exists for all $x \in G$ and T is additive. In this case, the equation

$$T(x) = \frac{1}{2}[f(x) - f(x^{-1})] + \frac{\hat{\varphi}(x^{-1}, x, x) - \hat{\varphi}(x^{-1}, x^{-1}, x)}{2}$$

holds for all $x \in G \setminus \{e\}$.

Proof. Let $f_2(x) = \frac{1}{2}[f(x) - f(x^{-1})]$. By the condition (3.3), f_2 satisfies

$$\begin{aligned} f_2(x*y*z) + f_2(x) + f_2(y) + f_2(z) - f_2(x*y) - f_2(y*z) - f_2(z*x) \\ = \frac{1}{2}[\varphi(x, y, z) - \varphi(z^{-1}, y^{-1}, x^{-1})] \end{aligned} \tag{3.21}$$

for all $x, y, z \in G \setminus \{e\}$ and $f_2(x) = -f_2(x^{-1}), f_2(e) = 0$ for all $x \in G$. Replacing x, y, z by x^{-1}, x, y in (3.21), we have

$$2f_2(y) - f_2(x*y) - f_2(y*x^{-1}) = \frac{1}{2}[\varphi(x^{-1}, x, y) - \varphi(y^{-1}, x^{-1}, x)] \tag{3.22}$$

for all $x, y \in G \setminus \{e\}$. Replacing y by x in (3.22), we have

$$2f_2(x) - f_2(x^2) = \frac{1}{2}[\varphi(x^{-1}, x, x) - \varphi(x^{-1}, x^{-1}, x)] \tag{3.23}$$

for all $x \in G \setminus \{e\}$.

Replacing x by x^{2^n} and dividing by 2^{n+1} in (3.23), we have
(3.24)
$$\frac{f_2(x^{2^n})}{2^n} - \frac{f_2(x^{2^{n+1}})}{2^{n+1}} = \frac{1}{2^{n+2}}[\varphi(x^{-2^n}, x^{2^n}, x^{2^n}) - \varphi(x^{-2^n}, x^{-2^n}, x^{2^n})]$$
for all $x \in G \setminus \{e\}$ and $n = 0, 1, 2, \cdots$. Replacing x by x^{2^n} and dividing by 2^n in (3.24), we have

$$\begin{aligned} f_2(x) - \frac{f_2(x^{2^{n+1}})}{2^{n+1}} &= \sum_{k=0}^{n} \frac{f_2(x^{2^k})}{2^k} - \frac{f_2(x^{2^{k+1}})}{2^{k+1}} \\ &= \sum_{k=0}^{n} \frac{\varphi(x^{-2^k}, x^{2^k}, x^{2^k}) - \varphi(x^{-2^k}, x^{-2^k}, x^{2^k})}{2 \cdot 2^{k+1}} \end{aligned}$$

for all $n \in N$ and $x \in G \setminus \{e\}$. Taking the limit in the above equation as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{f_2(x^{2^{n+1}})}{2^{n+1}} = f_2(x) + \frac{\hat{\varphi}(x^{-1}, x, x) - \hat{\varphi}(x^{-1}, x^{-1}, x)}{2}$$

for all $x \in G \setminus \{e\}$. Let $T : G \to X$ be a map defined by

$$T(x) = \lim_{n\to\infty} \frac{f_2(x^{2^n})}{2^n}$$

for all $x \in G$. From (3.22) and the definition of T, we easily obtain

$$T(x * y) + T(y^{-1} * x) - 2T(x) = T(x * y) + T(y^{-1} * x) - T(x^2) = 0$$

for all $x, y \in G \setminus \{e\}$. Replacing y by $x * y^2$ in the above equation, we have

$$T(x^2 * y^2) - T(y^2) - T(x^2) = 0$$

for all $x, y \in G \setminus \{e\}$. From (3.4), we know

$$\begin{aligned} T(x^2 * y^2) &= \lim_{n\to\infty} \frac{f_2((x^2 * y^2)^{2^n})}{2^n} = \lim_{n\to\infty} \frac{f_2(x^{2^{n+1}} * y^{2^{n+1}})}{2^n} \\ &= 2 \lim_{n\to\infty} \frac{f_2((x * y)^{2^{n+1}})}{2^{n+1}} = 2T(x * y) \end{aligned}$$

for all $x, y \in G$. From the above equation and $T(e) = 0$, we know that

$$T(x * y) - T(x) - T(y) = 0$$

for all $x \in G$. □

From Theorem 3.3 and Theorem 3.4, we obtain the following theorem.

THEOREM 3.5. *Let G be a group and let X be a topological vector space. Let $\varphi : G \setminus \{e\} \times G \setminus \{e\} \times G \setminus \{e\} \to X$ be a mapping satisfying the conditions (3.19) and (3.20) for all $x \in G \setminus \{e\}$. Suppose that the function $f : G \to X$ satisfies the conditions (3.3) and (3.4) in Theorem 3.1. Then there exist a quadratic function $Q : G \to X$ defined by $Q(x) = \lim_{n\to\infty}[f(x^{2^n}) + f(x^{-2^n})]/(2 \cdot 4^n)$ and an additive function $T : G \to X$ defined by $T(x) = \lim_{n\to\infty}[f(x^{2^n}) - f(x^{-2^n})]/2^{n+1}$ for all $x \in G$. In this case, the equation*

$$\begin{aligned} Q(x) + T(x) = f(x) - \frac{2}{3}f(e) - \frac{\tilde{\varphi}(x, x, x^{-1}) + \tilde{\varphi}(x, x^{-1}, x^{-1})}{2} \\ + \frac{\hat{\varphi}(x^{-1}, x, x) - \hat{\varphi}(x^{-1}, x^{-1}, x)}{2} \end{aligned}$$

holds for all $x \in G \setminus \{e\}$.

COROLLARY 3.6. *Let G be a group and let X be a topological vector space. Suppose that the function $f : G \to X$ satisfies the equation (1.2) for all $x, y, z \in G \setminus \{e\}$ and the condition (3.4) in Theorem 3.1. Then there exist a quadratic function $Q : G \to X$ defined by*

$$Q(x) = \begin{cases} [f(x) + f(x^{-1})]/2 - 2f(e)/3 & \text{for } x \in G \setminus \{e\} \\ 0 & \text{for } x = e \end{cases}$$

and an additive function $T : G \to X$ defined by $T(x) = [f(x) - f(x^{-1})]/2$ for all $x \in G$. In this case

$$Q(x) + T(x) = \begin{cases} f(x) - 2f(e)/3 & \text{for } x \in G \setminus \{e\} \\ 0 & \text{for } x = e. \end{cases}$$

Proof. Let $\varphi : G \setminus \{e\} \times G \setminus \{e\} \times G \setminus \{e\} \to X$ be a map defined by $\varphi(x) = 0$ for all $x \in G \setminus \{e\}$. In the proof of Theorem 3.3 and Theorem 3.4, we know that

$$Q(x) = \lim_{n\to\infty} [f(x^{2^n}) + f(x^{-2^n})]/(2 \cdot 4^n) = [f(x) + f(x^{-1})]/2 - 2f(e)/3$$

and

$$T(x) = \lim_{n\to\infty} [f(x^{2^n}) - f(x^{-2^n})]/(2 \cdot 2^n) = [f(x) - f(x^{-1})]/2$$

for all $x \in G \setminus \{e\}$. □

COROLLARY 3.7. *Let G be a group and let X be a topological vector space. Suppose that the function $f : G \to X$ satisfies the equation (1.2) for all $x, y, z \in G$ and the condition (3.4) in Theorem 3.1. Then there exist a quadratic function $Q : G \to X$ defined by $Q(x) = [f(x) + f(x^{-1})]/2$ and an additive function $T : G \to X$ defined by $T(x) = [f(x) - f(x^{-1})]/2$ for all $x \in G$. In this case, the equation $Q(x)+T(x) = f(x)$ holds for all $x \in G$.*

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