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# NILRADICALS OF SKEW POWER SERIES RINGS

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ABSTRACT. For a ring endomorphism $\sigma$ of a ring $R$, J. Krempa called $\sigma$ a rigid endomorphism if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is called rigid if there exists a rigid endomorphism of $R$. In this paper, we extend the $\sigma$-rigid property of a ring $R$ to the upper nilradical $N_r(R)$ of $R$. For an endomorphism $\sigma$ and the upper nilradical $N_r(R)$ of a ring $R$, we introduce the condition $(*)$: $N_r(R)$ is a $\sigma$-ideal of $R$ and $a\sigma(a) \in N_r(R)$ implies $a \in N_r(R)$ for $a \in R$. We study characterizations of a ring $R$ with an endomorphism $\sigma$ satisfying the condition $(*)$, and we investigate their related properties. The connections between the upper nilradical of $R$ and the upper nilradical of the skew power series ring $R[[x;\sigma]]$ of $R$ are also investigated.

Throughout this paper $R$ denotes an associative ring with identity. We use $\mathbf{P}(R)$, $N_r(R)$ and $\mathbf{N}(R)$ to represent the prime radical, the upper nilradical and the set of all nilpotent elements of $R$, respectively.

A ring $R$ is called *2-primal* [1] if $\mathbf{P}(R) = \mathbf{N}(R)$. Every reduced ring (i.e., $\mathbf{N}(R) = 0$) is obviously a 2-primal ring. Observe that $R$ is a 2-primal ring if and only if $\mathbf{P}(R) = N_r(R) = \mathbf{N}(R)$ if and only if $\mathbf{P}(R)$ is a completely semiprime ideal (i.e., $a^2 \in \mathbf{P}(R)$ implies $a \in \mathbf{P}(R)$ for $a \in R$) of $R$. Also, $N_r(R) = \mathbf{N}(R)$ if and only if $N_r(R)$ is completely semiprime. Hence the class of rings which satisfy $N_r(R) = \mathbf{N}(R)$ properly contains the class of 2-primal rings; while there exists a ring $R$ with $N_r(R) = \mathbf{N}(R)$ which is not 2-primal [2, Example 3.3]. We refer to [2, 3 and 4] for more details on 2-primal rings.

For an endomorphism $\sigma$ of a ring $R$, Krempa [6] called $\sigma$ a *rigid* endomorphism if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. We called $R$ a

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*$\sigma$-rigid* ring [4] if the endomorphism $\sigma$ of $R$ is rigid. Note that any rigid endomorphism is a monomorphism, and $\sigma$-rigid rings are reduced rings. But there exists an endomorphism of a reduced ring which is not rigid [4, Example 9]. However, if $\sigma$ is an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\sigma(r) = u^{-1}ru$ for any $r \in R$) of a reduced ring $R$, then $\sigma$ is rigid.

In this paper, we investigate the relationship between the upper nilradical $N_r(R)$ of a ring $R$ and the upper nilradical $N_r(R[[x;\sigma]])$ of the skew power series ring $R[[x;\sigma]]$ of $R$.

We shall always assume that $\sigma$ is an endomorphism of a given ring and it is a nonzero and non-identity endomorphism, unless especially noted.

## 1. $\sigma$-completely semiprime ideals

In this section, we introduce $\sigma$-completely semiprime ideals of a ring $R$, and then we investigate their equivalent conditions and related properties.

A ring $R$ is said to be *prime* if $AB \neq 0$ for any nonzero ideals $A$, $B$ of $R$. An ideal $P$ of $R$ is *prime* if $R/P$ is a prime ring. $R$ is said to be *strongly prime* if $R$ is prime with no nonzero nil ideals. An ideal $P$ of $R$ is *strongly prime* if $R/P$ is a strongly prime ring. An ideal $P$ of a ring $R$ is *minimal strongly prime* if $P$ is minimal among strongly prime ideals of $R$. We can show that there exists a minimal strongly prime ideal of a ring $R$ using Zorn's lemma. Observe that for a ring $R$, $N_r(R) = \{a \in R \mid (a) \text{ is a nil ideal of } R\} = \bigcap\{P \mid P \text{ is a strongly prime ideal of } R\} = \bigcap\{P \mid P \text{ is a minimal strongly prime ideal of } R\}$ [9].

An ideal $P$ of a ring $R$ is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Observe that every completely prime ideal of $R$ is strongly prime and every strongly prime ideal is prime, but the converses do not hold, in general.

Recall that an ideal $I$ of $R$ is called a *$\sigma$-ideal* if $\sigma(I) \subseteq I$, and $I$ is called *$\sigma$-invariant* if $\sigma^{-1}(I) = I$. Note that every $\sigma$-invariant ideal is a $\sigma$-ideal.

Let (m)Spec$_S(R)$ be the set of all (minimal) strongly prime ideals of a ring $R$.

PROPOSITION 1.1. *Let $R$ be a ring.*

(1) *If $P$ is $\sigma$-invariant for each $P \in$ (m)Spec$_S(R)$, then $N_r(R)$ is $\sigma$-invariant.*

(2) *If $P$ is a $\sigma$-ideal for each $P \in$ (m)Spec$_S(R)$, then $N_r(R)$ is a $\sigma$-ideal.*

(3) *If $R$ satisfies $N_r(R) = \mathbf{N}(R)$, then*

(i) *$N_r(R)$ is a $\sigma$-ideal of $R$, and*

(ii) *$N_r(R)$ is $\sigma$-invariant, where $\sigma$ is a monomorphism.*

*Proof.* (1) Let $a \in \sigma^{-1}(N_r(R))$. Then $\sigma(a) \in N_r(R) \subseteq P$ and so $a \in \sigma^{-1}(P) = P$ for all $P \in$ (m)Spec$_S(R)$. Thus $a \in N_r(R)$ and so $\sigma^{-1}(N_r(R)) \subseteq N_r(R)$. Now, if $a \in N_r(R)$, then $a \in P = \sigma^{-1}(P)$ and so $\sigma(a) \in P$ for all $P \in$ (m)Spec$_S(R)$. Thus $\sigma(a) \in N_r(R)$ and so $a \in \sigma^{-1}(N_r(R))$. Therefore $N_r(R) \subseteq \sigma^{-1}(N_r(R))$ and so $N_r(R)$ is $\sigma$-invariant.

(2) Let $a \in N_r(R)$. Then $a \in P$ for all $P \in$ (m)Spec$_S(R)$. Thus $\sigma(a) \in \sigma(P) \subseteq P$. Therefore $\sigma(a) \in N_r(R)$ and so $N_r(R)$ is a $\sigma$-ideal.

(3) Recall that a ring $R$ satisfies $N_r(R) = \mathbf{N}(R)$ if and only if $N_r(R)$ is a completely semiprime ideal of $R$.

(i) Let $a \in N_r(R)$. Then $a^n = 0$ for some positive integer $n$. Thus $(\sigma(a))^n = \sigma(a^n) = \sigma(0) = 0$, and so $\sigma(a) \in \mathbf{N}(R) = N_r(R)$. Therefore $N_r(R)$ is a $\sigma$-ideal.

(ii) By (i), we have $N_r(R) \subseteq \sigma^{-1}(N_r(R))$. Thus it suffices to show $\sigma^{-1}(N_r(R)) \subseteq N_r(R)$. Let $a \in \sigma^{-1}(N_r(R))$. Then $\sigma(a) \in N_r(R)$ and so $(\sigma(a))^n = 0$ for some positive integer $n$. Thus $\sigma(a^n) = 0 = \sigma(0)$ and so $a^n = 0$ because $\sigma$ is a monomorphism. Hence $a \in \mathbf{N}(R) = N_r(R)$. Thus $\sigma^{-1}(N_r(R)) \subseteq N_r(R)$. Consequently, $N_r(R)$ is $\sigma$-invariant. □

COROLLARY 1.2. *If $R$ is a 2-primal ring, then $\mathbf{P}(R)$ is a $\sigma$-ideal of $R$ and $\mathbf{P}(R)$ is $\sigma$-invariant when $\sigma$ is a monomorphism.*

In the next example, parts (1) and (2) show that the converses of Proposition 1.1(1) and (3)(i) do not hold, respectively; while part (3) illustrates that the converse of Proposition 1.1(2) does not hold, and that the condition "$\sigma$ is a monomorphism" in Proposition 1.1(3)(ii) is not superfluous.

EXAMPLE 1.3. (1) Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then $R$ is a commutative reduced ring. Define $\sigma : R \to R$ by $\sigma(a, b) = (b, a)$. Then $N_r(R) = \{(0, 0)\}$ is $\sigma$-invariant since $\sigma$ is an automorphism. However, $P = \{0\} \oplus \mathbb{Z}_2 \in$ (m)Spec$_S(R)$ is not $\sigma$-invariant: For $(0, 1) \in P$, $\sigma(0, 1) \notin P$. Hence $P$ is not a $\sigma$-ideal and so it is not $\sigma$-invariant.

(2) Let $R = \mathrm{Mat}_2(F)$ be the $2 \times 2$ full matrix ring over a field $F$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) =$

$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Then $N_r(R) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a strongly prime ideal of $R$ and $N_r(R)$ is a $\sigma$-ideal. But $N_r(R) \neq \mathbf{N}(R)$.

(3) Let $R = F[x]$ be the polynomial ring over a field $F$. Then $R$ is a commutative domain, and so it satisfies $N_r(R) = \mathbf{N}(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Clearly $N_r(R)$ is a $\sigma$-ideal. However, $N_r(R)$ is not $\sigma$-invariant. For, $ax \in \sigma^{-1}(N_r(R))$ but $ax \notin N_r(R)$, where $ax \in R$ and $0 \neq a \in F$. Moreover, for $P = \langle x+1 \rangle = \{g(x)(x+1) \mid g(x) \in R\} \in \mathrm{mSpec}_S(R)$, we have $x + 1 \in P$, but $\sigma(x+1) = 1 \notin P$. Thus $P$ is not a $\sigma$-ideal.

Now we extend the $\sigma$-rigid property of a ring $R$ to its upper nilradical $N_r(R)$ to study the connection of the upper nilradical $N_r(R)$ of a ring $R$ and the upper nilradical $N_r(R[[x;\sigma]])$ of the skew power series ring $R[[x;\sigma]]$ of $R$ as follows.

Definition. Let $\sigma$ be an endomorphism and $N_r(R)$ be a $\sigma$-ideal of a ring $R$. $N_r(R)$ is called to be *$\sigma$-completely semiprime* if $a\sigma(a) \in N_r(R)$ implies $a \in N_r(R)$ for $a \in R$.

Note that if $R$ is a $\sigma$-rigid ring, then $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$ but the converse does not hold by the next example.

Example 1.4. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where $F$ is a field. Then $N_r(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix} = \mathbf{N}(R)$. Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then obviously $\sigma$ is not a monomorphism. Thus $R$ is not $\sigma$-rigid.

Now we show that $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$: Clearly $N_r(R)$ is a $\sigma$-ideal of $R$. If $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) \in N_r(R)$, then $\begin{pmatrix} a^2 & bc \\ 0 & c^2 \end{pmatrix} \in N_r(R)$. Since $N_r(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, we obtain $a^2 = 0$ and $c^2 = 0$. Thus $a = 0 = c$ and so $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in N_r(R)$.

Observe that if $I$ is a $\sigma$-ideal, then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(a + I) = \sigma(a) + I$ for $a \in R$ is also an endomorphism of $R/I$.

Proposition 1.5. *If $N_r(R)$ is a $\sigma$-completely semiprime ideal of a ring $R$, then $N_r(R) = \mathbf{N}(R)$, i.e., $N_r(R)$ is a completely semiprime ideal of $R$.*

*Proof.* Note that $a\sigma(a) \in N_r(R)$ if and only if $\bar{a}\bar{\sigma}(\bar{a}) = \bar{0}$ for $\bar{a} = a + N_r(R) \in R/N_r(R)$. Then $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$ if and only if the factor ring $R/N_r(R)$ is a $\bar{\sigma}$-rigid ring, where $\bar{\sigma} : R/N_r(R) \to R/N_r(R)$ defined by $\bar{\sigma}(a + N_r(R)) = \sigma(a) + N_r(R)$. Thus $R/N_r(R)$ is reduced, and so $N_r(R) = \mathbf{N}(R)$. □

Example 1.3(3) illustrates that the converse of Proposition 1.5 does not hold. Indeed, for the ring $R = F[x]$ which satisfies $N_r(R) = \mathbf{N}(R) = \{0\}$ with $\sigma$ in Example 1.3(3). But $N_r(R)$ is not $\sigma$-completely semiprime : For, if $0 \neq a \in F$ and $f(x) = ax \in R$, then $f(x)\sigma(f(x)) \in N_r(R)$, but $f(x) \notin N_r(R)$.

Under certain conditions, the converse of Proposition 1.5 can be done as the next result shows:

LEMMA 1.6. [3, Theorem 5, Theorem 8 and Corollary 13] *Let $R$ be a ring. Then the following are equivalent:*
(1) $N_r(R) = \mathbf{N}(R)$.
(2) *$P$ is a completely prime ideal of $R$ for each $P \in \mathrm{mSpec_S}(R)$.*
(3) *$P = \{a \in R \mid ab \in N_r(R)$ for some $b \in R\backslash P\}$ for each $P \in \mathrm{mSpec_S}(R)$.*

PROPOSITION 1.7. *Assume that for each $P \in \mathrm{mSpec_S}(R)$, $P$ is a $\sigma$-invariant ideal of a ring $R$. Then the following are equivalent:*
(1) $N_r(R) = \mathbf{N}(R)$.
(2) *$N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$.*

*Proof.* It is enough to show that (1)⇒(2). Suppose that $N_r(R) = \mathbf{N}(R)$. Clearly, $N_r(R)$ is a $\sigma$-ideal of $R$ by Proposition 1.1(3). Let $a\sigma(a) \in N_r(R)$ then $a\sigma(a) \in P$ for all $P \in \mathrm{mSpec_S}(R)$. Since $P$ is completely prime and $\sigma$-invariant by Lemma 1.6 and hypothesis, $a \in P$ for all $P \in \mathrm{mSpec_S}(R)$ and so $a \in N_r(R)$. Thus $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$. □

There exists a ring $R$ with $N_r(R) \neq \mathbf{N}(R)$, even though every strongly prime ideal of $R$ is $\sigma$-invariant (Example 1.3(2)); while, the condition "$P$ is a $\sigma$-invariant ideal of a ring $R$ for each $P \in \mathrm{mSpec_S}(R)$" cannot be replaced by the condition "$N_r(R)$ is a $\sigma$-invariant ideal". In fact, for the $\sigma$-invariant ideal $N_r(R) = \{(0,0)\} = \mathbf{N}(R)$ in Example 1.3(1), $(0,1)\sigma(0,1) = (0,0) \in N_r(R)$ but $(0,1) \notin N_r(R)$.
However, we have the following:

THEOREM 1.8. *For a ring $R$, the following are equivalent:*
(1) *$N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$.*

(2) $N_r(R) = \mathbf{N}(R)$ *and* $P$ *is* $\sigma$*-invariant for each* $P \in \mathrm{mSpec_S}(R)$.

(3) $P$ *is a* $\sigma$*-ideal such that* $a\sigma(a) \in P$ *implies* $a \in P$ *for each* $P \in \mathrm{mSpec_S}(R)$.

*Proof.* (1)$\Rightarrow$(2): Observe that $N_r(R) = \mathbf{N}(R)$ by Proposition 1.5. Let $P \in \mathrm{mSpec_S}(R)$ and $a \in \sigma^{-1}(P)$. Then $\sigma(a) \in P$. By Lemma 1.6, there exists $b \in R \backslash P$ such that $\sigma(a)b \in N_r(R)$ since $N_r(R) = \mathbf{N}(R)$. Thus $b\sigma(a) \in N_r(R)$ and so $ab\sigma(ab) = ab\sigma(a)\sigma(b) \in N_r(R)$. Since $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$, $ab \in N_r(R)$ and so $ab \in P$. Thus $a \in P$ by Lemma 1.6 and therefore $\sigma^{-1}(P) \subseteq P$. Now, we show that $P \subseteq \sigma^{-1}(P)$. Let $a \in P$, then there exists $b \in R \backslash P$ such that $ab \in N_r(R)$ by Lemma 1.6 and $N_r(R)$ is a $\sigma$-ideal by Proposition 1.1(3). Thus $\sigma(a)\sigma(b) = \sigma(ab) \in N_r(R)$. Then $\sigma(a)\sigma(b) \in P$ and hence $\sigma(a) \in P$, by Lemma 1.6. Therefore $a \in \sigma^{-1}(P)$ and so $P$ is $\sigma$-invariant.

(2)$\Rightarrow$(3) follows from Lemma 1.6.

(3)$\Rightarrow$(1): Clearly $N_r(R)$ is a $\sigma$-ideal of $R$ by Proposition 1.1(2). Let $a\sigma(a) \in N_r(R)$ then $a\sigma(a) \in P$ for all $P \in \mathrm{mSpec_S}(R)$. By hypothesis, $a \in P$ for all $P \in \mathrm{mSpec_S}(R)$ and so $a \in N_r(R)$. Thus $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$. □

From Theorem 1.8, observe that if $R$ is a $\sigma$-rigid ring, then $P$ is $\sigma$-invariant for each $P \in \mathrm{mSpec_S}(R)$, equivalently, $P$ is a $\sigma$-ideal of $R$ for each $P \in \mathrm{mSpec_S}(R)$, and $\sigma$ is a monomorphism. Hence, we have the following.

Corollary 1.9. *The following are equivalent:*

(1) $R$ *is a* $\sigma$*-rigid ring.*

(2) $R$ *is a reduced ring,* $\sigma$ *is a monomorphism and* $P$ *is a* $\sigma$*-ideal for each* $P \in \mathrm{mSpec_S}(R)$.

*Proof.* It is enough to show (2)$\Rightarrow$(1). Let $P \in \mathrm{mSpec_S}(R)$ and $a\sigma(a) = 0$ for $a \in R$. Then $a\sigma(a) \in P$. Since $R$ is reduced, $P$ is a completely prime ideal of $R$. Thus $a \in P$ or $\sigma(a) \in P$. If $a \in P$, then $\sigma(a) \in P$ because $P$ is a $\sigma$-ideal. Thus $\sigma(a) \in N_r(R) = \{0\}$ and so $a = 0$ because $\sigma$ is a monomorphism. Therefore $R$ is a $\sigma$-rigid ring. □

## 2. The upper nilradical of the skew power series ring

For a ring $R$ with an endomorphism $\sigma$ of $R$, the *skew power series* ring $R[[x;\sigma]]$ of $R$ is the ring obtained by giving the power series ring over $R$ with the new multiplication: $xr = \sigma(r)x$ for all $r \in R$.

In this section, we characterize the upper nilradical $N_r(R[[x;\sigma]])$ of the skew power series ring $R[[x;\sigma]]$ of a ring $R$ using the upper nilradical $N_r(R)$ of $R$.

PROPOSITION 2.1. *Let $N_r(R)$ be a $\sigma$-completely semiprime ideal of a ring $R$. For $a, b \in R$ we have the following.*

*(1) If $ab \in N_r(R)$, then $a\sigma^n(b)$, $\sigma^n(a)b \in N_r(R)$ for any positive integer $n$.*

*(2) If $a\sigma^k(b)$ or $\sigma^k(a)b \in N_r(R)$ for some positive integer $k$, then $ab \in N_r(R)$.*

*Proof.* Note that $N_r(R)$ is completely semiprime, since $R$ satisfies $N_r(R) = \mathbf{N}(R)$ by Theorem 1.8.

(1) It is enough to show that $a\sigma(b) \in N_r(R)$ for $ab \in N_r(R)$. If $ab \in N_r(R)$, then $b\sigma(a)\sigma(b\sigma(a)) = b\sigma(ab)\sigma^2(a) \in N_r(R)$ by hypothesis. Since $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$, we have $b\sigma(a) \in N_r(R)$. Then $\sigma(a)b \in N_r(R)$ because $N_r(R)$ is completely semiprime. Similarly, using $ba \in N_r(R)$, we obtain $a\sigma(b) \in N_r(R)$.

(2) Suppose that $a\sigma^k(b) \in N_r(R)$ for some positive integer $k$. Then, by the previous part, we obtain $\sigma^k(ab) = \sigma^k(a)\sigma^k(b) \in N_r(R)$. Since $N_r(R)$ is $\sigma$-invariant by Theorem 1.8 and Proposition 1.1(2), $\sigma^{k-1}(ab) \in \sigma^{-1}(N_r(R)) = N_r(R)$ and so $\sigma^{k-2}(ab) \in \sigma^{-1}(N_r(R)) = N_r(R)$. Continuing this process, we have $ab \in N_r(R)$ by induction. Similarly, $\sigma^k(a)b \in N_r(R)$ for some positive integer $k$ implies $ab \in N_r(R)$. □

Note that if $N_r(R)$ is a $\sigma$-completely semiprime ideal of a ring $R$, then $N_r(R)[[x;\sigma]]$ is an ideal of the skew power series ring $R[[x;\sigma]]$ of $R$ by Proposition 2.1.

THEOREM 2.2. *Let $N_r(R)$ be a $\sigma$-completely semiprime ideal of a ring $R$. Assume that $p(x) = \sum_{i=0}^{\infty} a_i x^i$ and $q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\sigma]]$. Then the following are equivalent:*

*(1) $p(x)q(x) \in N_r(R)[[x;\sigma]]$.*

*(2) $a_i b_j \in N_r(R)$ for all $0 \le i$ and $0 \le j$.*

*Proof.* (1)⇒(2): Assume that $p(x)q(x) \in N_r(R)[[x;\sigma]]$. Then

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i\sigma^i(b_j)x^{j+j}\right) \in N_r(R)[[x;\sigma]]. \tag{1}$$

We claim that $a_ib_j \in N_r(R)$ for all $i, j$. We proceed by induction on $i + j$. Then we obtain $a_0b_0 \in N_r(R)$ and so this proves for $i + j = 0$. Now suppose that our claim is true for $i + j \leq n - 1$. From Eq.(1), we have

$$\sum_{i+j=n} a_i\sigma^i(b_j) \in N_r(R). \tag{2}$$

Multiplying $a_0$ to Eq.(2) from the right hand-side, we obtain $a_0b_na_0 = 0$ by Proposition 2.1. Since $N_r(R) = \mathbf{N}(R)$ by Theorem 1.8, $a_0b_n \in N_r(R)$. Now Eq.(2) becomes

$$\sum_{i+j=n,1\leq i\leq n} a_i\sigma^i(b_j) \in N_r(R). \tag{3}$$

Multiplying $a_1$ to Eq.(3) from the right hand-side, we obtain $a_1\sigma(b_{n-1})a_1 \in N_r(R)$ and so $a_1b_{n-1} \in N_r(R)$. Continuing this process, we can prove $a_ib_j \in N_r(R)$ for all $i, j$ with $i + j = n$. Therefore $a_ib_j = 0$ for all $i$ and $j$.

(2)⇒(1) follows directly from Proposition 2.1. □

COROLLARY 2.3. *If $N_r(R)$ is a $\sigma$-completely semiprime ideal of a ring $R$, then $N_r(R)[[x;\sigma]]$ is a completely semiprime ideal of $R[[x;\sigma]]$.*

*Proof.* Let $0 \neq (p(x))^2 \in N_r(R)[[x;\sigma]]$ where $p(x) = \sum_{i=0}^{\infty} a_ix^i \in R[[x;\sigma]]$. Then $a_m^2 \in N_r(R)$ for all $i$ by Theorem 2.2. Since $N_r(R) = \mathbf{N}(R)$ by Theorem 1.8, we have $a_m \in N_r(R)$ for all $i$ and so $p(x) \in N_r(R)[[x;\sigma]]$. Thus $N_r(R)[[x;\sigma]]$ is a completely semiprime ideal of $R[x;\sigma]$. □

Example 1.3(1) shows that the condition "$N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$" in Theorem 2.2 and Corollary 2.3 is not superfluous: Indeed, (i) $(1,0)\sigma(1,0) = (0,0) \in N_r(R)$, but $(1,0) \notin N_r(R)$. Thus $N_r(R)$ is not $\sigma$-completely semiprime. (ii) For $p(x) = (1,0)+(1,0)x$ and $q(x) = (0,1) + (1,0)x \in R[[x;\sigma]]$, we have $p(x)q(x) = 0 \in N_r(R)[[x;\sigma]]$ but $(1,0) \cdot (1,0) \notin N_r(R)$. Thus $N_r(R)[[x;\sigma]]$ does not satisfy the conclusion of Theorem 2.2(2). (iii) $((1,0)x)^2 = 0 \in N_r(R)[[x;\sigma]]$ but $(1,0)x \notin N_r(R)[[x;\sigma]]$ showing that the conclusions in Corollary 2.3 does not hold for $N_r(R)$. (iv) Moreover, $((1,0)x)^2 = 0 \in P[[x;\sigma]]$ but $(1,0)x \notin P[[x;\sigma]]$; this illustrates that not every completely semiprime ideal of a ring $R$ can be lifted to a completely semiprime ideal of the skew power series ring $R[[x;\sigma]]$ of $R$, in general. However, we have the following:

LEMMA 2.4. *If $N_r(R)$ is a $\sigma$-completely semiprime ideal of a ring $R$, then $P[[x;\sigma]]$ is a completely prime ideal of $R[[x;\sigma]]$ for each $P \in \mathrm{mSpec_S}(R)$.*

*Proof.* Note that $P$ is a $\sigma$-invariant ideal (as well as a completely prime ideal by Lemma 1.6) for each $P \in \mathrm{mSpec_S}(R)$ by Theorem 1.8. Let $p(x)q(x) \in P[[x;\sigma]]$ with $q(x) \notin P[[x;\sigma]]$, where $p(x) = \sum_{i=0}^{\infty} a_i x^i$ and $q(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\sigma]]$. Then

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i\sigma^i(b_j)x^{i+j}\right)$$
$$= c_0 + c_1x + c_2x^2 + \cdots \in P[[x;\sigma]].$$

(i) If $b_0 \notin P$, then $a_0b_0 \in P$ implies $a_0 \in P$ because $P$ is completely prime. Thus $c_1 \in P$ implies $a_1 \in P$ because $P$ is $\sigma$-invariant. By the same method, $c_2 \in P$ implies $a_2 \in P$. Continuing this process, we have $a_0, a_1, a_2, \ldots \in P$ and hence $p(x) \in P[[x;\sigma]]$.

(ii) If $b_0, b_1, \ldots, b_{n-1} \in P$ and $b_n \notin P$, then we have $p(x)(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) \in P[[x;\sigma]]$. Since $b_n \notin P$, by the same method of the above (i), we have $p(x) \in P[[x;\sigma]]$.

(iii) Continuing this process, $p(x) \in P[[x;\sigma]]$ and so $P[[x;\sigma]]$ is a completely prime ideal of $R[[x;\sigma]]$. □

For the skew power series ring $R[[x;\sigma]]$ of a ring $R$, if $N_r(R[[x;\sigma]])$ is a completely semiprime ideal of $R[[x;\sigma]]$, i.e., $N_r(R[[x;\sigma]]) = \mathbf{N}(R[[x;\sigma]])$, then $N_r(R) = \mathbf{N}(R)$: For, if $a \in \mathbf{N}(R)$, then $a \in \mathbf{N}(R[[x;\sigma]]) = N_r(R[[x;\sigma]])$. Thus $\langle a\rangle$ is a nil ideal of $R[[x;\sigma]]$ generated by $a$ and so $\langle a\rangle \cap R$ is a nil ideal of $R$. Since $N_r(R)$ is the sum of all nil ideals of $R$, we have $a \in \langle a\rangle \cap R \subseteq N_r(R)$ and so $\mathbf{N}(R) = N_r(R)$. Therefore $R$ satisfies $N_r(R) = \mathbf{N}(R)$. But the converse does not hold by the next example.

EXAMPLE 2.5. ([5, Example 1.3] and [9, Example 2.7.38]) Let $F$ be a field and let $V$ be an infinite dimensional left vector space over $F$ with $\{v_1, v_2, \ldots\}$ a basis. For the endomorphism ring $A = \mathrm{End}_F(V)$, define

$$I = \{f \in A \mid rank(f) < \infty \text{ and } f(v_i) \in \sum_{j<i} Fv_j\}.$$

Let $R$ be the $F$-subalgebra of $A$ generated by $I$ and the identity $1_A$ of $A$. Note that $I = \mathbf{N}(R) = N_r(R)$, and so $N_r(R)$ is a $1_R$-completely semiprime ideal of $R$ where $1_R$ is the identity endomorphism of $R$. Moreover $I[x] \subseteq R[[x]]$ since every element in $I$ is strongly nilpotent in $R[[x]]$, where $I[x]$ and $R[[x]]$ denote the polynomial ring and the formal power series ring over $I$ and $R$, respectively. Let

$$p(x) = \sum_{i=0}^{\infty} e_{(2i+1)(2i+2)}x^i \text{ and } q(x) = \sum_{i=0}^{\infty} e_{(2i+2)(2i+3)}x^i,$$

where $e_{ij}$ is the infinite matrix unit over $F$ with $(i, j)$-entry 1 and 0 elsewhere. Then $p(x), q(x) \in N_r(R)[[x]]$ and $p(x)^2 = 0 = q(x)^2$. However, the coefficients of $(p(x)+q(x))^k$ are $e_{1(k+1)}, e_{2(k+2)}, \dots, e_{n(k+n)}, \dots$ for $k = 2, 3, \dots$ and so it is not nilpotent. Hence $p(x) \notin N_r(R[[x]])$, or $q(x) \notin N_r(R[[x]])$. Therefore $N_r(R)[[x]] \nsubseteq N_r(R[[x]])$.

We note that Example 2.5 also shows that

$$N_r(R[[x;\sigma]]) \neq N_r(R)[[x;\sigma]]$$

even if $N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$, in general.

However, we have the following:

THEOREM 2.6. *Let $N_r(R)$ be a $\sigma$-completely semiprime ideal of a ring $R$. Then the following are equivalent:*

(1) *$N_r(R[[x;\sigma]])$ is a completely semiprime ideal of $R[[x;\sigma]]$.*

(2) *$N_r(R)[[x;\sigma]] = N_r(R[[x;\sigma]])$.*

*Proof.* (1)⇒(2): Suppose that $N_r(R[[x;\sigma]])$ is a completely semiprime ideal of $R[[x;\sigma]]$. It is enough to show that $N_r(R)[[x;\sigma]] \subseteq N_r(R[[x;\sigma]])$ by Lemma 2.4. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i \in N_r(R)[[x;\sigma]]$. Since $R$ satisfies $\mathbf{N}(R) = N_r(R)$ as a subring, $a_i$ is a nilpotent element in $R[[x;\sigma]]$, and thus $a_i \in \mathbf{N}(R[[x;\sigma]]) = N_r(R[[x;\sigma]])$ for every $i$. Then $a_i x^i \in N_r(R[[x;\sigma]])$ and hence $f(x) \in N_r(R[[x;\sigma]])$. Therefore $N_r(R)[[x;\sigma]] \subseteq N_r(R[[x;\sigma]])$.

(2)⇒(1) follows from Corollary 2.3. □

Observe that if $R$ is a $\sigma$-rigid ring, then $R[[x;\sigma]]$ is a reduced ring by [4, Corollary 18] and thus $N_r(R[[x;\sigma]]) = \{0\}$. Moreover, we have the following consequence of Theorem 2.5.

COROLLARY 2.7. *If $R$ is a $\sigma$-rigid ring, then $N_r(R[[x;\sigma]])$ is a completely semiprime ideal of $R[[x;\sigma]]$ if and only if $N_r(R[[x;\sigma]]) = \{0\}$.*

The following example shows that the condition "$N_r(R)$ is a $\sigma$-completely semiprime ideal of $R$" in Theorem 2.6 is not superfluous.

EXAMPLE 2.8. (1) Consider the commutative domain $R = F[x]$, where $F$ is a field, and the endomorphism $\sigma$ of $R$ defined by $\sigma(f(x)) = f(0)$ in Example 1.3(3). Note that $N_r(R)$ is not a $\sigma$-completely semiprime ideal of $R$ and $N_r(R)[[y;\sigma]] = \{0\}$. We claim that $N_r(R[[y;\sigma]])$ is a completely semiprime ideal of $R[[y;\sigma]]$ with $N_r(R)[[y;\sigma]] \neq N_r(R[[y;\sigma]])$: Let $p(y) = \sum_{i=0}^{\infty} f_i y^i$ be a nonzero nilpotent in $R[[y;\sigma]]$, where $f_i = a_{i0} + a_{i1}x + \cdots + a_{in_i}x^{n_i} \in R$ for $i = 0, 1, \ldots$. Then $p(y)^t = 0$ for some positive integer $t$. Thus $f_0^t = 0$ and so $f_0 = 0$. Hence $p(y) = \sum_{i=s}^{\infty} f_i y^i$, where $s \geq 1$ and $f_s \neq 0$. Now, we show that $\sigma(f_i) = f_i(0) = 0$ for all $i = s, s+1, \ldots$. Suppose that $f_k(0) \neq 0$ and $f_i(0) = 0$ for $s \leq i \leq k-1$. Then $0 = p(y)^t = (\sum_{i=s}^{\infty} f_i y^i)^t$ implies $0 =$ the term of degree $(t-1)k + s$ in the expansion of the right hand-side. So

$$
\begin{aligned}
0 &= \sum_{i=s}^{\infty} f_i y^i (\sum f_{i_2} y^{i_2} f_{i_3} y^{i_3} \cdots f_{i_t} y^{i_t}) \\
&= (\sum_{i=s}^{\infty} \sum f_i \cdot f_{i_2}(0) f_{i_3}(0) \cdots f_{i_t}(0)) y^{i+i_2+\cdots i_t}
\end{aligned}
$$

where $i + i_2 + \cdots i_t = (t-1)k + s$. Note that $s \leq i$ and $s \leq i_j$ for $j = 2, \ldots, t$. If $k \leq i_j$ for all $j = 2, \ldots, t$, then $(t-1)k \leq i_2 + \cdots + i_t = (t-1)k + s - i$. Thus $i = s$ and so $i_2 = \cdots = i_t = k$. Hence, for $i \geq s+1$, there exists $i_j$ such that $2 \leq j \leq t$ and $i_j \leq k-1$, i.e., $f_{i_j}(0) = 0$. Thus, we obtain that $0 =$ the term of degree $(t-1)k + s$ in the expansion of the right hand-side $= f_s \cdot f_k(0)^{t-1}$. Since $f_s \neq 0$, we have $f_k(0)^{t-1} = 0$ and so $f_k(0) = 0$; which is a contradiction. Therefore, $p(y) = \sum_{i=s}^{\infty} f_i y^i$ with $f_i(0) = 0$ for all $i = s, s+1, \ldots$. For each $i = s, s+1, \ldots$, we get $f_i y^i R[[y;\sigma]] f_i y^i = 0$ because $f_i(0) = 0$. Hence $f_i y^i \in \mathbf{P}(R[[y;\sigma]])$ and so $p(y) \in \mathbf{P}(R[[y;\sigma]])$, where $\mathbf{P}(R[[y;\sigma]])$ is the prime radical of $R[[y;\sigma]]$. Thus $R[[y;\sigma]]$ is 2-primal, and so $N_r(R[[y;\sigma]])$ is completely semiprime, but $N_r(R[[y;\sigma]]) \nsubseteq N_r(R)[[y;\sigma]]$ since $0 \neq xy \in N_r(R[[y;\sigma]])$.

(2) Consider the $2 \times 2$ full matrix ring $R = \mathrm{Mat}_2(F)$ over a field $F$ and the automorphism $\sigma$ of $R$ is defined by $\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ in Example 1.3(2). Clearly $N_r(R) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not $\sigma$-completely

semiprime, and $N_r(R)[[x;\sigma]] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We claim that $N_r(R[[x;\sigma]])$ is not a completely semiprime ideal of $R[[x;\sigma]]$, even though $N_r(R)[[x;\sigma]] = N_r(R[[x;\sigma]])$. First, we show that $N_r(R[[x;\sigma]]) = N_r(R)[[x;\sigma]]$. Assume on the contrary that $N_r(R[[x;\sigma]]) \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $0 \neq p(x) = \sum_{i=s}^{\infty} a_i x^i \in N_r(R[[x;\sigma]])$, where $s \geq 0$ and $a_s \neq 0$. Note that $\sigma^s(R) = R$ and $Ra_sR = R$. Then $Rp(x)R = Ra_sRx^s + \cdots$, and so there exists $q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^s + \cdots \in N_r(R[[x;\sigma]])$. But $q(x)$ is not a nilpotent; which is a contradiction. Thus $N_r(R[[x;\sigma]]) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = N_r(R)[[x;\sigma]]$. However, $\mathbf{N}(R[[x;\sigma]]) \neq N_r(R[[x;\sigma]])$ since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \in \mathbf{N}(R[[x;\sigma]])$. Thus $N_r(R[[x;\sigma]])$ is not a completely semiprime ideal of $R[[x;\sigma]]$.

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