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EXISTENCE, MULTIPLICITY AND UNIQUENESS RESULTS FOR A SECOND ORDER M-POINT BOUNDARY VALUE PROBLEM

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Abstract. Let $f : [0,1] \times [0,\infty) \to [0,\infty)$ be continuous and $a \in C([0,1],[0,\infty))$,and let $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, a_i, b_i \in [0,\infty)$ with $0 < \sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$.This paper is concerned with the following m-point boundary value problem:

$$x''(t) + a(t)f(t,x(t)) = 0, t \in (0,1),$$
$$x'(0) = \sum_{i=1}^{m-2} b_i x'(\xi_i), x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

The existence, multiplicity and uniqueness of positive solutions of this problem are discussed with the help of two fixed point theorems in cones, respectively.

1. Introductions

In this paper we consider the second order m-point boundary value problem as follows.

$$x''(t) + a(t)f(t,x(t)) = 0, t \in (0,1), \tag{1.1}$$

$$x'(0) = \sum_{i=1}^{m-2} b_i x'(\xi_i), x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i) \tag{1.2}$$

where $\xi_i \in (0,1)$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$.

The following conditions will be assumed throughout this paper.

(H_1) $a_i, b_i \in [0,\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$ and $\sum_{i=1}^{m-2} b_i < 1$;

(H_2) $f \in C([0,1] \times [0,\infty), [0,\infty))$;

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(H_3) $a \in C([0,1],[0,\infty))$ and there exists $t_0 \in [0,1]$ such that $a(t_0) > 0$.

There has been much research on the m-point boundary value problems (see e.g. [1], [2] and references therein).

Very recently in [4], Ruyun Ma and Nelson Castaneda studied the existence of positive solutions of the m-point boundary value problem

$$x^{''}(t) + a(t)f(x(t)) = 0, t \in (0,1),$$

$$x^{'}(t) = \sum_{i=1}^{m-2} b_i x^{'}(\xi_i), x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

Under the assumptions $(H_1), (H_3)$ and $(H_2^{'})$ $f \in C([0,\infty),[0,\infty))$, an existence theorem was established when f is either sublinear (i.e. $\lim_{x\to 0^+} \frac{f(x)}{x} = \infty$ and $\lim_{x\to\infty} \frac{f(x)}{x} = 0$) or suplinear (i.e. $\lim_{x\to 0^+} \frac{f(x)}{x} = 0$ and $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$). See [4, Theorem 1].

The purpose of this paper is to give an existence result for the more general case that f need not to be either sublinear or suplinear, and present criteria that ensure the multiplicity and uniqueness of positive solutions, respectively.

In section 2, we provide some background results, and state two fixed point theorems. Then in section 3, we impose growth conditions on which allow us to apply Krasnosel'skii's fixed point theorem in obtaining the existence and multiplicity results. Section 4 concerns with the uniqueness of positive solution. As applications of our results, two examples are given in the last section.

2. Preliminaries

In this section, we provide some background material from the theory of cones in Banach space.

Let E be a Banach space,$K \subset E$ be a cone, and the order induced by K on E be $'' \leq''$.

DEFINITION 2.1. Let$X$ be a convex subset of E, an operator$A : X \to E$ is said to be convex if

$$A(tx + (1-t)y) \leq tAx + (1-t)Ay$$

for all $t \in [0,1]$ and all $x, y \in X$ with $x \leq y$.

DEFINITION 2.2. Let$X$ be a convex subset of E, an operator$A : X \to E$ is said to be decresing if

$$Ax \leq Ay$$

for all $x, y \in X$ with $x \leq y$.

In obtaining existence, multiplicity and uniqueness results of BVP (1.1)-(1.2), the following two fixed point theorems are crucial.

THEOREM 2.1. [10] *Let X be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$ and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that
(i) $\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_1$,and $\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2$; or
(ii) $\|Au\| \geq \|u\|, u \in K \cap \partial\Omega_1$,and $\|Au\| \leq \|u\|, u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

THEOREM 2.2. [6] *Let K be a normal cone in Banach space E, $A : K \to E$ be a convex decreasing operator, and there exists a positive constant ϵ and nature number m, such that*

$$\theta \leq A\theta, \epsilon A\theta \leq A^2\theta, \frac{1}{2}A\theta \leq A^{2m}\theta$$

Then operator A has a unique fixed point x^* in K, and for arbitrary initial $x_0 \in [\theta, A\theta]$ given and iterative sequence $x_n = Ax_{n-1}, n = 1, 2, \cdots$, one has $x_n \to x^*$.

3. Existence and multiplicity results

In this section, we will establish the existence and multiplicity theorems for BVP (1.1)-(1.2).

It is known by [4, Theorem 1] that $u \in C[0,1]$ is a solution of BVP (1.1)-(1.2) if and only if u is a fixed point of the operator $A : C[0,1] \to C[0,1]$, where

$$\begin{aligned} Ay(t) \quad = \quad & -\int_0^t (t-s)a(s)f(s,y(s))ds + t\frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(s,y(s))ds}{\sum_{i=1}^{m-2} b_i - 1} \\ & + \frac{1}{1-\sum_{i=1}^{m-2} a_i}\Big(\int_0^1 (1-s)a(s)f(s,y(s))ds \\ & - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)a(s)f(s,y(s))ds \\ & - \frac{\sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(s,y(s))ds}{\sum_{i=1}^{m-2} b_i - 1}\Big(1 - \sum_{i=1}^{m-2} a_i\xi_i\Big)\Big) \end{aligned}$$

The above formula can be rewritten as follows

$$\begin{aligned} Ay(t) &= (1-t)\int_0^t a(s)f(s,y(s))ds + \int_t^1 (1-s)a(s)f(s,y(s))ds \\ &+ \frac{1-t-\sum_{i=1}^{m-2} a_i(\xi_i - t)}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(s,y(s))ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i (\int_0^{\xi_i} (1-\xi_i)a(s)f(s,y(s))ds \\ &+ \int_{\xi_i}^1 (1-s)a(s)f(s,y(s))ds). \end{aligned}$$

A solution u of BVP (1.1)-(1.2) is positive if $u(t) > 0$ for $t \in (0,1)$.

As a direct consequence of Lemma 2 and Lemma 4 in [4],we have

LEMMA 3.1. *Assume $(H_1 - H_3)$ hold, let $y \in C[0,1]$ and $y(t) \geq 0$ for all $t \in [0,1]$, then $Ay(t)$ satisfies*

(1) *$Ay(t) \geq 0$ for all $t \in [0,1]$;*

(2) *$Ay(t)$ is decreasing on $[0,1]$;*

(3) $Ay(1) = \min_{t\in[0,1]} Ay(t) \geq \gamma \max_{t\in[0,1]} Ay(t) = \gamma Ay(0)$, *where* $\gamma = \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}$.

Denoted by E the space $C[0,1]$, the norm endowed on E is $\|.\| : \|y\| = \max_{t\in[0,1]} |y(t)|$ for $y \in E$.

Let $K = \{y \in E \mid y(t) \geq 0, \forall t \in [0,1], \min_{t\in[0,1]} y(t) \geq \gamma\|y\|\}$, where

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{1-\sum_{i=1}^{m-2} a_i}.$$

It is easy to check that K is a normal cone in E. By Ascoli-Arzela's theorem and Lemma 3.1, we have

LEMMA 3.2. *Assume $(H_1 - H_3)$ hold, then $A(K) \subset K$ and $A : K \to K$ is completely continuous.*

Denote

$$\begin{aligned} P = (\int_0^1 (1-s)a(s)ds &+ \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)ds \\ &+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i (\int_0^{\xi_i}(1-\xi_i)a(s)ds + \int_{\xi_i}^1 (1-s)a(s)ds))^{-1} \end{aligned}$$

$$\Phi(l) = \max\{f(t,x) : 0 \leq t \leq 1, \gamma l \leq x \leq 1\}$$

$$\Psi(l) = \min\{f(t,x) : 0 \leq t \leq 1, \gamma l \leq x \leq 1\}.$$

We now present our results of this section.

THEOREM 3.1. *Assume $(H_1 - H_3)$ hold. If there exist two positive numbers $a, b (a \neq b)$, such that $\Phi(a) \leq aP$ and $\Psi(b) \geq bP$,then BVP (1.1)-(1.2) has at least one positive solution $u \in K$ satisfying*

$$\min\{a,b\} \leq \|u\| \leq \max\{a,b\}.$$

Proof. Without loss of generality, we assume $a < b$.Let $\Omega_1 = \{y \in E \mid \|y\| < a\}, \Omega_2 = \{y \in E \mid \|y\| < b\}$.

For arbitrary $y \in K \cap \partial\Omega_1$, we have

$$
\begin{aligned}
(3.1)\quad & \|Ay(t)\| \\
= \;& Ay(0) \\
= \;& \int_0^1 (1-s)a(s)f(s,y(s))ds \\
& + \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(s,y(s))ds \\
& + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i(\int_0^{\xi_i} (1-\xi_i)a(s)f(s,y(s))ds \\
& + \int_{\xi_i}^1 (1-s)a(s)f(s,y(s))ds) \\
\leq \;& \Phi(a)(\int_0^1 (1-s)a(s)ds \\
& + \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)ds \\
& + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i(\int_0^{\xi_i} (1-\xi_i)a(s)ds + \int_{\xi_i}^1 (1-s)a(s)ds)) \\
= \;& \Phi(a)P^{-1} \\
\leq \;& a \\
= \;& \|y\|.
\end{aligned}
$$

For $y \in K \cap \partial\Omega_2$, we get

$$
\begin{aligned}
(3.2)\quad & \|Ay(t)\| \\
= \;& Ay(0) \\
= \;& \int_0^1 (1-s)a(s)f(s,y(s))ds \\
& + \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(s,y(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i\left(\int_0^{\xi_i}(1-\xi_i)a(s)f(s,y(s))ds\right. \\
&\left.+\int_{\xi_i}^1(1-s)a(s)f(s,y(s))ds\right) \\
\geq\ & \Psi(b)\left(\int_0^1(1-s)a(s)ds\right. \\
&+\frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)ds \\
&+\frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i\left(\int_0^{\xi_i}(1-\xi_i)a(s)ds+\int_{\xi_i}^1(1-s)a(s)ds\right)\Bigg) \\
=\ & \Psi(b)P^{-1} \\
\geq\ & b \\
=\ & \|y\| .
\end{aligned}
$$

From (3.1) and (3.2), we show that the condition (i) of Theorem 2.1 is satisfied. Hence, an application of Theorem 2.1 completes the proof. □

THEOREM 3.2. *Assume $(H_1 - H_3)$ hold. If there exist $n+1$ positive numbers $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n+1}$ such that*
(i) $\Phi(\alpha_{2k-1}) < \alpha_{2k-1}P$ and $\Psi(\alpha_{2k}) > \alpha_{2k}P, k = 1, 2, \cdots, [\frac{n+1}{2}]$;or
(ii) $\Psi(\alpha_{2k-1}) < \alpha_{2k-1}P$ and $\Phi(\alpha_{2k}) > \alpha_{2k}P, k = 1, 2, \cdots, [\frac{n+1}{2}]$.
where$[\lambda]$ denotes the integer part of $\lambda \in R$. Then BVP (1.1)-(1.2) has at least n positive solutions $u_1, u_2, \cdots, u_n$, which satisfy

$$\alpha_k < \|u\| < \alpha_{k+1}, k = 1, 2, \cdots, n.$$

Proof. Suppose case (i) holds. Assumption (H_2) implies $\Phi, \Psi : [0, \infty) \to [0, \infty)$ are continuous, thus for every pair (α_k, α_{k+1}), there exists (b_k, c_k) such that $\alpha_k < b_k < c_k < \alpha_{k+1}$, and

$$\Phi(b_{2k-1}) = b_{2k-1}P, \Psi(c_{2k-1}) = c_{2k-1}P, k = 1, 2, \cdots, [\frac{n+1}{2}],$$

$$\Psi(b_{2k}) = b_{2k}P, \Phi(c_{2k}) = c_{2k}P, k = 1, 2, \cdots, [\frac{n+1}{2}].$$

According to Theorem 3.1, every pair (b_k, c_k) gives a positive solution of BVP (1.1)-(1.2), such that

$$b_k < \|u\| < c_k, k = 1, 2, \cdots, n$$

When case (ii) holds, we can show the conclusion is still true. This completes the proof. □

4. Uniqueness of positive solution

In this section, we will impose growth conditions on, which allow us to obtain unique positive solution of BVP (1.1)-(1.2) with the help of Theorem 2.1.

Now we present the main result of this section.

THEOREM 4.1. *Assume* $(H_1 - H_3)$ *hold, and* f *satisfies*

$(H_4) f(t,.)$ *is convex and decreasing for each* $t \in [0,1]$.

(H_5) *there exists* $M > 0$ *such that* $f(t,0) \leq MP$ *and* $f(t,M) \geq \frac{1}{2}MQ$ *for all* $t \in [0,1]$,

where P *is defined in section 3, and*

$$Q = (\frac{\sum_{i=1}^{m-2} a_i(1-\xi_i)}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)ds$$
$$+ \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i(\int_0^{\xi_i} (1-\xi_i)a(s)ds + \int_{\xi_i}^1 (1-s)a(s)ds))^{-1}$$

Then BVP (1.1)-(1.2) *has a unique positive solution* $u^* \in K$.

Proof. By Lemma 3.1, we obtain

$$\begin{aligned}
A\theta(t) &\leq A\theta(0) \\
&= \int_0^1 (1-s)a(s)f(s,0)ds \\
&\quad + \frac{1-\sum_{i=1}^{m-2} a_i\xi_i}{(1-\sum_{i=1}^{m-2} a_i)(1-\sum_{i=1}^{m-2} b_i)} \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} a(s)f(s,0)ds \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i} \sum_{i=1}^{m-2} a_i(\int_0^{\xi_i} (1-\xi_i)a(s)f(s,0)ds \\
&\quad + \int_{\xi_i}^1 (1-s)a(s)f(s,0)ds) \\
&\leq \max_{t\in[0,1]}\{f(t,0)\}(\int_0^1 (1-s)a(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\sum_{i=1}^{m-2}a_i\xi_i}{(1-\sum_{i=1}^{m-2}a_i)(1-\sum_{i=1}^{m-2}b_i)}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}a(s)ds\\
&+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i(\int_0^{\xi_i}(1-\xi_i)a(s)ds+\int_{\xi_i}^1(1-s)a(s)ds)\\
=&\max_{t\in[0,1]}\{f(t,0)\}P^{-1}\\
\le&\ M.
\end{aligned}
$$

Let $h(t)\equiv M, t\in[0,1]$, by condition (H_4), A is a decreasing convex operator, then

$$
\begin{aligned}
A^2\theta(t) &= A(A\theta(t))\\
&\ge A(h(t))\\
&\ge Ah(1)\\
&=\frac{\sum_{i=1}^{m-2}a_i(1-\xi_i)}{(1-\sum_{i=1}^{m-2}a_i)(1-\sum_{i=1}^{m-2}b_i)}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}a(s)f(s,M)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i(\int_0^{\xi_i}(1-\xi_i)a(s)f(s,M)ds\\
&\quad+\int_{\xi_i}^1(1-s)a(s)f(s,M)ds)\\
&\ge\min_{t\in[0,1]}\{f(t,M)\}(\frac{\sum_{i=1}^{m-2}a_i(1-\xi_i)}{(1-\sum_{i=1}^{m-2}a_i)(1-\sum_{i=1}^{m-2}b_i)}\sum_{i=1}^{m-2}b_i\int_0^{\xi_i}a(s)ds\\
&\quad+\frac{1}{1-\sum_{i=1}^{m-2}a_i}\sum_{i=1}^{m-2}a_i(\int_0^{\xi_i}(1-\xi_i)a(s)ds\\
&\quad+\int_{\xi_i}^1(1-s)a(s)ds))\\
&=\min_{t\in[0,1]}\{f(t,M)\}Q^{-1}\\
&\ge\frac{1}{2}M.
\end{aligned}
$$

By the decreasing of A, we have $A^2\theta\le A\theta$. Hence $\theta\le\frac{1}{2}A\theta\le A^2\theta\le A\theta$.

According to Theorem 2.2, A has a unique fixed point; therefore BVP (1.1)-(1.2) has a unique positive solution. □

5. Examples and remarks

In this context, we present two examples as applications of above results.

EXAMPLE 5.1. Let $m = 3, \xi = \frac{1}{2}, a = \frac{1}{2}, b = 0, a(t) \equiv 1$, then one can compute $P = \frac{7}{8}, \gamma = \frac{1}{3}$. Let

$$f(t,u) = f(u) = \begin{cases} u, & 0 \le u \le 1 \\ 1 + 16(u-1)^2, & 1 < u \le 2 \\ 17 + \frac{6}{19}(u-2), & 2 < u \le 21 \\ 23(u-20) + \sqrt{u-21}, & u > 21. \end{cases}$$

Choosing $\alpha_1 = 1, \alpha_2 = 6, \alpha_3 = 21, \alpha_4 = 90$,we have

$$\Phi(1) = 1 < 1P = \frac{7}{8}, \Psi(6) = 17 > 6P = \frac{48}{7}$$

$$\Phi(21) = 23 < 21P = 24, \Psi(90) = 233 > 90P = \frac{720}{7}.$$

Theorem 3.2 implies that the following three-point boundary value problem

$$u''(t) + f(u) = 0, u'(0) = 0, u(1) = \frac{1}{2}u(\frac{1}{2})$$

has at least three positive solutions, which satisfy

$$1 < \|u_1\| < 6 < \|u_2\| < 21 < \|u_3\| < 90.$$

REMARK 1. In the above example $\lim_{u\to 0^+} \frac{f(u)}{u} = 1$ and $\lim_{x\to\infty} \frac{f(u)}{u} = 23$, i.e.$f$ is neither sublinear nor suplinear, thus the conditions of Theorem 1 in [4] do not hold. However, by Theorem 3.2 of this paper, we obtain not only existence but also multiplicity results.

REMARK 2. When $f(t,u) = f(u)$ and f is either sublinear (i.e. $\lim_{x\to 0^+} \frac{f(x)}{x} = \infty$ and $\lim_{x\to\infty} \frac{f(x)}{x} = 0$) or suplinear (i.e. $\lim_{x\to 0^+} \frac{f(x)}{x} = 0$ and $\lim_{x\to 0^+} \frac{f(x)}{x} = \infty$), it is easy to check that there always exist two positive numbers $a, b (a \ne b)$ such that $\Phi(a) \le aP$ and $\Psi(b) \ge bP$. Then, the existence results can be obtained according to Theorem 3.1 of this paper. Hence, Theorem 3.1 is an extension of Theorem 1 of [4].

EXAMPLE 5.2. Let $m = 3, \xi = \frac{1}{2}, a = \frac{2}{3}, b = 0, a(t) \equiv 1, f(t,u) = f(u) = 19 + e^{-u}$, then f is decreasing and convex, and, we can calculate

$P = \frac{18}{19}, Q = \frac{18}{10}, \gamma = \frac{1}{2}$. Choosing $M = \frac{190}{19}$, then

$$f(0) = 19 < MP = \frac{190}{19} \times \frac{18}{19} = 20,$$

$$f(M) = 19 + e^{-\frac{190}{19}} > \frac{1}{2}MQ = \frac{1}{2} \times \frac{190}{19} \times \frac{18}{10} = 19.$$

By theorem 4.1, we know the following three-point boundary value problem

$$u''(t) + 19 + e^{-u} = 0, u'(0) = 0, u(1) = \frac{2}{3}u(\frac{1}{2})$$

has a unique positive solution.

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