

**LOCALLY NILPOTENT GROUPS
WITH THE MAXIMAL CONDITION
ON INFINITE NORMAL SUBGROUPS**

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ABSTRACT. A group G is said to satisfy the maximal condition on infinite normal subgroups if there does not exist an infinite properly ascending chain of infinite normal subgroups. We characterize the structure of locally nilpotent groups satisfying this chain condition. We then show how to construct locally nilpotent groups with the maximal condition on infinite normal subgroups, but not the maximal condition on subgroups.

1. Introduction

The first appearance of chain conditions in algebra was in the work of E. Noether and E. Artin in the 1920s and 1930s. The maximal and minimal conditions on ideals of a ring were considered. Subsequently the maximal and minimal conditions on classes of subgroups of a group were studied by R. Baer, S. N. Chernikov, K. A. Hirsch, O. J. Schmidt and others. Recently there has been interest by group theorists in weaker forms of the maximal and minimal conditions on classes of subgroups.

A group G is said to satisfy the weak maximal condition on normal subgroups if there does not exist an infinite properly ascending chain of normal subgroups $G_1 < G_2 < \cdots$ such that $|G_{i+1} : G_i| = \infty$ for each i . Similarly a group G is said to satisfy the weak minimal condition on normal subgroups if there does not exist an infinite properly descending chain of normal subgroups $G_1 > G_2 > \cdots$ such that $|G_i : G_{i+1}| = \infty$ for each i . In the late 1970s and early 1980s Kurdachenko considered groups satisfying the weak maximal or the weak minimal conditions on normal subgroups and their connection with each other and with other finiteness

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conditions, particularly within the class of locally nilpotent groups. Kurdachenko [1] proved that a torsion(torsion-free) locally nilpotent group satisfies the weak maximal condition on normal subgroups if and only if it is Chernikov (minimax respectively). In addition, in [2] he gave necessary and sufficient conditions for a locally nilpotent group to satisfy the weak minimal condition on normal subgroups.

A group is said to satisfy $max-\infty$ (the maximal condition on infinite subgroups) if there is no infinite properly ascending chain of infinite subgroups. A group satisfying the conditions $max-\infty s$ (the maximal condition on infinite subnormal subgroups) or $max-\infty n$ (the maximal condition on infinite normal subgroups) can be defined similarly by imposing the conditions on infinite subnormal or infinite normal subgroups respectively. A group G is said to satisfy $min-\infty$ (the minimal condition on infinite subgroups of infinite index) if there does not exist an infinite properly descending chain of subgroups with infinite index in G . Again, we can define groups satisfying $min-\infty s$ (the minimal condition on infinite subnormal subgroups of infinite index) or $min-\infty n$ (the minimal condition on infinite normal subgroups of infinite index) analogously.

Groups satisfying $max-\infty$ or $min-\infty$ and the structure of solvable groups with $max-\infty s$ or $min-\infty s$ were investigated in [3, 4]. Also in [5] locally nilpotent groups satisfying $min-\infty n$ were considered. In this paper we focus on the structure of locally nilpotent groups satisfying $max-\infty n$.

It is clear that if a group satisfies $max-\infty n$ then it satisfies the weak maximal condition on normal subgroups. The additive group of p -adic rationals \mathbb{Q}_p satisfies both the weak maximal and weak minimal conditions, but it satisfies neither $max-\infty$ nor $min-\infty$. Thus the condition $max-\infty n$ is stronger than the weak maximal condition on normal subgroups. A motivation for our study is to estimate the influence of the condition $max-\infty n$ on the subgroup lattice of the infinite normal subgroups.

2. Preliminary results

We first note that if G is a group satisfying $max-\infty n$ and N is an infinite normal subgroup of G , then G/N satisfies $max-n$ (the maximal condition on normal subgroups). It is also clear that the property $max-\infty n$ is quotient closed.

We write $\gamma_k(G)$ for the k th term in the lower central series of a group G . Recall that G' is the derived subgroup of G , being generated by all

commutators in G : thus $G' = [G, G]$. The following simple result is well-known.

LEMMA 2.1 ([6], Lemma 2.22). *Let $X \leq G$, let $N \triangleleft G$ and suppose that $G = XN'$. Then $G = X\gamma_i(N)$ for every positive integer i .*

The next result is very useful in investigating the structure of locally nilpotent groups satisfying $\text{max-}\infty n$.

LEMMA 2.2. *Let G be a locally nilpotent group such that G/G' is finitely generated. Then $\gamma_c(G) = \gamma_{c+1}(G) = \dots$ for some c and $G/\gamma_c(G)$ is nilpotent.*

Proof. Let $G = XG'$ for some finitely generated subgroup X . Then X is nilpotent of class $c - 1$, say. Now $G = X\gamma_n(G)$ for all positive n by Lemma 2.1. Let “bars” denote quotient groups modulo $\gamma_{c+1}(G)$. Then, since $G = X\gamma_{c+1}(G)$, we have $\overline{G} = \overline{X}$, which means that \overline{G} has nilpotent class at most $c - 1$ and $\gamma_c(G) = \gamma_{c+1}(G)$. Hence $\gamma_c(G) = [\gamma_c(G), G]$ and so $\gamma_c(G) = \gamma_{c+1}(G) = \dots$. Of course $G/\gamma_c(G)$ is nilpotent. \square

Recall that a group which is an extension of a finite direct product of quasicyclic groups by a finite group is called a Chernikov group. We now consider necessary and sufficient conditions for a Chernikov group to satisfy $\text{max-}\infty$.

LEMMA 2.3 ([4], Theorem 2.7). *A Chernikov group satisfies $\text{max-}\infty$ if and only if it is Prüfer-by-finite.*

The following result classifies all abelian groups satisfying $\text{max-}\infty$.

LEMMA 2.4 ([4], Theorem 3.1). *An abelian group satisfies $\text{max-}\infty$ if and only if it is either finitely generated or Prüfer-by-finite*

3. Locally nilpotent groups with $\text{Max-}\infty n$

We first need to characterize nilpotent groups satisfying $\text{max-}\infty n$. We write *max* and *min* for the maximal and minimal conditions on subgroups respectively. We recall well-known results of Baer: (i) if G is a nilpotent group and G_{ab} is finitely generated, then G satisfies *max* and (ii) a nilpotent groups G satisfies *min* if and only if $Z(G)$ satisfies *min* and $G/Z(G)$ is finite.

THEOREM 3.1. *A nilpotent group G satisfies $\text{max-}\infty n$ if and only if it is either finitely generated or central Prüfer-by-finite.*

Proof. Sufficiency follows from Lemma 2.3. Suppose that G satisfies $\text{max-}\infty n$, but not max . Since $G_{ab} = G/G'$ does not satisfy max , it is Prüfer-by-finite by Lemma 2.4. Hence G_{ab} has min and so does G . Thus $Z(G)$ satisfies min and $G/Z(G)$ is finite. Now $Z(G)$ is a direct sum of finitely many Prüfer groups and finite cyclic groups. Thus $Z(G)$ is Chernikov and so it is Prüfer-by-finite by Lemma 2.3. Therefore G has a central Prüfer subgroup of finite index. \square

Let G and N be groups with G acting on N , then a subgroup H of N is a G -invariant group if $H^g = H$ for all $g \in G$. We proceed now to identify the structure of locally nilpotent groups satisfying $\text{max-}\infty n$.

THEOREM 3.2. *Let G be a locally nilpotent group with $\text{max-}\infty n$, but not max . Then G has a unique minimal infinite normal subgroup N ; further*

1. G/N is finitely generated nilpotent;
2. G is hypercentral;
3. $C_G(N)$ is torsion;
4. N is either a divisible abelian p -group of finite rank or an infinite elementary abelian p -group.

Proof. Suppose that G satisfies $\text{max-}\infty n$, but not max . If G is torsion, then it is hypercentral Chernikov (see [1], Theorem 1 and Corollary 1). Now if G is nilpotent, then it is central Prüfer-by-finite by Theorem 3.1. Hence we can assume that G is neither torsion nor nilpotent. Thus G' is infinite and so G/G' is finitely generated; hence, by Lemma 2.2, G has an infinite normal subgroup N such that $N = \gamma_c(G) = \gamma_{c+1}(G) = \dots$ for some c and G/N is finitely generated.

Let L be an infinite normal subgroup of G . Then $G/L \cap N$ has max and so $L \cap N$ is infinite. Put $M = L \cap N$. If $M \neq N$, then N/M has $\text{max-}G$ (the maximal condition for G -invariant subgroups). So there is a G -principal factor N/M_1 where $M \leq M_1$. But then, since G is locally nilpotent, N/M_1 is G -central and $N = [N, G] \leq M_1 < N$. By this contradiction $M = N$. Hence N is the unique minimal infinite normal subgroup of G .

Since every proper G -invariant subgroup of N is finite, the upper hypercenter of G contains N . It follows that G is hypercentral.

Let A be a maximal normal abelian subgroup of G . Then $A = C_G(A)$ (see [7], 5.2.3) and so A is infinite; for otherwise G would be finite since G/A is isomorphic to a subgroup of $\text{Aut}(A)$. Hence $N \leq A$ and so N is abelian.

Suppose that $C = C_G(N)$ has an element a of infinite order. Then $N \leq \langle a \rangle^G$. Since $G = \langle g_1, g_2, \dots, g_m, N \rangle$ for some g_1, g_2, \dots, g_m ,

$$G = \langle g_1, g_2, \dots, g_m, a^g \mid g \in G \rangle.$$

For any $g \in G$ we can write $g = nw$ where $n \in N$ and $w \in \langle g_1, g_2, \dots, g_m \rangle$. Hence

$$a^g = a^{nw} = a^w \in \langle g_1, g_2, \dots, g_m, a \rangle.$$

Thus $G = \langle g_1, g_2, \dots, g_m, a \rangle$, a contradiction. Hence $C_G(N)$ must be torsion.

Since N is torsion abelian, it is the direct sum of its primary components. Moreover it has only finitely many non-trivial primary components by the minimality of N . Clearly N is a p -group. Now if $N[p] = \{a \in N \mid a^p = 1\}$, then either $N[p]$ is finite, and N has finite rank, or $N = N[p]$ is infinite. If N has finite rank, then it is a divisible p -group of finite rank. On the other hand, if $N = N[p]$ has infinite rank, then it is an infinite elementary abelian p -group. \square

The converse is next.

THEOREM 3.3. *Let G be a locally nilpotent group with a minimal infinite normal subgroup N such that G/N is finitely generated nilpotent and $C_G(N)$ is torsion. Then G satisfies $\max\text{-}\infty$, but not \max .*

Proof. Suppose that G has the structure indicated. Assume that $G_1 < G_2 < \dots$ is an infinite ascending chain of infinite normal subgroups of G . We consider the following two cases.

Case: $G_i \cap N$ is finite for some i . Then $G_i N/N \simeq G_i/G_i \cap N$ is infinite and so $G_i N/N$ is not torsion; for otherwise G_i is finite. Hence $G_i N/N$ has an element xN of infinite order. Since $\langle x \rangle N \leq G_i N$, we can assume that $x \in G_i$. Note that

$$[N, x^j] \leq G_i \cap N$$

for any $j > 0$. If $[N, x^j] = 1$, then $x^j \in C_G(N)$, a contradiction. Hence $[N, x^j]$ is finite with bounded order. Since $[N, x^j]^{(x)} = [N, x^j]$, it follows that

$$[N, x^j, x^k] = 1$$

for some $k > 0$ and all $j > 0$. Hence, in particular, $[N, x^k, x^k] = 1$. Let l be the order of $[N, x^k]$. Then

$$[N, x^{lk}] \leq [N, x^k]^l [N, x^k, x^k] = 1$$

by repeated use of the identity: $[a, bc] = [a, c][a, b][a, b, c]$. Hence $x^{lk} \in C_G(N)$, which is impossible.

Case: $G_i \cap N$ is infinite for all i . Then $G_i \cap N = N$ by minimality of N . Hence $N \leq G_i$ for all i . Thus $G_1/N < G_2/N < \dots$ is an infinite ascending chain of normal subgroups of G/N , a contradiction. Therefore G has $\text{max-}\infty n$.

Now if G has max , then G is finitely generated nilpotent. Hence $C_G(N)$ is finite. But $Z(G) \leq C_G(N)$ implies that $Z(G)$ is finite, and so is G , a contradiction. \square

Next we give two examples to show that the minimal infinite normal subgroup in Theorem 3.2 can be either an elementary abelian p -group or a divisible abelian p -group of finite rank.

EXAMPLE 3.4. Let $N = A_1 \times A_2 \times \dots$ be an infinite elementary abelian p -group, where $A_i = \langle a_i \rangle$ is cyclic of order p and let $X = \langle x \rangle$ be an infinite cyclic group acting on N via

$$\begin{aligned} a_1^x &= a_1; \\ a_{i+1}^x &= a_{i+1}a_i \end{aligned}$$

for all i . Thus $[a_1, x] = 1$ and $[a_{i+1}, x] = a_i$. Let G be the corresponding semidirect product $X \ltimes N$. We note that $N \leq C = C_G(N)$. Suppose $N \neq C$. Then $x^m \in C$ for some $m > 0$. We may assume that m is prime. Now

$$1 = [a_2, x^m] = [a_2, x]^m = a_1^m$$

since $[a_2, x, x] = 1$. Therefore $m = p$ and so $a^m = 1$ for all $a \in N$. On the other hand, by use of the binomial identity,

$$a^{(x-1)^m} = a^{x^m-1} = 1$$

for all $a \in N$, which is not true for a_i for $i > m$. Hence C/N must be trivial, so $C = N$ and therefore C is torsion.

Let L be an infinite normal subgroup of G with $L \leq N$. Suppose $L \neq N$. Then $a_i \notin L$ for some least $i > 0$, and $\langle a_1, a_2, \dots, a_{i-1} \rangle \leq L$. Since L is infinite, $L \not\leq \langle a_1, a_2, \dots, a_{i-1} \rangle$. Hence L has an element u such that

$$u = a_1^{k_1} a_2^{k_2} \dots a_r^{k_r}$$

with $r \geq i$ and $k_r \not\equiv 0 \pmod{p}$. Now inductive computation shows that

$$[u, r-i x] = a_1^{k_{r-i+1}} a_2^{k_{r-i+2}} \dots a_i^{k_r}.$$

Let $v = a_1^{k_{r-i+1}} a_2^{k_{r-i+2}} \dots a_{i-1}^{k_{r-1}}$. Then $v \in L$ and $[u, r-i x] = v a_i^{k_r} \in L$ since $L \triangleleft G$. Hence $a_i^{k_r} \in L$ and $a_i \in L$ since $k_r \not\equiv 0 \pmod{p}$, a contradiction. Consequently $L = N$ and so N is a minimal infinite normal subgroup of G . Therefore G satisfies $\text{max-}\infty n$, but not max by Theorem 3.3. Note that G is a locally nilpotent group.

Next we consider the divisible case.

EXAMPLE 3.5. First we give an example where the minimal infinite normal subgroup N is a divisible abelian group of rank 1. Let $G = X \ltimes N$ where N is of type p^∞ and $X = \langle x \rangle$ is an infinite cyclic group acting on N via $a^x = a^\alpha$ for all $a \in N$ where α is a p -adic unit of infinite order such that $\alpha \equiv 1 \pmod{p}$ or $\alpha \equiv 1 \pmod{4}$ if $p = 2$. Then $C_G(N) = N$ and G satisfies $\text{max-}\infty n$, but not max by Theorem 3.3.

Next we construct an example where N has arbitrary rank $k > 1$. Let $N = N_1 \oplus N_2 \oplus \dots \oplus N_k$, each N_i of type p^∞ . Then $\text{Aut}(N) \simeq GL(k, R_p)$ where R_p is the ring of p -adic integers. We need to find $\alpha \in GL(k, R_p)$ which acts irreducibly on N in the sense that N will have no proper infinite $\langle \alpha \rangle$ -invariant subgroups. Also we need $(\alpha - 1)^k \equiv 0 \pmod{p}$ to ensure that G is locally nilpotent. Here we choose a specific α as follows.

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & p \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

Notice that $\det(\alpha) \equiv 1 \pmod{p}$, so $\alpha \in GL_k(R_p)$ and $\alpha^m \neq 1$ if $m \neq 0$. Now the characteristic polynomial of α is

$$f(x) = (-1)^k \{(x - 1)^k - px^{k-2}\}.$$

We show that $f(x)$ is irreducible over R_p . It is sufficient to show that $f(x + 1)$ is irreducible over R_p . We note that R_p is a principal ideal domain, and so it is a unique factorization domain. Hence $f(x + 1)$ is irreducible over R_p by Eisenstein's Criterion. Hence α acts irreducibly on N . Thus N is the minimal infinite $\langle \alpha \rangle$ -invariant subgroup. Furthermore $C_G(N) = N$ and $(\alpha - 1)^k \equiv 0 \pmod{p}$ since $(\alpha - 1)^k - p\alpha^{k-2} = 0$. Thus $G = \langle \alpha \rangle \ltimes N$ is a locally nilpotent group with $\text{max-}\infty n$, but not max .

Next we show how to construct locally nilpotent groups with $\text{max-}\infty n$, but not max . Let X be a finitely generated nilpotent group and let M be a minimal infinite X -module (that is, an infinite $\mathbb{Z}X$ -module whose proper submodules are finite). Form the semidirect product

$$G(X, M) = X \ltimes M.$$

Then we have

THEOREM 3.6. *Suppose that G is a locally nilpotent group with $\max\text{-}\infty n$, but not \max . Then there is a finite normal subgroup F such that G/F is isomorphic with some $G(X, M)$. Conversely, if G is a locally nilpotent group which is an extension of a finite group by some $G(X, M)$, then G has $\max\text{-}\infty n$, but not \max .*

Proof. Suppose that G has $\max\text{-}\infty n$, but not \max . Then, by Theorem 3.2, G has a minimal infinite normal subgroup N which is either a divisible abelian p -group of finite rank or an infinite elementary abelian p -group, G/N is a finitely generated nilpotent group, and $C_G(N)$ is torsion. Write $G = XN$ where X is a finitely generated subgroup. Since X is finitely generated nilpotent, it satisfies \max . Hence $X \cap N$ is finitely generated and so it is finite. We note that $X \cap N \triangleleft XN = G$. Put $F = X \cap N$. We now pass to the group $\overline{G} = G/F$. Then $\overline{X} \overline{N} = \overline{G}$ and $\overline{X} \cap \overline{N} = \overline{1}$; hence $\overline{G} = \overline{X} \rtimes \overline{N} = G(\overline{X}, \overline{N})$.

Conversely, suppose that M is a minimal infinite X -module where X is a finitely generated nilpotent group. Then $G(X, M)$ satisfies $\max\text{-}\infty n$, but not \max and the rest is routine by Theorem 3.3. \square

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