Bull. Korean Math. Soc. **41** (2004), No. 3, pp. 457–464

HEREDITARY PROPERTIES OF CERTAIN IDEALS OF COMPACT OPERATORS

CHONG-MAN CHO AND EUN JOO LEE

ABSTRACT. Let X be a Banach space and Z a closed subspace of a Banach space Y. Denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X,Y)$ its subspace of compact linear operators. Using Hahn-Banach extension operators corresponding to ideal projections, we prove that if either X^{**} or Y^* has the Radon-Nikodým property and $\mathcal{K}(X,Y)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,Y)$, then $\mathcal{K}(X,Z)$ is also an M-ideal (resp. HB-subspace) in $\mathcal{L}(X,Z)$. If $\mathcal{K}(X,Y)$ has property SU instead of being an M-ideal in $\mathcal{L}(X,Y)$ in the above, then $\mathcal{K}(X,Z)$ also has property SU in $\mathcal{L}(X,Z)$. If X is a Banach space such that X^* has the metric compact approximation property with adjoint operators, then M-ideal (resp. HB-subspace) property of $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$ is inherited to $\mathcal{K}(X,Z)$ in $\mathcal{L}(X,Z)$.

1. Introduction

A closed subspace E of a Banach space X is called an *ideal* in X if $E^{\perp}$, the annihilator of E in X^*, is the kernel of a norm one projection P on X^*. In this case P is called the *ideal projection*. The notion of an ideal in a Banach space was introduced by Godefroy, Kalton and Shaper [4] in 1993.

Let E be an ideal in X with the ideal projection P on X^*, let $x^* \in X^*$ and consider the following norm conditions ;

$$(1.1) \qquad \|x^*\| = \|Px^*\| + \|(I-P)x^*\|,$$

$$(1.2) \qquad \|x^*\| > \|Px^*\| \quad \text{if } x^* \neq Px^*,$$

$$(1.3) \qquad \|x^*\| \geq \|x^* - Px^*\|,$$

Received September 9, 2003.

2000 Mathematics Subject Classification: 46B20, 46B28.

Key words and phrases: ideal, M-ideal, HB-subspace, property SU, compact operator.

The first named author was supported by Hanyang University, Korea, in the program year of 2002.

$$(1.4) \qquad \|x^*\| \;\geq\; \|x^* - 2Px^*\|.$$

An ideal E is called an M-ideal if the condition (1.1) holds for all $x^* \in X^*$. An M-ideal was introduced by Alfsen and Effros [1] in 1972 and has been studied seriously by many authors [5].

Following Hennefeld [6], an ideal E is called an HB-subspace if conditions (1.2) and (1.3) hold for all $x^* \in X^*$. It is easy to see that an HB-subspace has property U in the sense of Phelps. According to Phelps [17], a subspace E of a Banach space X is said to have *property* U in X if every $e^* \in E^*$ has a unique norm-preserving extension $x^* \in X^*$. E. Oja [15] defined property SU which is an intermediate property between property U and HB-subspace. A subspace E is said to have *property* SU in X if E is an ideal in X and the condition (1.2) holds for all $x^* \in X^*$.

An ideal E is called a u-ideal if condition (1.4) holds. A u-ideal was introduced by Casazza and Kalton [2].

An ideal is closely linked with a Hahn-Banach extension operator. For a closed subspace E of a Banach space X a linear operator $\phi : E^* \to X^*$ is called a Hahn-Banach extension operator if $\phi(e^*)$ is a norm preserving extension of e^* for all $e^* \in E^*$. It is well known that there exists a Hahn-Banach extension operator $\phi : E^* \to X^*$ if and only if E is an ideal in X. In this case, the Hahn-Banach extension operator ϕ and the corresponding ideal projection $P : X^* \to X^*$ are related by $Px^* = \phi(x^*|_E)$, where $x^*|_E$ is the restriction of x^* to E. Therefore, if a subspace E is an ideal with property U in X, then the ideal projection is unique.

Let X and Y be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X,Y)$ its subspace of compact operators.

In 1994, Lima, Oja, Rao and Werner [14] proved a sort of hereditary property of an M-ideal for $\mathcal{K}(X,Y)$. More specifically, they proved the following results.

THEOREM 1.1. *Suppose that X^{**} or Y^* has the Radon-Nikodým property and that $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$.*

(a) *If X^* has the bounded compact approximation property with adjoint operators and Z is a closed subspace of Y, then $\mathcal{K}(X,Z)$ is an M-ideal in $\mathcal{L}(X,Z)$.*
(b) *If Y^* has the bounded compact approximation property with adjoint operators and E is a closed subspace of X, then $\mathcal{K}(X/E,Y)$ is an M-ideal in $\mathcal{L}(X/E,Y)$.*

In this paper, we will investigate various ideal properties of $\mathcal{K}(X,Z)$ in $\mathcal{L}(X,Z)$ inherited from those of $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$ for a closed subspace Z of Y.

In Theorem 3.3, we will assume that X and Y are Banach spaces such that either X^{**} or Y^* has the Radon-Nikodým property and show that if $\mathcal{K}(X,Y)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,Y)$, then $\mathcal{K}(X,Z)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,Z)$. If $\mathcal{K}(X,Y)$ has property SU in $\mathcal{L}(X,Y)$, then $\mathcal{K}(X,Z)$ also has property SU in $\mathcal{L}(X,Z)$. The idea of proofs is using suitable Hahn-Banach extension operators corresponding to ideal projections and using Feder-Saphar representation of the dual space of certain space of compact operators (Theorem 2.1).

In Theorem 3.5 we prove that if X^* has the metric compact approximation property with adjoint operators, then M-ideal (resp. HB-subspace) property of $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$ is inherited to $\mathcal{K}(X,F)$ in $\mathcal{L}(X,F)$, where F is a closed subspace of Y. The same properties are inherited to $\mathcal{K}(X/E,Y)$ in $\mathcal{L}(X/E,Y)$ if Y has the metric compact approximation property, where E is a closed subspace of X.

2. Preliminaries

A Banach space X is said to have the *compact approximation property* if there exists a net (K_α) in $\mathcal{K}(X)$ such that $K_\alpha x \to x$ for all $x \in X$. If the net (K_α) in $\mathcal{K}(X)$ above can be chosen to be $\|K_\alpha\| \le 1$ for all α, then we say that X has the *metric compact approximation property.* The dual space X^* of X is said to have the *compact approximation property with adjoint operators* if there exists a net (K_α) in $\mathcal{K}(X)$ such that $K_\alpha^* x^* \to x^*$ for all $x^* \in X^*$. We say that X^* has the *metric compact approximation property with adjoint operators* if the net (K_α) above can be taken to be $\|K_\alpha\| \le 1$ for all α.

Let $X\widehat{\otimes}Y$ be the projective tensor product of Banach spaces X and Y. If $v \in X\widehat{\otimes}Y$, then there exist sequences (x_n) in X and (y_n) in Y such that $v = \sum_{n=1}^\infty x_n \otimes y_n$, and $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$. Moreover, $\|v\| = \inf\{\sum_{n=1}^\infty \|x_n\|\|y_n\|\}$ with infimum being taken over all representations $v = \sum_{n=1}^\infty x_n \otimes y_n$, $x_n \in X$, $y_n \in Y$.

Let $v = \sum_{n=1}^\infty y_n^* \otimes x_n^{**} \in Y^*\widehat{\otimes}X^{**}$ with $\sum_{n=1}^\infty \|y_n^*\|\|x_n^{**}\| < \infty$. For any $T \in \mathcal{L}(X,Y)$, we define $T^{**}v = \sum_{n=1}^\infty y_n^* \otimes T^{**}x_n^{**}$. Then $T^{**}v \in Y^*\widehat{\otimes}Y^{**}$ and the map $T \to \text{trace}(T^{**}v) = \sum_{n=1}^\infty (T^{**}x_n^{**})(y_n^*)$ defines a bounded linear functional on $\mathcal{L}(X,Y)$ with norm no larger than $\|v\|$.

The following theorem is originally due to M. Feder and P. Sapher [3] and slightly modified by Lima, Nygaard and Oja [9].

THEOREM 2.1. *Let X and Y be Banach spaces such that either X^{**} or Y^* has the Radon-Nikodým property. Let $\Psi : Y^*\widehat{\otimes}X^{**} \to \mathcal{L}(X,Y)^*$ be defined by*

$$(\Psi v)(T) = trace(T^{**}v)$$

for all $T \in \mathcal{L}(X,Y)$ and $v \in Y^*\widehat{\otimes}X^{**}$. Then $\tau^*\Psi : Y^*\widehat{\otimes}X^{**} \to \mathcal{K}(X,Y)^*$ is a quotient map, where $\tau : \mathcal{K}(X,Y) \to \mathcal{L}(X,Y)$ is the inclusion map. Moreover, for each $f \in \mathcal{K}(X,Y)^*$, there exists $v \in Y^*\widehat{\otimes}X^{**}$ such that $f = (\Psi v)|_{K(X,Y)}$ and $\|f\| = \|\Psi v\|$.

3. Hereditary properties of ideals of compact operators.

Let X and Y be Banach spaces, and let $E \subset X$ and $F \subset Y$ be closed subspaces. In this section we will investigate various ideal properties of $\mathcal{K}(X/E,F)$ in $\mathcal{L}(X/E,F)$ inherited from the corresponding ideal properties of $\mathcal{K}(X,Y)$ in $\mathcal{L}(X,Y)$.

Let $\pi : X \to X/E$ be the canonical projection and $i : F \to Y$ the inclusion mapping. Define $I : \mathcal{L}(X/E,F) \to \mathcal{L}(X,Y)$ and $J : \mathcal{K}(X/E,F) \to \mathcal{K}(X,Y)$ by $I(T) = i \circ T \circ \pi$ and $J(K) = i \circ K \circ \pi$ for $T \in \mathcal{L}(X/E,F)$ and $K \in \mathcal{K}(X/E,F)$, respectively. Then I and J are isometries into $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$, respectively. By a diagram chase we can easily check the following Lemma.

LEMMA 3.1. *Let X and Y be Banach spaces and let $E \subset X$ and $F \subset Y$ be closed subspaces. If $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ is a Hahn-Banach extension operator and $\phi_2 : \mathcal{K}(X/E,F)^* \to \mathcal{L}(X/E,F)^*$ is a linear operator such that*

$$I^* \circ \phi_1 = \phi_2 \circ J^*$$

then ϕ_2 is a Hahn-Banach extension operator.

PROPOSITION 3.2. *Let X and Y be Banach spaces and let $E \subset X$ and $F \subset Y$ be closed subspaces. Suppose that $\mathcal{K}(X,Y)$ is an M-ideal (resp. an HB-subspace, or a u-ideal) in $\mathcal{L}(X,Y)$ with an ideal projection P. If $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ is the Hahn-Banach extension operator associated with P and $\phi_2 : \mathcal{K}(X/E,F)^* \to \mathcal{L}(X/E,F)^*$ is a linear operator such that*

$$I^* \circ \phi_1 = \phi_2 \circ J^*,$$

then $\mathcal{K}(X/E,F)$ is an M-ideal (resp. an HB-subspace, or a u-ideal) in $\mathcal{L}(X/E,F)$. If $\mathcal{K}(X,Y)$ has property SU in $\mathcal{L}(X,Y)$, then $\mathcal{K}(X/E,F)$ also has property SU in $\mathcal{L}(X/E,F)$.

Proof. By Lemma 3.1 ϕ_2 is a Hahn-Banach extension operator, and so $\mathcal{K}(X/E,F)$ is an ideal in $\mathcal{L}(X/E,F)$. Let Q be the ideal projection on $\mathcal{L}(X/E,F)^*$ induced by ϕ_2. Then for $f \in \mathcal{L}(X,Y)^*$ and $g \in \mathcal{L}(X/E,F)^*$ we have $Pf = \phi_1(\bar{f})$ and $Qg = \phi_2(\bar{g})$, where $\bar{f}$ and $\bar{g}$ are the restrictions of f and g to $\mathcal{K}(X,Y)$ and $\mathcal{K}(X/E,F)$, respectively.

If $g \in \mathcal{L}(X/E,F)^*$, then there exists $f \in \mathcal{L}(X,Y)^*$ such that $\|g\| = \|f\|$, $g = I^*f$, and so $\bar{g} = J^*\bar{f}$. Therefore, $Qg = \phi_2(\bar{g}) = \phi_2(J^*\bar{f}) = I^*\phi_1\bar{f} = I^*Pf$.

If $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$, then

$$\begin{aligned}\|g\| &\leq \|I^*Pf\| + \|I^*f - I^*Pf\| \\ &\leq \|Pf\| + \|f - Pf\| \\ &= \|f\| = \|g\|,\end{aligned}$$

and hence

$$\begin{aligned}\|g\| &= \|I^*Pf\| + \|I^*f - I^*Pf\| \\ &= \|Qg\| + \|g - Qg\|.\end{aligned}$$

Therefore, $\mathcal{K}(X/E,F)$ is an M-ideal in $\mathcal{L}(X/E,F)$.

If $\mathcal{K}(X,Y)$ has property SU in $\mathcal{L}(X,Y)$ and $g \neq Qg$, then $f \neq Pf$ and so

$$\|g\| = \|f\| > \|Pf\| \geq \|I^*Pf\| = \|Qg\|.$$

Therefore, $\mathcal{K}(X/E,F)$ has property SU in $\mathcal{L}(X/E,F)$.

The inequalities,

$$\|g\| = \|f\| \geq \|f - Pf\| \geq \|I^*f - I^*Pf\| = \|g - Qg\|$$

and

$$1 \geq \|f - 2Pf\| \geq \|I^*f - 2I^*Pf\| = \|g - 2Qg\|$$

prove HB-subspaces and a u-ideal cases. □

In the above Proposition the existence of a linear operator $\phi_2 : \mathcal{K}(X/E,F)^* \to \mathcal{L}(X/E,F)^*$ such that $I^* \circ \phi_1 = \phi_2 \circ J^*$ plays a key role. An interesting question is when such an operator exists. Theorem 3.3 and Theorem 3.5 are cases in which such operators exist.

THEOREM 3.3. *Let X and Y be Banach spaces such that either X^{**} or Y^* has the Radon-Nikodým property. If $\mathcal{K}(X,Y)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,Y)$, then for every closed subspace Z of Y $\mathcal{K}(X,Z)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,Z)$. If $\mathcal{K}(X,Y)$*

has property SU in $\mathcal{L}(X,Y)$, *then* $\mathcal{K}(X,Z)$ *also has property SU in* $\mathcal{L}(X,Z)$ *for every closed subspace* Z *of* Y.

Proof. Let Z be a closed subspace of Y. Observe that if Y^* has the Radon-Nikodým property, then Z^* also has the Radon-Nikodým property. Let $\Psi : Z^*\widehat{\otimes}X^{**} \to \mathcal{L}(X,Z)^*$ be defined as in Theorem 2.1. Since $\mathcal{K}(X,Z)^* = Z^*\widehat{\otimes}X^{**}/\ker\Psi$, there exists a bounded linear operator $\phi_2 : \mathcal{K}(X,Z)^* \to \mathcal{L}(X,Z)^*$ such that $\Psi = \phi_2 \circ \rho$, where $\rho : Z^*\widehat{\otimes}X^{**} \to Z^*\widehat{\otimes}X^{**}/\ker\Psi$ is the canonical projection. Then we have that

$$\phi_2(z^* \otimes x^{**}) = z^* \otimes x^{**} \quad \text{for all } x^{**} \in X^{**} \text{ and } z^* \in Z^*.$$

Let $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ be the Hahn-Banach extension operator corresponding to the ideal projection P on $\mathcal{L}(X,Y)^*$ with $\ker P = \mathcal{K}(X,Y)^\perp$. By Proposition 3.2, it suffices to show that $I^* \circ \phi_1 = \phi_2 \circ J^*$. Since either X^{**} or Y^* has the Radon-Nikodým property, by Theorem 2.1, every $f \in \mathcal{K}(X,Y)^*$ has a representation

$$f = \sum_n y_n^* \otimes x_n^{**}, \quad \sum_n \|x_n^{**}\|\|y_n^*\| < \infty, \; x_n^{**} \in X^{**}, \; y_n^* \in Y^*.$$

Therefore, it is sufficient to show that for each $x^{**} \in X^{**}$ and $y^* \in Y^*$,

$$I^* \circ \phi_1(y^* \otimes x^{**}) = \phi_2 \circ J^*(y^* \otimes x^{**}).$$

Since $\mathcal{K}(X,Y)$ has property U in $\mathcal{L}(X,Y)$, we have

$$\phi_1(y^* \otimes x^{**}) = y^* \otimes x^{**} \quad \text{for all } x^{**} \in X^{**} \text{ and } y^* \in Y^*.$$

Therefore, we have

$$\begin{aligned} I^* \circ \phi_1(y^* \otimes x^{**}) &= I^*(y^* \otimes x^{**}) \\ &= i^*y^* \otimes x^{**} \\ &= \phi_2(i^*y^* \otimes x^{**}) \\ &= \phi_2 \circ J^*(y^* \otimes x^{**}). \end{aligned}$$

□

Since a reflexive Banach space has the Radon-Nikodým property, we have the following corollary.

Corollary 3.4. *Let* X *be a reflexive Banach space and* Z *a closed subspace of a Banach space* Y. *If* $\mathcal{K}(X,Y)$ *is an* M*-ideal (resp. an* HB*-subspace) in* $\mathcal{L}(X,Y)$, *then* $\mathcal{K}(X,Z)$ *is an* M*-ideal (resp. an* HB*-subspace) in* $\mathcal{L}(X,Z)$. *If* $\mathcal{K}(X,Y)$ *has property SU in* $\mathcal{L}(X,Y)$, *then* $\mathcal{K}(X,Z)$ *also has property SU in* $\mathcal{L}(X,Z)$.

In 1979, J. Johnson [7] proved that if X and Y are Banach spaces, and Y has the metric compact approximation property, then there exists a Hahn Banach extension operator $\phi : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$. His construction of ϕ easily proves the following Theorem (cf. [14, Proposition 2.9, Corollary 2.10]).

THEOREM 3.5. *Let X and Y be Banach spaces. Suppose that $\mathcal{K}(X, Y)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,Y)$.*

(a) *If X^* has the metric compact approximation property with adjoint operators, then $\mathcal{K}(X,F)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X,F)$ for every closed subspace F of Y.*
(b) *If Y has the metric compact approximation property, then $\mathcal{K}(X/E, Y)$ is an M-ideal (resp. an HB-subspace) in $\mathcal{L}(X/E,Y)$ for every closed subspace E of X.*

Proof. (a) Let (K_α) be a net in $K(X)$ such that $\|K_\alpha\| \leq 1$ for all α and $K_\alpha^* x^* \to x^*$ for all $x^* \in X^*$. Then, by passing to a subnet of (K_α), which we still denote by (K_α), we can define Hahn-Banach extension operators [14, Lemma 1] $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ and $\phi_2 : \mathcal{K}(X,F)^* \to \mathcal{L}(X,F)^*$ by

$$\phi_1(f)(T) = \lim_\alpha f(T \circ K_\alpha), \quad f \in \mathcal{K}(X,Y)^*, T \in \mathcal{L}(X,Y)$$

and

$$\phi_2(g)(T) = \lim_\alpha g(T \circ K_\alpha), \quad g \in \mathcal{K}(X,F)^*, T \in \mathcal{L}(X,F).$$

Then we can easily check that $I^* \circ \phi_1 = \phi_2 \circ J^*$. Now we appeal to Proposition 3.2 to finish the proof.

(b) We choose a suitable net (S_α) of compact operators on Y such that $\|S_\alpha\| \leq 1$ for all α and $S_\alpha y \to y$ for all $y \in Y$. we can define Hahn-Banach extension operators [14, Lemma 1] $\phi_1 : \mathcal{K}(X,Y)^* \to \mathcal{L}(X,Y)^*$ and $\phi_2 : \mathcal{K}(X/E,Y)^* \to \mathcal{L}(X/E,Y)^*$ by

$$\phi_1(f)(T) = \lim_\alpha f(S_\alpha \circ T), \quad f \in \mathcal{K}(X,Y)^*, T \in \mathcal{L}(X,Y)$$

and

$$\phi_2(g)(T) = \lim_\alpha g(S_\alpha \circ T), \quad g \in \mathcal{K}(X/E,Y)^*, T \in \mathcal{L}(X/E,Y).$$

Then $I^* \circ \phi_1 = \phi_2 \circ J^*$ and another appeal to Proposition 3.2 finishes the proof. □

References

[1] E. M. Alfsen and E. G. Effros, *Structure in real Banach spaces*, Ann. of Math. **96** (1972), 98–173.

[2] P. G. Casazza and N. J. Kalton, *Notes on approximation properties in separable Banach spaces*, In Geometry of Banach spaces, Proc. Conf. Strobl. (1989) (eds. P.F.X. Müller and W. Schachermayer). London Math. Soc. Lecture Note Ser. **158** (1990), 49–63.

[3] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. **21** (1975), 38–49.

[4] G. Godefroy, N. J. Kalton and P. D. Saphar, *Unconditional ideals in Banach spaces*, IBID Press. **104** (1993), 13–59.

[5] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach Algebras,* Lecture Notes in Math. **1547** (Springer Berlin-Heiderberg-New York 1993).

[6] J. Hennefeld, *M-ideals, HB-subspace, and compact operator*, Indiana Univ. Math. J. **28** (1979), 927–934.

[7] J. Johnson, *Remarks on Banach spaces of compact operators*, J. Funct. Anal. **32** (1979), 304–311.

[8] Å. Lima, *Uniqueness of Hahn-Banach extensions and liftings of linear dependences*, Math. Scand. **53** (1983), 97–113.

[9] Å. Lima, O. Nygaard and E. Oja, *Isometric factorization of weakly compact operators and the approximation property*, Israel J. Math. **119** (2000), 325–348.

[10] Å. Lima and E. Oja, *Ideals of finite rank operators, intersection properties of balls, and the approximation property*, Studia Math. **133** (1999), no. 2, 175–186.

[11] ———, *Ideals of compact operators*, preprint.

[12] ———, *Hahn-Banach extension operators and spaces of operators*, Proc. Amer. Math. Soc. **130** (2002), 3631–3640.

[13] ———, *Ideals of operators, approximality in the strong operator toporogy, and the approximation property*, preprint.

[14] Å. Lima, E. Oja, T.S.S.R.K. Rao and D. Werner, *Geometry of operator spaces*, Michigan Math. J. **41** (1994), 473–490.

[15] E. F. Oja, *Strong uniqueness of the extension of linear continuous functionals according to the Hahn-Banach theorem*, Mat. Zametki **43** (1988), 237–246 (in Russian) Math. Notes **43** (1988), 134–139.

[16] ———, *HB-subspaces and Godun sets of subspaces in Banach spaces*, Matematika **44** (1997), 120–132.

[17] R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximation*, Trans. Amer. Math. Soc. **95** (1960), 238–255.

Department of Mathematics, Hanyang University, Seoul 133-791, Korea
E-mail: cmcho@hanyang.ac.kr
woorimir@kornet.net