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ON THE PRESSURE OF CONTINUOUS TRANSFORMATIONS

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Abstract. The purpose of the paper is to correct erroneous proofs of two theorems of Walters ([1]) about the pressure of continuous transformations.

1. Introduction

In this paper we provide the proofs of the following:

Theorem A. *If $T : X \to X$ is continuous and $k > 0$, then*

$$Q(T, \Phi, \mathcal{A}_0^k) = Q(T, \Phi, \mathcal{A}), \quad P(T, \Phi, \mathcal{A}_0^k) = P(T, \Phi, \mathcal{A}).$$

If T is a homeomorphism and $m, k > 0$, then

$$Q(T, \Phi, \mathcal{A}_{-m}^k) = Q(T, \Phi, \mathcal{A}), \quad P(T, \Phi, \mathcal{A}_{-m}^k) = P(T, \Phi, \mathcal{A}).$$

Theorem B. *Let $T : X \to X$ be continuous and $\Phi \in C(X, \mathbb{R})$. If $m > 0$, then*

$$P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) = mP(T, \Phi).$$

These theorems were proved by Walters in [1, Lemma 1.8] and [1, Theorem 2.2(i)], respectively. However, his proofs have partial defects.

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In the proof of Theorem A, he used the fact that any subcover of $(\mathcal{A}_0^k)_0^{n-1}$ becomes a subcover of $\mathcal{A}_0^{k+n-1}$ in proving that

$$Q_n(T,\Phi,\mathcal{A}_0^k)$$
$$= \inf\Big\{\sum_{A\in\alpha}\inf\{\exp\sum_{i=0}^{n-1}\Phi(T^ix) : x\in A\} : \alpha \text{ is a subcover of } (\mathcal{A}_0^k)_0^{n-1}\Big\}.$$

But this statement is not true in general. Any element A of $(\mathcal{A}_0^k)_0^{n-1}$ is of the form $A = \bigcap_{i=0}^{n-1} T^{-i}\bigcap_{j=0}^{k} T^{-j}A_j^i$ where $A_j^i \in \mathcal{A}$. But, $A = \cap_{l=0}^{k+n-1}T^{-l}\cap_{i+j=l}A_j^i$ does not imply $\cap_{i+j=l}A_j \in \mathcal{A}$. In general, any subcover of $(\mathcal{A}_0^k)_0^{n-1}$ is not a subcover of $\mathcal{A}_0^{k+n-1}$.

In the proof of Theorem B, he proved that $Q_n(T^m, \sum_{i=0}^{m-1}\Phi\circ T^i,\delta) \geq Q_{nm}(T,\Phi,\varepsilon)$ using

$$\limsup_{n\to\infty}\frac{1}{nm}\log Q_{nm}(T,\Phi,\varepsilon) = \limsup_{n\to\infty}\frac{1}{n}\log Q_n(T,\Phi,\varepsilon),$$

and then he claimed that $Q(T^m,\sum_{i=0}^{m-1}\Phi\circ T^i,\delta) \geq mQ(T,\Phi,\varepsilon)$. However, this does not hold in general. For a counterexample, we take a sequence $x_n = (-1)^{n+1}$. Then $\limsup_{n\to\infty} x_n = 1$, but $\limsup_{n\to\infty} x_{2n} = -1$.

The purpose of this paper is to provide new proofs and thereby to show that his theorems are still true.

2. Preliminaries

We begin by recalling some definitions and known results from [1]. Let (X,d) be a compact metric space and let $T : X \to X$ be continuous. Let $C(X,\mathbb{R})$ denote the Banach space of real valued continuous functions on X equipped with the supermum norm. Logarithms will be to the natural base e. $\mathbb{Z}^+$ denotes the non-negative integers. There are some standard definitions and results of the pressure.

A. *Pressure using (n,ε) separated sets*

For $\varepsilon > 0$ and $n \in \mathbb{Z}^+$, the finite subset E of X is an (n,ε) *separated set* if for $x \neq y$ in E there is a $0 \leq j < n$ such that $d(T^jx, T^jy) > \varepsilon$. For $\Phi \in C(X,\mathbb{R})$ define

$$P_n(T,\Phi,\varepsilon) = \sup\{\sum_{x\in E}\exp\sum_{i=0}^{n-1}\Phi(T^ix) : E \text{ is a } (n,\varepsilon) \text{ separated set}\}.$$

Since the exponential function is positive, it suffices to take the supremum over maximal (n, ε) separated sets. Put

$$P(T, \Phi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, \Phi, \varepsilon).$$

Then $\varepsilon_1 < \varepsilon_2$ implies $P(T, \Phi, \varepsilon_1) \geq (T, \Phi, \varepsilon_2)$. Hence

$$P(T, \Phi) = \lim_{\varepsilon \to 0} P(T, \Phi, \varepsilon)$$

exists or is ∞. The map $P(T, \cdot) : C(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is called the *pressure* of T.

B. *Pressure using (n, ε) spanning sets*

For $\varepsilon > 0$ and $n \in \mathbb{Z}^+$, the finite subset E of X is an (n, ε) *spanning set* if for each $x \in X$ there is a $y \in E$ such that $d(T^i x, T^i y) \leq \varepsilon$ for all i with $0 \leq i \leq n - 1$. Let

$$Q_n(T, \Phi, \varepsilon) = \inf\{\sum_{x \in E} \exp \sum_{i=0}^{n-1} \Phi(T^i x) : E \text{ is an } (n, \varepsilon) \text{ spanning sets}\}.$$

It suffices to take the infimum over minimal (n, ε) spanning sets. If $Q(T, \Phi, \varepsilon) = \limsup_{n \to \infty} 1/n \log Q_n(T, \Phi, \varepsilon)$, then $Q(T, \Phi, \varepsilon)$ increases as ε decreases and therefore $Q(T, \Phi) = \lim_{\varepsilon \to 0} Q(T, \Phi, \varepsilon)$ exists or is ∞. The map $Q(T, \cdot) : C(X, \mathbb{R}) \to \mathbb{R} \cup \{\infty\}$ is again the *pressure* of T. This follows from the following ([1]):

LEMMA 1. $Q(T, \Phi) = P(T, \Phi)$.

C. *Pressure using open covers*

If $\mathcal{A}$ is an open cover of X, then the *diameter* of $\mathcal{A}$ is defined by

$$\operatorname{diam}(\mathcal{A}) = \sup_{A \in \mathcal{A}} \sup_{x, y \in A} d(x, y).$$

Clearly $T^{-1}\mathcal{A} = \{T^{-1}(A) : A \in \mathcal{A}\}$ is an open cover of X. For open covers $\mathcal{A}$ and $\mathcal{B}$ of X, we define $\mathcal{A} \bigvee \mathcal{B}$ to be the open cover $\{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ of X. We will use $\mathcal{A}_{-m}^{n}$ to denote $\bigvee_{i=-m}^{n} T^{-i}\mathcal{A}$. Then it is clear that $\mathcal{A}_0^{k+n} \subset (\mathcal{A}_0^k)_0^n$.

For any open cover $\mathcal{A}$ of X, put

$$Q_n(T,\Phi,\mathcal{A}) = \inf\{\sum_{A\in\alpha} \inf_{x\in A} \exp \sum_{i=0}^{n-1} \Phi(T^i x) : \alpha \text{ is a finite subcover of } \mathcal{A}_0^{n-1}\}.$$

Let $Q(T,\Phi,\mathcal{A}) = \limsup_{n\to\infty} \frac{1}{n} \log Q_n(T,\Phi,\mathcal{A})$.

The following are proved in [1].

LEMMA 2. *$P(T,\Phi) = \sup\{Q(T,\Phi,\mathcal{A}) : \mathcal{A}$ is an open cover of $X\}$.*

LEMMA 3. *$Q(T,\Phi,\mathcal{A}) \to P(T,\Phi)$ as $\operatorname{diam}(\mathcal{A}) \to 0$.*

To give another definition, let $\mathcal{A}$ be an open cover of X and put

$$P_m(T,\Phi,\mathcal{A}) = \inf\{\sum_{A\in\alpha} \sup_{x\in A} \exp \sum_{i=0}^{n-1} \Phi(T^i x) : \alpha \text{ is a finite subcover of } \mathcal{A}_0^{n-1}\}.$$

LEMMA 4. *Let $\mathcal{A}$ be an open cover of X. Then*

$$P(T,\Phi,\mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log P_n(T,\Phi,\mathcal{A})$$

exists and equals $\inf\{1/n \log P_n(T,\Phi,\mathcal{A}) : n \in \mathbb{Z}^+\}$.

LEMMA 5. *$P(T,\Phi,\mathcal{A}) \to P(T,\Phi)$ as $\operatorname{diam}(\mathcal{A}) \to 0$.*

LEMMA 6. *Let $T : X \to X$ be surjective. Then $Q(T,\Phi,T^{-1}\mathcal{A}) = Q(T,\Phi,\mathcal{A})$ and $P(T,\Phi,T^{-1}\mathcal{A}) = P(T,\Phi,\mathcal{A})$.*

3. Proofs of Theorems A and B

Proof of Theorem A. Suppose $Q(T,\Phi,\mathcal{A}) < Q(T,\Phi,\mathcal{A}_0^k)$. Choose real numbers a, b so that $Q(T,\Phi,\mathcal{A}) < a < b < Q(T,\Phi,\mathcal{A}_0^k)$. There exists an integer N such that if $n > N$ then $1/n \log Q_n(T,\Phi,\mathcal{A}) < a$. Let

$$M = \max_{x\in X} |\Phi(x)|.$$

Choose n_0 such that $n_0 \geq \max\{N, \frac{(a+M)k}{b-a}\}$. Since $b < Q(T, \Phi, \mathcal{A}_0^k)$, we may assume that $b < 1/n_0 \log Q_{n_0}(T, \Phi, \mathcal{A}_0^k)$. This reduces to

$$e^{n_0 b} < Q_{n_0}(T, \Phi, \mathcal{A}_0^k).$$

Since $n_0+k > N$, we have $1/(n_0+k)\log Q_{n_0+k}(T, \Phi, \mathcal{A}) < a$. Therefore, this gives

$$Q_{n_0+k}(T, \Phi, \mathcal{A}) < e^{(n_0+k)a}.$$

Thus there exists a finite subcover α of $\mathcal{A}_0^{n_0+k-1}$ such that

$$\sum_{A \in \alpha} \inf\{\exp \sum_{i=0}^{n_0+k-1} \Phi(T^i x) : x \in A\} < e^{(n_0+k)a}.$$

From the fact that α is also a finite subcover of $(\mathcal{A}_0^k)_0^{n_0-1}$, we have

$$\begin{aligned} e^{(n_0+k)a} &> \sum_{A \in \alpha} \inf\{\exp \sum_{i=0}^{n_0+k-1} \Phi(T^i x) : x \in A\} \\ &\geq e^{-kM} \sum_{A \in \alpha} \inf\{\exp \sum_{i=0}^{n_0-1} \Phi(T^i x) : x \in A\} \\ &\geq e^{-kM} Q_{n_0}(T, \Phi, \mathcal{A}_0^k) \\ &> e^{-kM} e^{n_0 b} = e^{n_0 b - kM}. \end{aligned}$$

This implies that $(n_0+k)a > n_0 b - kM$, which contradicts our choice of $n_0 \geq \frac{(a+M)k}{b-a}$. Hence $Q(T, \Phi, \mathcal{A}_0^k) \leq Q(T, \Phi, \mathcal{A})$.

Consider any finite subcover α of $(\mathcal{A}_0^k)_0^{n-1}$. Every $A \in \alpha$ is of the form $A = \bigcap_{i=0}^{n-1} T^{-i} \bigcap_{j=0}^{k} T^{-j} A_j^i$ where $A_j^i \in \mathcal{A}$. Setting $B_A = \bigcap_{i=0}^{n-1} T^{-i} A_0^i$ and $\gamma_\alpha = \{B_A : A \in \alpha\}$, we see that $A \subset B_A$, $B_A \in \mathcal{A}_0^{n-1}$ and γ_α is a finite subcover of $\mathcal{A}^{n-1}$. Furthermore,

$$\begin{aligned} &\sum_{A \in \alpha} \inf\{\exp \sum_{i=0}^{n-1} \Phi(T^i x) : x \in A\} \\ &\geq \sum_{A \in \alpha} \inf\{\exp \sum_{i=0}^{n-1} \Phi(T^i x) : x \in B_A\} \\ &= \sum_{B \in \gamma_\alpha} \inf\{\exp \sum_{i=0}^{n-1} \Phi(T^i x) : x \in B\} \\ &\geq Q_n(T, \Phi, \mathcal{A}). \end{aligned}$$

This induces $Q_n(T,\Phi,\mathcal{A}_0^k) \geq Q_n(T,\Phi,\mathcal{A})$ for all $n > 0$. Hence

$$Q(T,\Phi,\mathcal{A}_0^k) \geq Q(T,\Phi,\mathcal{A}).$$

Consequently, $Q(T,\Phi,\mathcal{A}_0^k) = Q(T,\Phi,\mathcal{A})$.

If $T : X \to X$ is a homeomorphism, then

$$Q(T,\Phi,\mathcal{A}_{-m}^k) = Q(T,\Phi,\mathcal{A}_0^{k+m}) = Q(T,\Phi,\mathcal{A})$$

by Lemma 6 and the above result. The results for the case P are proved similarly. □

Proof of Theorem B. If $P(T,\Phi) = \infty$, then there is nothing to show. Thus we may assume that $P(T,\Phi) < \infty$. To show that $P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) \leq mP(T,\Phi)$, suppose $P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) > mP(T,\Phi)$. Choose a real number η so that $mP(T,\Phi) < \eta < P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i)$. Since, by Lemma 1, $P(T,\Phi) = Q(T,\Phi) = \lim_{\epsilon\to 0} Q(T,\Phi,\epsilon)$, there exists $\delta > 0$ such that if $0 < \varepsilon < \delta$, then

$$mQ(T,\Phi,\varepsilon) < \eta < Q(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \varepsilon).$$

These inequalities guarantee that there exists K such that if $n > K$, then $1/n \log Q_n(T,\Phi,\varepsilon) < \eta/m$, and that there exists $n_0 > K$ such that $\eta < 1/n_0 \log Q_{n_0}(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \varepsilon)$. Since $mn_0 > K$ we have $1/mn_0 \log Q_{mn_0}(T,\Phi,\varepsilon) < \eta/m$ and hence $Q_{mn_0}(T,\Phi,\varepsilon) < e^{n_0\eta}$. This implies that there exists an (mn_0,ε) spanning set F for T such that $\sum_{x\in F} \exp \sum_{i=0}^{mn_0-1} \Phi(T^i x) < e^{n_0\eta}$. Since F is also an (n_0,ε) spanning set for T^m, we obtain

$$\begin{aligned} e^{n_0\eta} &> \sum_{x\in F} \exp \sum_{i=0}^{mn_0-1} \Phi(T^i x) \\ &= \sum_{x\in F} \exp \sum_{j=0}^{n_0-1} (\sum_{i=0}^{m-1} \Phi \circ T^i)((T^m)^j x) \\ &\geq Q_{n_0}(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \varepsilon) > e^{n_0\eta}. \end{aligned}$$

This is impossible. Thus $P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) \leq mP(T,\Phi)$.

On the other hand, suppose that $P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) < mP(T, \Phi)$. We can choose real numbers a, b so that $P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) < a < b < mP(T, \Phi)$. These inequalities imply that there exists $\eta > 0$ such that if $0 < \varepsilon < \eta$, then $Q(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \varepsilon) < a$ and $b/m < Q(T, \Phi, \varepsilon)$. For $0 < \varepsilon < \eta$, since $b/m < Q(T, \Phi, \varepsilon) = \limsup_{n\to\infty} 1/n \log Q_n(T, \Phi, \varepsilon)$, there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that

$$b/m < \frac{1}{n_k} \log Q_{n_k}(T, \Phi, \varepsilon).$$

Put $n_k = mp_k + q_k$, where $0 \le q_k < m$. Since the sequence $\{n_k\}$ is strictly increasing, the sequence $\{p_k\}$ tends to ∞. Uniform continuity guarantees that there exists $0 < \delta < \varepsilon$ such that if $d(x, y) \le \delta$, then $d(T^i x, T^i y) \le \varepsilon$ for all $0 \le i \le m - 1$. Since $0 < \delta < \eta$, $Q(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \delta) < a$ and this implies that there exists N such that $1/n \log Q_n(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \delta) < a$ for all $n > N$. Choose k such that $p_k > \max\{N, a/(b - a)\}$. Since $Q_{p_k+1}(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i, \delta) < e^{(p_k+1)a}$, there is a $(p_k + 1, \delta)$ spanning set F for T^m such that $\sum_{x\in F} \exp \sum_{j=0}^{p_k} (\sum_{i=0}^{m-1} \Phi \circ T^i)((T^m)^j x) < e^{(p_k+1)a}$. Since $m(p_k + 1) - 1 > n_k - 1$, we have

$$\begin{aligned} e^{(p_k+1)a} &> \sum_{x\in F} \exp \sum_{j=0}^{p_k} (\sum_{i=0}^{m-1} \Phi \circ T^i)((T^m)^j x) \\ &= \sum_{x\in F} \exp \sum_{i=0}^{m(p_k+1)-1} \Phi(T^i x) \\ &> \sum_{x\in F} \exp \sum_{i=0}^{n_k-1} \Phi(T^i x). \end{aligned}$$

Noting also that F is an (n_k, ε) spanning set for T, we obtain

$$e^{(p_k+1)a} > \sum_{x\in F} \exp \sum_{i=0}^{n_k-1} \Phi(T^i x) \ge Q_{n_k}(T, \Phi, \varepsilon) > e^{bn_k/m}.$$

But this contradicts our choice of $p_k > a/(b-a)$. Therefore $mP(T, \Phi) \le P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i)$. Consequently, we obtain $P(T^m, \sum_{i=0}^{m-1} \Phi \circ T^i) = mP(T, \Phi)$. □

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