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# INVERTIBLE AND ISOMETRIC COMPOSITION OPERATORS ON VECTOR-VALUED HARDY SPACES

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Abstract. Invertible and isometric composition operators acting on vector-valued Hardy space $H^2(E)$ are characterized.

## 1. Introduction

If $\phi$ is an analytic self-map of the unit disc $D$, the composition operator $C_\phi$ is defined by $C_\phi f = fo\phi$ for $f$ analytic in $D$. It is well known that every composition operator is bounded on scalar-valued Hardy Spaces $H^P$ as well as on other spaces of analytic functions. For detailed study of these operators on $H^P$ and other spaces of analytic functions, consult Schwartz [7], Shapiro and Taylor [8], Nordgren [5] and Cowen and MacCluer [1]. In this paper we study invertible and isometric composition operators on vector-valued Hardy space $H^2(E)$.

Let $(X, ||.||_X)$ and $(E, \langle,\rangle)$ denote a complex Banach space and a Hilbert space respectively. For $0 < p < \infty$, the vector-valued Hardy space $H^P(X)$ consists of functions $f : D \to X$ such that $x^* of$ is holomorphic in $D$ for every $x^* \in X^*$, the dual of $X$ and

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} ||f(re^{i\theta})||_X^P d\theta < \infty,$$

where $D$ is the open unit disc in the complex plane $\mathbb{C}$ with boundary $\partial D$ . For $1 \leq p < \infty$, $H^P(X)$ becomes a Banach space with norm $|||.|||_P$ defined as

$$|||f|||_P^P = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} ||f(re^{i\theta})||_X^P d\theta < \infty.$$

When $X = \mathbb{C}$, we drop $X$ and writer simply $H^P$ for $H^P(X)$ and $||.||_P$ for $|||.|||_P$.

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In terms of harmonic majorants $H^P(X)$ consists of those holomorphic functions $f : D \to X$ for which $||f(.)||_X^P$ has harmonic majorant and in this case (cf. [6, Theorem A, p.74])

$$|||f|||_P^P = h_f(0),$$

where $h_f$ is the least harmonic majorant of $||f(.)||_X^P$.

A more detailed discussion of vector-valued analytic functions and Hardy spaces can be found in Hille and Philips [4], Rosenblum and Rovnyak [6], Hensgen [3] and a convenient reference for classical Hardy spaces is Duren [2].

We now prove the following lemma.

LEMMA 1.1. *Let $f \in H^P(X)$. Then*

$$||f(z)||_X^P \leq \frac{2|||f|||_P^P}{1-|z|} \text{ for every } z \in D.$$

*Proof.* Let $f \in H^P(X)$. Then $||f(z)||_X^P$ has the least harmonic majorant, say $h_f$. Therefore, by Harnack's inequality,

$$\begin{aligned} ||f(z)||_X^P &\leq \frac{1+|z|}{1-|z|} h_f(0) \\ &\leq \frac{2}{1-|z|} |||f|||_P^P. \end{aligned}$$

□

A painless verification, using theorem C of Rosenblum and Rovnyak [6, p.76] and Harnack's inequality shows that if $\phi : D \to D$ is analytic, then $C_\phi : H^P(X) \to H^P(X)$ is bounded and

$$||C_\phi||^P \leq \frac{1+|\phi(0)|}{1-|\phi(0)|}. \tag{1.1}$$

In case $p = 2$ and $X = E, H^P(E)$ becomes a Hilbert space with the inner product $\langle\langle , \rangle\rangle$ defined as

$$\langle\langle f, g\rangle\rangle = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \langle f(re^{i\theta}), g(re^{i\theta})\rangle d\theta.$$

Let $\{e_j : j \in J\}$ be an orthonormal basis for $E$, where $J$ is an index set of any size and let $N = \{0, 1, 2, \ldots\}$. For $(n, j) \in N \times J$, we define $e_{nj} : D \to E$ as

$$e_{nj}(z) = z^n e_j \text{ for every } z \in D.$$

Then $\{e_{nj} : (n,j) \in N \times J\}$ is an orthonormal basis for $H^P(E)$. For $z \in D$ and $j \in J$, we define $E_z^j : H^2(E) \to \mathbb{C}$ as

$$E_z^j f = \langle f(z), e_j \rangle,$$

for every $f \in H^P(E)$. Then, by Lemma 1.1, $E_z^j$ is a bounded linear functional on $H^2(E)$. Hence by Riesz-representation theorem, there exists $k_z^j \in H^2(E)$ such that

$$E_z^j f = \langle\langle f, k_z^j \rangle\rangle,$$

for every $f \in H^2(E)$ . We designate $k_z^j$'s as generalized reproducing kernels or simply kernel functions whenever there is no confusion. A straight forward calculation, using Parseval's identity, shows that

$$k_z^j(w) = \frac{\mathrm{e_j}}{1 - \bar{\mathrm{z}}\mathrm{w}},$$

for every $w \in D$ and

$$|||k_z^j|||_2^2 = \frac{1}{1 - |\mathrm{z}|^2}.$$

The invertibility of $C_\phi$ on $H^2(E)$ in terms of the invertibility of inducing map $\phi$ is characterized in Section 2. We also present a necessary and sufficient condition for $C_\phi$ to be an isometry.

## 2. Invertible and isometric composition operators

Schwartz [7] proved that $C_\phi$ is invertible on $H^P$ if and only if $\phi$ is a conformal automorphism of the open unit disc. In the following theorem we generalize this criterion for invertibility of $C_\phi$ to vector-valued Hardy space $H^2(E)$ . The techniques applied to prove this result are different from those applied by Schwartz.

THEOREM 2.1. *$C_\phi$ is invertible on $H^2(E)$ if and only if $\phi$ is invertible.*

Before we prove this theorem we first note that if $C_\phi$ is a composition operator on $H^2(E)$, then $C_\phi^* k_z^j = k_{\phi(z)}^j$ , where $C_\phi^*$ is the adjoint of $C_\phi$. In fact,

$$\begin{aligned} \langle\langle f, C_\phi^* k_z^j \rangle\rangle &= \langle\langle C_\phi f, k_z^j \rangle\rangle \\ &= \langle f(\phi(z)), e_j \rangle \\ &= \langle\langle f, k_{\phi(z)}^j \rangle\rangle \text{ for every } f \in H^2(E). \end{aligned}$$

This implies that

$$C_\phi^* k_z^j = k_{\phi(z)}^j.$$

*Proof of Theorem 2.1.* If $\phi$ is invertible, then $C_\phi$ is also invertible with $C_\phi^{-1} = C_{\phi^{-1}}$.

Conversely, suppose $C_\phi$ is invertible. Let $\phi(z) = \phi(w)$ for some $z, w \in D$. Then

$$\begin{aligned} C_\phi^* k_z^j &= k_{\phi(z)}^j \\ &= k_{\phi(w)}^j \\ &= C_\phi^* k_w^j. \end{aligned}$$

Since $C_\phi$ and hence $C_\phi^*$ is invertible, so $k_z^j = k_{\phi(w)}^j$, which implies that $z = w$. Hence $\phi$ is univalent. Again, since $C_\phi^*$ is invertible and so it is bounded below. Hence there exists $\alpha > 0$ such that

$$|||C_\phi^* f|||_2 \geq \alpha |||f|||_2$$

for every $f \in H^2(E)$. In particular,

$$|||C_\phi^* k_z^j|||_2 \geq \alpha |||k_z^j|||_2$$

for every $(z, j) \in D \times J$, and so by the above remark

$$\text{(2.1)} \qquad \frac{|||k_{\phi(z)}^j|||_2}{|||k_z^j|||_2} \geq \alpha \text{ for every } (z, j) \in D \times J.$$

If $\phi$ is not onto, then we can find $w \in \partial\phi(D) \cap D$ and a sequence $\{z_n\} \subset D$ such that

$$\lim_n \phi(z_n) = w.$$

By open mapping theorem $|z_n| \to 1$. Since

$$|||k_{\phi(z_n)}^j|||_2 \to |||k_w^j|||_2, \; |||k_{(w)}^j|||_2 < \infty,$$

and $|||k_{(z_n)}^j|||_2 \to \infty$ as $n \to \infty$, we conclude that

$$\frac{|||k_{\phi(z_n)}^j|||_2}{|||k_{z_n}^j|||_2} \to 0 \text{ as } n \to \infty,$$

a contradiction to (2.1). Hence $\phi$ must be onto. This completes the proof of the theorem. □

We next present a necessary and sufficient condition for $C_\phi$ to be an isometry.

THEOREM 2.2. *$C_\phi$ is an isometric on $H^2(E)$ if and only if $\phi$ is inner and $\phi(0) = 0$.*

To prove this theorem we need the following lemmas.

LEMMA 2.3. $||C_\phi||^2 \geq \frac{1}{1-|\phi(0)|^2}$.

*Proof.* Since $C_\phi^* k_z^j = k_{\phi(z)}^j$ for every $(z,j) \in D \times N$ and $|||k_o^j|||_2 = 1$, we have $\frac{1}{1-|\phi(0)|^2} = |||k_{\phi(o)}^j|||_2 = |||C_\phi^* k_o^j|||_2 \leq ||C_\phi||^2$. This completes the proof. □

LEMMA 2.4. $||C_\phi|| = 1$ *if and only if* $\phi(0) = 0$.

Proof follows from the inequality (1.1) and Lemma 2.3.

*Proof of theorem 2.2.* Suppose $C_\phi$ is an isometry on $H^2(E)$. Then $|||C_\phi f|||_2 = |||f|||_2$ for every $f \in H^2(E)$. In particular, taking $f = e_{ij}$, we get, $\frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta = 1$ and so $\phi$ is inner. Also, since $C_\phi$ is an isometry $|C_\phi| = 1$ and so by Lemma2.4 $\phi(0) = 0$.

Conversely, suppose $\phi$ is inner and $\phi(0) = 0$. Then $\overline{\phi(e^{i\theta})} = [\phi(e^{i\theta})]^{-1}$ a.e. Further, if $f \in H^2(E)$, then $f(z) = \sum_{n=1}^{\infty} a_n z^n (a_n \in E)$ and $|||f|||_2 = \sum_{n=1}^{\infty} ||a_n||_E^2$ see ([6, section 1.15] and [4, chapter III]).

$$\begin{aligned}
|||C_\phi f|||_2^2 &= |||fo\phi|||_2^2 \\
&= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \langle \sum_{m=0}^{\infty} a_m \phi^m(re^{i\theta}), \sum_{n=0}^{\infty} a_n \phi^n(re^{i\theta}) d\theta\rangle \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle a_m, a_n\rangle \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} \phi^{m-n}(re^{i\theta}) d\theta \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \langle a_m, a_n\rangle \frac{1}{2\pi}\int_0^{2\pi} \phi^{m-n}(e^{i\theta}) d\theta \\
&= \sum_{m,n\in N, m>n} \langle a_m, a_n\rangle \phi^{m-n}(0) + \sum_{m,n\in N, m<n} \langle a_m, a_n\rangle \overline{\phi^{m-n}}(0) \\
&\quad + \sum_{n=0}^{\infty} \langle a_n, a_n\rangle \frac{1}{2\pi}\int_0^{2\pi} \phi^n |(e^{i\theta})|^2 d\theta. \ (*)
\end{aligned}$$

Since $\phi^{m-n}(0) = 0$ for $m > n$, from (*) we have

$$|||C_\phi f|||_2^2 = \sum_{n=0}^{\infty} \langle a_n, a_n\rangle$$

$$= \sum_{n=0}^{\infty} ||a_n||_E^2$$
$$= |||f|||_2^2.$$

Hence $C_\phi$ is an isometry. □

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