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FOURIER INVERSION OF DISTRIBUTIONS ON THE SPHERE

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ABSTRACT. We show that the Fourier-Laplace series of a distribution on the sphere is uniformly Cesàro-summable to zero on a neighborhood of a point if and only if this point does not belong to the support of the distribution. Similar results on the ball and on the real projective space are also proved.

1. Introduction

In [4] Kahane and Salem used the support of distributions to characterize the closed sets of uniqueness in the unit circle $\mathbf{S}^1$. For this they proved that, given a distribution T on $\mathbf{S}^1$ whose Fourier transform $\mathcal{F}T$ vanishes at infinity and E a closed set in $\mathbf{S}^1$, the support of T is in E if and only if for all $x \in \mathbf{S}^1 \setminus E$

$$\lim_{N\to+\infty} \sum_{k=-N}^{N} \mathcal{F}T(k)\, \mathrm{e}^{2\pi i x k} = 0.$$

Later Walter proved that the Fourier series

$$\sum_{k=-\infty}^{\infty} \mathcal{F}T(k)\, \mathrm{e}^{2\pi i x k}$$

of a general distribution T on $\mathbf{S}^1$ is Cesàro-summable to zero for all x out of the support of T [10]. However, this is not sufficient to characterize the support of T, since, as Walter himself remarks, the Fourier series of the first derivative of the Dirac measure at a point $s \in \mathbf{S}^1$, δ'_s, is summable in Cesàro means of order 2 to zero everywhere on $\mathbf{S}^1$. In fact a point x is out of the support of T if and only if the Fourier series of T is

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uniformly Cesàro-summable to zero on a neighborhood of x. In Section 4 we establish this result for the general case of a distribution T on $\mathbf{S}^{n-1}$ ($n \geq 2$) and its Fourier-Laplace series. We obtain as corollaries similar results on the ball (Section 6) and on the real projective space (Section 7). In Section 5 we study in more detail some particular distributions on $\mathbf{S}^{n-1}$. The necessary facts about Cesàro-summation and Fourier-Laplace series are recalled in sections 2 and 3, respectively.

2. Summability

Let $\sum_{m\geq 0} b_m$ be a series of complex numbers. Define, for all $m \in \mathbf{N}_0$, $k \geq 0$,

$$B_m^k := \sum_{\nu=0}^{m} \binom{\nu+k}{k} b_{m-\nu} \quad \text{and} \quad \binom{\nu+k}{k} := (k+1)(k+2)\dots(k+\nu)/\nu!.$$

The series $\sum_{m\geq 0} b_m$ is said to be *(C, k)-summable to $B \in \mathbf{C}$* if

$$\lim_{m\to\infty} B_m^k \binom{m+k}{k}^{-1} = B,$$

and we write in this case

$$\sum_{m=0}^{+\infty} b_m = B \quad (C, k).$$

When the numbers b_m depend on a parameter t taken in a set T, the series $\sum_{m\geq 0} b_m(t)$ is said to be *uniformly (C, k)-summable to $B(t) \in \mathbf{C}$ on T* if

$$\lim_{m\to\infty} B_m^k(t) \binom{m+k}{k}^{-1} = B(t)$$

uniformly in $t \in T$. The numbers

$$B_m^k \binom{m+k}{k}^{-1} = \sum_{l=0}^{m} \binom{m-l+k}{k} \binom{m+k}{k}^{-1} b_l$$

are called *Cesàro means of order k* of the series [3, p.97]. The series $\sum_{m\geq 0} b_m(t)$ converges to B uniformly on T if and only if it is uniformly $(C, 0)$-summable to B on T; if it is uniformly (C, k)-summable to B on T, it is uniformly (C, k')-summable to B on T for all $k' \geq k$ [3, p.101]. A basic result is the following [3, pp.136–139]:

LEMMA 1. *Let $l \geq -1$, $k \geq 0$ and $\theta \in]0, 2\pi[$. Then the series $\sum_{m\geq 1} m^l \,e^{mi\theta}$ is (C,k)-summable if and only if $k > l$. Moreover, when $k > l$, $\sum_{m\geq 1} m^l \,e^{mi\theta}$ is uniformly (C,k)-summable on every compact subinterval of $]0, 2\pi[$.*

3. Fourier-Laplace expansions

We write $\mathbf{S}^{n-1}$ the unit sphere in $\mathbf{R}^n$ $(n \geq 2)$ and $d\sigma_{n-1}$ the measure on $\mathbf{S}^{n-1}$ induced by the Lebesgue measure on $\mathbf{R}^n$, so that

$$\omega_{n-1} := \int_{\mathbf{S}^{n-1}} d\sigma_{n-1}(\eta) = 2\pi^{n/2}/\Gamma(n/2).$$

We define a distance d on $\mathbf{S}^{n-1}$ by $d(\zeta,\eta) := 1 - (\zeta|\eta)$, where $(\,.\,|\,.\,)$ is the euclidean scalar product in $\mathbf{R}^n$. We have $0 \leq d(\zeta,\eta) \leq 2$ for all $\zeta, \eta \in \mathbf{S}^{n-1}$ and $d(\zeta,\eta) = 2$ if and only if $\zeta = -\eta$. A *spherical harmonic of degree l on* $\mathbf{S}^{n-1}$ $(l \in \mathbf{N}_0)$ is the restriction to $\mathbf{S}^{n-1}$ of a polynomial on $\mathbf{R}^n$ which is harmonic and homogeneous of degree l. We write $\mathcal{S}H_l(\mathbf{S}^{n-1})$ the vector space of spherical harmonics of degree l; its dimension is

$$\begin{aligned} d_l = d_l^n &:= \dim_{\mathbf{C}} \mathcal{S}H_l(\mathbf{S}^{n-1}) \\ &= \frac{(2l+n-2)(n+l-3)!}{(n-2)!\,l!} = \frac{2l^{n-2}}{(n-2)!} + O(l^{n-3}). \end{aligned}$$

Two spherical harmonics of different degrees are orthogonal with respect to the scalar product $(\,.\,|\,.\,)_2$ of $L^2(\mathbf{S}^{n-1}, d\sigma_{n-1})$. The space

$$\cup_{l\geq 0}\mathcal{S}H_l(\mathbf{S}^{n-1})$$

is total in $L^2(\mathbf{S}^{n-1})$: if $(E_1^l, \ldots, E_{d_l}^l)$ is an orthonormal basis of

$$\mathcal{S}H_l(\mathbf{S}^{n-1})$$

then, for every $f \in L^2(\mathbf{S}^{n-1})$, the series

$$\sum_{l=0}^{+\infty} \sum_{j=1}^{d_l} (f|E_j^l)_2 \, E_j^l,$$

called *Fourier-Laplace series of f*, converges to f in square mean; it converges uniformly to f on $\mathbf{S}^{n-1}$ for $f \in C^\infty(\mathbf{S}^{n-1})$ (see Section 4). We put

$$\Pi_l(f) := \sum_{j=1}^{d_l} (f|E_j^l)_2 \, E_j^l;$$

it is the orthogonal projection of f on $\mathcal{S}H_l(\mathbf{S}^{n-1})$. We have, for $\zeta \in \mathbf{S}^{n-1}$,

$$\Pi_l(f)(\zeta) = \int_{\mathbf{S}^{n-1}} Z_l(\zeta,\eta) f(\eta) d\sigma_{n-1}(\eta),$$

where

$$Z_l(\zeta,\eta) := \sum_{j=1}^{d_l} E_j^l(\zeta)\overline{E_j^l(\eta)}$$

(with $\eta \in \mathbf{S}^{n-1}$) is the *zonal with pole* ζ *of degree* l. If f is a function defined on $\mathbf{S}^{n-1}$, we write $f\!\uparrow$ the homogeneous function of degree 0 defined on $\mathbf{R}^n\backslash\{0\}$ by $(f\!\uparrow)(x) := f(x/\|x\|)$. Conversely, if g is a function defined on $\mathbf{R}^n \setminus \{0\}$ we write $g\!\downarrow$ its restriction to $\mathbf{S}^{n-1}$. We say that a function f on $\mathbf{S}^{n-1}$ is in $C^l(\mathbf{S}^{n-1})$ (where $l \in \mathbf{N}_0$) if $f\!\uparrow \in C^l(\mathbf{R}^n \setminus \{0\})$. When $f \in C^l(\mathbf{S}^{n-1})$ we can define, for every multiindex $\alpha \in \mathbf{N}_0^n$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq l$, $D_{\mathbf{S}}^\alpha f \in C^{l-|\alpha|}(\mathbf{S}^{n-1})$ by

$$D_{\mathbf{S}}^\alpha f := (D^\alpha(f\!\uparrow))\!\downarrow = \left(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}(f\!\uparrow)\right)\!\downarrow.$$

In this way we can obtain from the Laplacian $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$ on $\mathbf{R}^n$ the *Laplace-Beltrami operator on* $\mathbf{S}^{n-1}$, $\Delta_{\mathbf{S}}$. We write $\mathcal{D}(\mathbf{S}^{n-1})$ the space of functions $C^\infty(\mathbf{S}^{n-1})$ with the topology given by the family of seminorms

$$p_m(\varphi) := \sup_{|\alpha|\leq m} \sup_{\eta\in\mathbf{S}^{n-1}} |D_{\mathbf{S}}^\alpha \varphi(\eta)|$$

($m \in \mathbf{N}_0$). Its dual, $\mathcal{D}'(\mathbf{S}^{n-1})$, is the set of *distributions on* $\mathbf{S}^{n-1}$. The *Fourier-Laplace series of a distribution* T on $\mathbf{S}^{n-1}$ is

$$\sum_{l=0}^{+\infty}\sum_{j=1}^{d_l} T(\overline{E_j^l})\, E_j^l = \sum_{l=0}^{+\infty} \Pi_l(T),$$

with, for $\zeta \in \mathbf{S}^{n-1}$,

$$\Pi_l(T)(\zeta) := T[\eta \mapsto Z_l(\zeta,\eta)];$$

it converges to T in the sense of distributions. The support of $T \in \mathcal{D}'(\mathbf{S}^{n-1})$ will be written $\operatorname{supp} T$. Because $\mathbf{S}^{n-1}$ is compact, every distribution on it is of finite order.

4. Fourier inversion on the sphere

THEOREM 1. *Let $T \in \mathcal{D}'(\mathbf{S}^{n-1})$ be of order $m \in \mathbf{N}_0$.* i) *If there exist $k \geq 0$ and U an open subset of $\mathbf{S}^{n-1}$ on which*

$$\sum_{l=0}^{+\infty} \Pi_l(T)(\zeta) = 0 \quad (C, k) \tag{1}$$

uniformly (in ζ), then T is zero on U. ii) *Conversely, if $k > n-2+2m$, then (1) holds uniformly on every closed subset of $\mathbf{S}^{n-1} \setminus \operatorname{supp} T$.*

Proof. First, we suppose there exist $k \geq 0$ and U open subset of $\mathbf{S}^{n-1}$ on which (1) holds uniformly. We take $\varphi \in \mathcal{D}(\mathbf{S}^{n-1})$ with $\operatorname{supp}\varphi \subset U$. We have

$$\begin{aligned} 0 &= \lim_{N\to+\infty} \int_{\mathbf{S}^{n-1}} \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \\ &\quad \times \Pi_l(T)(\zeta)\varphi(\zeta)d\sigma_{n-1}(\zeta) \\ &= \lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \\ &\quad \times \int_{\mathbf{S}^{n-1}} T[\eta \mapsto Z_l(\zeta,\eta)]\varphi(\zeta)d\sigma_{n-1}(\zeta) \\ &= \lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \\ &\quad \times T[\eta \mapsto \int_{\mathbf{S}^{n-1}} Z_l(\zeta,\eta)\varphi(\zeta)d\sigma_{n-1}(\zeta)] \\ &= \lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} T[\eta \mapsto \Pi_l(\varphi)(\eta)] \\ &= \lim_{N\to+\infty} T[\eta \mapsto \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \Pi_l(\varphi)(\eta)], \end{aligned}$$

by using successively (1), the definition of $\Pi_l(T)$, [5, theorem III.2.b p.208] on the tensor product of two distributions, the definition of $\Pi_l(\varphi)$ and the linearity of T. We will now show that

$$\lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \Pi_l(\varphi) = \varphi$$

in the topology of $\mathcal{D}(\mathbf{S}^{n-1})$, or, equivalently, that, for all $\alpha \in \mathbf{N}_0^n$,

$$\lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k} \binom{N+k}{k}^{-1} D_{\mathbf{S}}^{\alpha} \Pi_l(\varphi) = D_{\mathbf{S}}^{\alpha} \varphi$$

uniformly on $\mathbf{S}^{n-1}$. According to [2, 3.6.5 p.129] there exists a constant c_q depending only on $q \in \mathbf{N}_0$ and n such that, for all $Y \in \mathcal{S}H_l(\mathbf{S}^{n-1})$ and $\alpha \in \mathbf{N}_0^n$ with $|\alpha| = q$,

$$\sup_{\eta\in\mathbf{S}^{n-1}} |D_{\mathbf{S}}^{\alpha} Y(\eta)| \le c_q\, l^{n/2+q-1} \|Y\|_2.$$

On the other hand, if $\psi \in C^{2p}(\mathbf{S}^{n-1})$ ($p \in \mathbf{N}_0$), the Fourier-Laplace expansion of $\Delta_s^p \psi$ is $(-1)^p \sum_{l=0}^{+\infty} l^p(l+n-2)^p \Pi_l(\psi)$ [2, 3.2.11 p.75]. Hence, by Parseval,

$$l^p(l+n-2)^p \|\Pi_l(\psi)\|_2 \le \|\Delta_s^p \psi\|_2.$$

Combining these two inequalities, we deduce the existence of a constant $C_{q,p}$ depending only on q, $p \in \mathbf{N}_0$ and n such that, for all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| = q$,

$$\sup_{\eta\in\mathbf{S}^{n-1}} |D_{\mathbf{S}}^{\alpha} \Pi_l(\varphi)(\eta)| \le C_{q,p}\, l^{n/2+q-1-2p} \|\Delta_s^p \varphi\|_2.$$

Taking $p > n/4 + q/2$, we see that, for all $\alpha \in \mathbf{N}_0^n$, the series

$$\sum_{l=0}^{+\infty} D_{\mathbf{S}}^{\alpha} \Pi_l(\varphi)$$

converges uniformly on $\mathbf{S}^{n-1}$. This has two consequences; firstly, for all $\alpha \in \mathbf{N}_0^n$ and $k \in \mathbf{N}_0$,

$$\lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k} \binom{N+k}{k}^{-1} D_{\mathbf{S}}^{\alpha} \Pi_l(\varphi) = \sum_{l=0}^{+\infty} D_{\mathbf{S}}^{\alpha} \Pi_l(\varphi)$$

uniformly on $\mathbf{S}^{n-1}$. Secondly, for all $\alpha \in \mathbf{N}_0^n$,

$$\sum_{l=0}^{+\infty} D_{\mathbf{S}}^{\alpha} \Pi_l(\varphi) = D_{\mathbf{S}}^{\alpha} \sum_{l=0}^{+\infty} \Pi_l(\varphi).$$

Indeed, the analogous result on a parallelepiped in $\mathbf{R}^n$ is well known; from it we can deduce the result on $\mathbf{S}^{n-1}$ by noting that $D^{\alpha}(\psi\uparrow)$ is homogeneous of degree $-|\alpha|$ and that a sequence of functions on $\mathbf{R}^n\setminus\{0\}$ homogeneous of the same degree which converges uniformly on $\mathbf{S}^{n-1}$

converges uniformly on every annulus $\{x \in \mathbf{R}^n : r \leq \|x\| \leq R\}$ with $0 < r < 1 < R$. Therefore

$$\lim_{N\to+\infty} \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} D^\alpha_{\mathbf{S}}\, \Pi_l(\varphi) = D^\alpha_{\mathbf{S}} \sum_{l=0}^{+\infty} \Pi_l(\varphi) = D^\alpha_{\mathbf{S}}\varphi$$

uniformly on $\mathbf{S}^{n-1}$. From the continuity of T follows

$$0 = \lim_{N\to+\infty} T[\eta \mapsto \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \Pi_l(\varphi)(\eta)] = T[\varphi].$$

Hence T is zero on U.

We will now prove the second part of the theorem. Since

$$\begin{aligned} &\sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \Pi_l(T)(\zeta) \\ = &\sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} T[\eta \mapsto Z_l(\zeta,\eta)] \\ = &\; T[\eta \mapsto \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} Z_l(\zeta,\eta)], \end{aligned}$$

we must study the function of $(\zeta,\eta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$

$${}_kL^n_N(\zeta,\eta) := \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} Z_l(\zeta,\eta).$$

For that we will use two properties of the zonals. Firstly, for every $l \in \mathbf{N}_0$ there exists a polynomial of degree l in one variable, written $P_l^{(n-2)/2}$, such that, for all ζ, $\eta \in \mathbf{S}^{n-1}$,

$$Z_l(\zeta,\eta) = \frac{d_l^n}{\omega_{n-1}} P_l^{(n-2)/2}((\zeta|\eta)) \tag{2}$$

[7, theorem 2.14 p.149]. Secondly, we have, for all ζ, $\eta \in \mathbf{S}^{n-1}$ and $0 \leq r < 1$,

$$\frac{1}{\omega_{n-1}} \frac{1-r^2}{(1-2r(\zeta|\eta)+r^2)^{n/2}} = \sum_{l=0}^{+\infty} r^l\, Z_l(\zeta,\eta)$$

[7, theorem 2.10 p.145], from which we deduce, by comparison with [6, formula (7) p.112], that the series $\sum_{l=0}^{+\infty} Z_l(\zeta,\eta)$, seen as a function of $(\zeta|\eta) \in [-1,1]$, is uniformly (C,k)-summable to 0 on $[-1, 1-\delta]$ (where $0 < \delta < 1$ is arbitrary) if and only if $k > n-2$ [6, pp.113–114 and

145–146]. Hence, for a given $\zeta \in \mathbf{S}^{n-1}$, ${}_kL_N^n(\zeta,\eta)$ converges, when N tends to $+\infty$, to 0 uniformly on every closed set of the form $\{\eta \in \mathbf{S}^{n-1} : d(\zeta,\eta) \geq \delta\}$ (where $0 < \delta < 2$) if $k \geq n-2$. We fix $\zeta \in \mathbf{S}^{n-1}$. We want to differentiate the function $\eta \mapsto {}_kL_N^n(\zeta,\eta)$. We first note that from the identity

$$\frac{1-r^2}{(1-2rt+r^2)^{n/2}} = \sum_{l=0}^{+\infty} r^l\, d_l^n\, P_l^{(n-2)/2}(t)$$

for all $t \in [-1,1]$ and $0 \leq r < 1$ follows, differentiating with respect to t,

$$nr\,\frac{1-r^2}{(1-2rt+r^2)^{(n+2)/2}} = \sum_{l=0}^{+\infty} r^l\, d_l^n\, \frac{d}{dt}\, P_l^{(n-2)/2}(t),$$

that is,

$$nr\sum_{q=0}^{+\infty} r^q d_q^{n+2} P_q^{n/2}(t) = \sum_{l=0}^{+\infty} r^l\, d_l^n\, \frac{d}{dt}\, P_l^{(n-2)/2}(t),$$

and so, by identifying the coefficients of r^{q+1}, $q \in \mathbf{N}_0$,

$$\frac{d}{dt}\, P_{q+1}^{(n-2)/2}(t) = n\,\frac{d_q^{n+2}}{d_{q+1}^n}\, P_q^{n/2}(t).$$

Then, if e_j is the multiindex given by $(e_j)_l = \delta_{jl}$ $(1 \leq j, l \leq n)$,

$$\begin{aligned}
&D_{\mathbf{S}}^{e_j}\,\frac{d_{q+1}^n}{\omega_{n-1}}\, P_{q+1}^{(n-2)/2}((\zeta|\eta)) \\
= \;& \frac{d_{q+1}^n}{\omega_{n-1}}\, D^{e_j}[P_{q+1}^{(n-2)/2}((\zeta|x/\|x\|))]_{x=\eta} \\
= \;& \frac{d_{q+1}^n}{\omega_{n-1}}\, n\,\frac{d_q^{n+2}}{d_{q+1}^n} [P_q^{n/2}((\zeta|x/\|x\|))D^{e_j}(\zeta|x/\|x\|)]_{x=\eta} \\
= \;& \frac{n}{\omega_{n-1}}\, d_q^{n+2} P_q^{n/2}((\zeta|\eta)) D_{\mathbf{S}}^{e_j}[\eta \mapsto (\zeta|\eta)] \\
= \;& 2\pi\,\frac{d_q^{n+2}}{\omega_{n+1}}\, P_q^{n/2}((\zeta|\eta)) D_{\mathbf{S}}^{e_j}(\zeta|\eta).
\end{aligned}$$

Hence

$$
\begin{aligned}
& D_{\mathbf{S}}^{e_j}\,{}_kL_N^n(\zeta,\eta) \\
= \ & D_{\mathbf{S}}^{e_j}\sum_{l=0}^{N}\binom{N-l+k}{k}\binom{N+k}{k}^{-1}\frac{d_l^n}{\omega_{n-1}}P_l^{(n-2)/2}((\zeta|\eta)) \\
= \ & D_{\mathbf{S}}^{e_j}\sum_{q=-1}^{N-1}\binom{N-q-1+k}{k}\binom{N+k}{k}^{-1}\frac{d_{q+1}^n}{\omega_{n-1}}P_{q+1}^{(n-2)/2}((\zeta|\eta)) \\
= \ & \sum_{q=0}^{N-1}\binom{N-q-1+k}{k}\binom{N+k}{k}^{-1}2\pi\,\frac{d_q^{n+2}}{\omega_{n+1}}P_q^{n/2}((\zeta|\eta))D_{\mathbf{S}}^{e_j}(\zeta|\eta) \\
= \ & 2\pi\,\frac{N}{k+N}\,D_{\mathbf{S}}^{e_j}(\zeta|\eta) \\
& \sum_{q=0}^{N-1}\binom{N-q-1+k}{k}\binom{N-1+k}{k}^{-1}\frac{d_q^{n+2}}{\omega_{n+1}}P_q^{n/2}((\zeta|\eta)) \\
= \ & 2\pi\,\frac{N}{k+N}\,D_{\mathbf{S}}^{e_j}(\zeta|\eta)\,{}_kL_{N-1}^{n+2}(\zeta,\eta)
\end{aligned}
$$

for $j=1,\ldots,n$. (Since ${}_kL_N^n(\zeta,\eta)$ depends only on $(\zeta|\eta)$, it can be defined on every space $\mathbf{S}^q\times\mathbf{S}^q$, $q\in\mathbf{N}$.) In a similar way we get, for every multiindex $\alpha\neq 0$,

$$
\begin{aligned}
& D_{\mathbf{S}}^{\alpha}\,{}_kL_N^n(\zeta,\eta) \\
= \ & \sum_{j=1}^{|\alpha|}(2\pi)^j\frac{N(N-1)\cdots(N-j+1)}{(k+N)(k+N-1)\cdots(k+N-j+1)} \\
& \times\, Q_j(\zeta,\eta)\,{}_kL_{N-j}^{n+2j}(\zeta,\eta)
\end{aligned}
$$

if $N\geq|\alpha|$, where $Q_j(\zeta,\eta)$ is a linear combination of products of $D_{\mathbf{S}}^{\beta}(\zeta|\eta)$, $\beta\leq\alpha$. But for all $k\in\mathbf{N}_0$ and $j\in\mathbf{N}$ we have

$$
\lim_{N\to+\infty}\frac{N(N-1)\cdots(N-j+1)}{(k+N)(k+N-1)\cdots(k+N-j+1)}=1.
$$

Therefore, given a $\zeta\in\mathbf{S}^{n-1}$, $D_{\mathbf{S}}^{\alpha}\,{}_kL_N^n(\zeta,\eta)$ converges to 0 uniformly on $\{\eta\in\mathbf{S}^{n-1}:d(\zeta,\eta)\geq\delta\}$ (with $0<\delta<2$) if we suppose $k\geq n+2j-2$ for $j=1,\ldots,|\alpha|$, that is, if $k>n-2+2|\alpha|$. We take now F closed in $\mathbf{S}^{n-1}\setminus\operatorname{supp}T$, $k>n-2+2m$ and $\varepsilon>0$. Let $r:=d(F,\operatorname{supp}T)$ (so $0<r\leq 2$); we put

$$
K:=\{\eta\in\mathbf{S}^{n-1}:d(\eta,\operatorname{supp}T)\leq r/4\}.
$$

Since T is of order m, there exists $C > 0$ such that, if $\varphi \in \mathcal{D}(\mathbf{S}^{n-1})$,

$$|T(\varphi)| \leq C \sup_{|\alpha|\leq m} \sup_{\eta\in K} |D^{\alpha}_{\mathbf{S}}\varphi(\eta)|.$$

There exists $N_0 > 0$ such that $N \geq N_0$ implies

$$\sup_{|\alpha|\leq m} \sup_{d(\zeta,\eta)>r/4} |D^{\alpha}_{\mathbf{S}}\, {}_kL^n_N(\zeta,\eta)| < \varepsilon/C$$

(where we take $D^{\alpha}_{\mathbf{S}}$ with respect to the variable η). We note that, if $\zeta \in F$ and $\eta \in K$, $d(\zeta,\eta) > r/4$. Then, for all $\zeta \in F$ and $N \geq N_0$,

$$\begin{aligned}
&\left|\sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \Pi_l(T)(\zeta)\right| \\
&= \left|T[\eta \mapsto \sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} Z_l(\zeta,\eta)]\right| \\
&= |T[\eta \mapsto {}_kL^n_N(\zeta,\eta)]| \\
&\leq C \sup_{|\alpha|\leq m} \sup_{\eta\in K} |D^{\alpha}_{\mathbf{S}}\, {}_kL^n_N(\zeta,\eta)| \\
&\leq C \sup_{|\alpha|\leq m} \sup_{d(\zeta,\eta)>r/4} |D^{\alpha}_{\mathbf{S}}\, {}_kL^n_N(\zeta,\eta)| \\
&< \varepsilon.
\end{aligned}$$

The theorem is proved. □

REMARK 1. A similar result can be obtained with the summation in Abel means instead of in Cesàro means; the order of T does not come up in this case.

REMARK 2. If the support of T is not reduced to a single point, then (1) holds uniformly on every closed subset of $\mathbf{S}^{n-1} \setminus \operatorname{supp} T$ as soon as $k > n/2 - 1 + m$; this follows from the fact that $\sum_{l=0}^{+\infty} Z_l(\zeta,\eta)$, seen as a function of $(\zeta|\eta) \in [-1,1]$, is uniformly (C,k)-summable to 0 on $[-1+\delta, 1-\delta]$ (where $0 < \delta < 1$ is arbitrary) if and only if $k > n/2 - 1$ [6, pp.113–114 and 145–146].

EXAMPLES. First we consider the Dirac measure on a point $s \in \mathbf{S}^{n-1}$, δ_s. We have

$$\sum_{l=0}^{N} \binom{N-l+k}{k}\binom{N+k}{k}^{-1} \Pi_l(\delta_s)(\zeta) = \delta_s[\eta \mapsto {}_kL^n_N(\zeta,\eta)] = {}_kL^n_N(\zeta,s).$$

In $\zeta \notin \{s,-s\}$ this converges (to 0) only if $k > n/2 - 1$; in $\zeta = -s$ this converges (to 0) only if $k > n - 2$; in $\zeta = s$ this does not converge when

$N \to +\infty$, since $P_l^{(n-2)/2}(1) = 1$ for all $l \in \mathbf{N}_0$ [2, (3.3.13) p.82]. Then we look at a derivative of order 1 of δ_s: $D_{\mathbf{S}}^{e_1}\,\delta_s$. For $k > n-1$,

$$\sum_{l=0}^{+\infty} \Pi_l(D_{\mathbf{S}}^{e_1}\,\delta_s)(\zeta) = 0 \quad (C,k)$$

for all $\zeta \in \mathbf{S}^{n-1}$. Indeed it suffices to note that

$$\begin{aligned} D_{\mathbf{S}}^{e_1}\,\delta_s[\eta \mapsto {}_kL_N^n(\zeta,\eta)] &= -\delta_s[D_{\mathbf{S}}^{e_1}\,{}_kL_N^n(\zeta,\eta)] \\ &= -\delta_s[\eta \mapsto 2\pi\,\frac{N}{N+k}\,D_{\mathbf{S}}^{e_1}(\zeta|\eta)\,{}_kL_{N-1}^{n+2}(\zeta,\eta)] \end{aligned}$$

and that $D_{\mathbf{S}}^{e_1}(\zeta|\eta) = \zeta_1 - \eta_1(\zeta|\eta)$, which implies

$$D_{\mathbf{S}}^{e_1}\,\delta_s[\eta \mapsto {}_kL_N^n(\zeta,\eta)] = 0$$

if $\zeta = s$.

Remark 3. If (1) holds uniformly on a subset A of $\mathbf{S}^{n-1}$, it holds uniformly on the closure of A. Therefore, when the interior of $\operatorname{supp} T$ is empty, (1) does not hold uniformly on $\mathbf{S}^{n-1} \setminus \operatorname{supp} T$; this is the case in the examples above.

5. Some rotation-invariant distributions

For this paragraph we fix $s \in \mathbf{S}^{n-1}$ arbitrarily. We write $SO(n)_s$ the stabilizer of s in $SO(n)$; it is isomorphic to $SO(n-1)$. We write $S(s,1)$ the sphere of $\mathbf{S}^{n-1}$ with centre s and radius 1; it is the intersection of $\mathbf{S}^{n-1}$ with the subspace of $\mathbf{R}^n$ orthogonal to s and can be identified to $\mathbf{S}^{n-2}$. We say that a distribution T on $\mathbf{S}^{n-1}$ is *rotation-invariant* if $T(\varphi \circ g) = T(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbf{S}^{n-1})$ and $g \in SO(n)_s$. We note that if T is rotation-invariant, then $\Delta_{\mathbf{S}} T$ is also rotation-invariant. The vector space of all rotation-invariant distributions on $\mathbf{S}^{n-1}$ supported by $S(s,1)$ and of order less or equal to $m \in \mathbf{N}_0$ has the basis $\{\Delta_{\mathbf{S}}^q\,\chi_{n-1} : 2q-1 \leq m\} \cup \{\Delta_{\mathbf{S}}^q\,\mu_{n-1} : 2q \leq m\}$, where χ_{n-1} is the indicator function of the hemisphere $\{\eta \in \mathbf{S}^{n-1} : d(s,\eta) \leq 1\}$ and μ_{n-1} the measure defined by

$$\mu_{n-1}(\varphi) := \int_{S(s,1)} \varphi(\eta)\,d\sigma_{n-2}(\eta).$$

We will need a result on χ_{n-1} and another one on μ_{n-1} which are consequences of the Funk-Hecke theorem. On the one hand, if $Y \in$

$\mathcal{S}H_l(\mathbf{S}^{n-1})$, $l \geq 1$, then

$$\int_{\mathbf{S}^{n-1}} \chi_{n-1}(\eta) Y(\eta) d\sigma_{n-1}(\eta) = v_{n-1} P_{l-1}^{n/2}(0) Y(s), \tag{3}$$

where v_{n-1} is the volume of the unit ball in $\mathbf{R}^{n-1}$ [2, 3.4.6 p.102]. On the other hand, if $Y \in \mathcal{S}H_l(\mathbf{S}^{n-1})$, $l \geq 0$, then

$$\mu_{n-1}(Y) = \int_{S(s,1)} Y(\eta) d\sigma_{n-2}(\eta) = \omega_{n-2} P_l^{(n-2)/2}(0) Y(s), \tag{4}$$

[2, 3.4.7 p.103]. Finally we will need the asymptotic behavior of

$$P_l^{(n-2)/2}(t)$$

when l tends to $+\infty$ depending on $t \in [-1, 1]$. First we have

$$P_l^{(n-2)/2}(1) = 1 \quad \text{and} \quad P_l^{(n-2)/2}(-1) = (-1)^l$$

for all $l \in \mathbf{N}_0$. Then, for $0 < \theta < \pi$,

$$P_l^{(n-2)/2}(\cos\theta) \sim \Gamma((n-1)/2)\, l^{-(n-2)/2} \sum_{r=0}^{+\infty} \frac{C_r(\theta)\, \mathrm{e}^{il\theta} + D_r(\theta)\, \mathrm{e}^{-il\theta}}{l^r}$$

when $l \to +\infty$, where

$$C_0(\theta)\, \mathrm{e}^{il\theta} + D_0(\theta)\, \mathrm{e}^{-il\theta} = \kappa \cos(l\theta + (n-2)\theta/2 - \pi(n-2)/4),$$

κ being a non-zero constant [8, p.79], [9, p.194].

PROPOSITION 1. *Let $T \in \mathcal{D}'(\mathbf{S}^{n-1})$ be of the form $\Delta_{\mathbf{S}}^q \mu_{n-1}$ ($q \geq 0$) or $\Delta_{\mathbf{S}}^q \chi_{n-1}$ ($q \geq 1$) and let $m \in \mathbf{N}_0$ be the order of T. Take $s \in \mathbf{S}^{n-1}$ and $k \geq 0$. i) For $\zeta \in \{s, -s\}$, the Fourier-Laplace series of T in ζ, is (C, k)-summable to 0 if and only if $k > n/2 - 1 + m$. ii) For $\zeta \notin S(s, 1) \cup \{s, -s\}$, the Fourier-Laplace series of T in ζ is (C, k)-summable to 0 if and only if $k > m$. iii) For $\zeta \in S(s, 1)$, the Fourier-Laplace series of T in ζ is equal to 0 in the case $T = \Delta_{\mathbf{S}}^q \chi_{n-1}$ but is not (C, k)-summable to 0 for any $k \geq 0$ in the case $T = \Delta_{\mathbf{S}}^q \mu_{n-1}$.*

Proof. We first note that the order of $\Delta_{\mathbf{S}}^q \mu_{n-1}$ is $2q$ and the order of $\Delta_{\mathbf{S}}^q \chi_{n-1}$, $2q - 1$. Since we already know that the Fourier-Laplace series of T is (C, k)-summable to 0 in $\zeta \notin S(s, 1)$ if $k > n/2 - 1 + m$, it will suffice, when $\zeta \notin S(s, 1)$, to find for which k it is (C, k)-summable. The function $\eta \mapsto Z_l(\zeta, \eta)$ is a spherical harmonic of degree l [7, p.143] and therefore an eigenfunction of $\Delta_{\mathbf{S}}$ with eigenvalue $-l(n+l-2)$ [2, 3.2.11 p.74]. Hence

$$\begin{aligned}\Pi_l(\Delta_{\mathbf{S}}^q\,\mu_{n-1})(\zeta) &= \Delta_{\mathbf{S}}^q\,\mu_{n-1}[\eta\mapsto Z_l(\zeta,\eta)]\\ &= \mu_{n-1}[\eta\mapsto \Delta_{\mathbf{S}}^q\,Z_l(\zeta,\eta)]\\ &= \mu_{n-1}[\eta\mapsto (-l(n+l-2))^q\,Z_l(\zeta,\eta)]\\ &= (-l)^q(n+l-2)^q\,\mu_{n-1}[\eta\mapsto Z_l(\zeta,\eta)]\\ &= (-l)^q(n+l-2)^q\,\Pi_l(\mu_{n-1})(\zeta);\end{aligned}$$

and for the same reason

$$\Pi_l(\Delta_{\mathbf{S}}^q\,\chi_{n-1})(\zeta) = (-l)^q(n+l-2)^q\,\Pi_l(\chi_{n-1})(\zeta).$$

So we only have to study $\Pi_l(\mu_{n-1})$ and $\Pi_l(\chi_{n-1})$. From (4) we get

$$\begin{aligned}\Pi_l(\mu_{n-1})(\zeta) &= \mu_{n-1}[\eta\mapsto Z_l(\zeta,\eta)]\\ &= \omega_{n-2}\,P_l^{(n-2)/2}(0)Z_l(\zeta,s)\\ &= \omega_{n-2}\,P_l^{(n-2)/2}(0)\,\frac{d_l^n}{\omega_{n-1}}\,P_l^{(n-2)/2}((\zeta|s)).\end{aligned}$$

First we take $\zeta\in S(s,1)$, which means $(\zeta|s)=0$. When l tends to $+\infty$,

$$\begin{aligned}\Pi_l(\mu_{n-1})(\zeta) &= \frac{\omega_{n-2}}{\omega_{n-1}}\,d_l^n\,[P_l^{(n-2)/2}(0)]^2\\ &\approx \frac{\omega_{n-2}}{\omega_{n-1}}\frac{2}{(n-2)!}\,l^{n-2}[\Gamma((n-1)/2)l^{-(n-2)/2}]^2\\ &= \frac{\omega_{n-2}}{\omega_{n-1}}\frac{2}{(n-2)!}\,[\Gamma((n-1)/2)]^2.\end{aligned}$$

The assertion iii) is established for every distribution $\Delta_{\mathbf{S}}^q\,\mu_{n-1}$. Next we take $\zeta\notin S(s,1)\cup\{s,-s\}$, which means $(\zeta|s)\notin\{-1,0,1\}$. We get, taking $\theta\in]0,\pi[$ such that $\cos\theta=(\zeta|s)$,

$$\begin{aligned}\Pi_l(\mu_{n-1})(\zeta) \sim{}& \frac{\omega_{n-2}}{\omega_{n-1}}\frac{2}{(n-2)!}\,[\Gamma((n-1)/2)]^2\\ &\times\sum_{r=0}^{+\infty}\frac{1}{l^r}\Big[A_r(\theta)\,\mathrm{e}^{il(\theta+\pi/2)}+B_r(\theta)\,\mathrm{e}^{il(\theta-\pi/2)}\\ &\qquad+C_r(\theta)\,\mathrm{e}^{il(-\theta+\pi/2)}+D_r(\theta)\,\mathrm{e}^{il(-\theta-\pi/2)}\Big]\end{aligned}$$

when $l\to+\infty$. The assertion ii) follows for every distribution $\Delta_{\mathbf{S}}^q\,\mu_{n-1}$, using lemma 1, since $\pm\theta\pm\pi/2\notin\{0,\pi\}$. Finally we take $\zeta=s$, which means $(\zeta|s) = 1$. We get

$$\begin{aligned}\Pi_l(\mu_{n-1})(s) &\sim \frac{\omega_{n-2}}{\omega_{n-1}} \frac{2}{(n-2)!} [\Gamma((n-1)/2)] l^{(n-2)/2} \\ &\quad \times \sum_{r=0}^{+\infty} \frac{C_r(0)\,\mathrm{e}^{il\pi/2} + D_r(0)\,\mathrm{e}^{-il\pi/2}}{l^r}\end{aligned}$$

when $l \to +\infty$. The assertion i) follows for every distribution $\Delta_{\mathbf{S}}^q \mu_{n-1}$, using lemma 1 and the symmetry between s and $-s$. The case $\Delta_{\mathbf{S}}^q \chi_{n-1}$ can be handled similarly, (3) allowing to start with

$$\begin{aligned}\Pi_l(\chi_{n-1})(\zeta) &= \int_{\mathbf{S}^{n-1}} \chi_{n-1}(\eta) Z_l(\zeta, \eta) d\sigma_{n-1}(\eta) \\ &= v_{n-1} P_{l-1}^{n/2}(0) Z_l(\zeta, s) \\ &= v_{n-1} P_{l-1}^{n/2}(0) \frac{d_l^n}{\omega_{n-1}} P_l^{(n-2)/2}((\zeta|s)).\end{aligned}$$

We only treat the case $\zeta \in S(s,1)$. Here we have $\Pi_l(\chi_{n-1})(\zeta) = 0$ for all $l \in \mathbf{N}$, since $P_r^{(n-2)/2}(0) = 0$ if r is odd [2, 3.3.8 p.85]. It follows

$$\begin{aligned}\sum_{l=0}^{+\infty} \Pi_l(\Delta_{\mathbf{S}}^q \chi_{n-1})(\zeta) &= \sum_{l=0}^{+\infty} (-l)^q (n+l-2)^q \, \Pi_l(\chi_{n-1})(\zeta) \\ &= (-0)^q (n+0-2)^q \, \Pi_0(\chi_{n-1})(\zeta) \\ &= 0,\end{aligned}$$

which is the assertion iii). □

REMARK 4. The difference in behavior between $\zeta \in \{s, -s\}$ and $\zeta \notin S(s,1) \cup \{s,-s\}$ for the Fourier inversion of these rotation-invariant distributions is parallel to the one for the Fourier inversion of the function χ_{n-1} (see [1]).

6. Fourier inversion on the ball

We fix d, $m \in \mathbf{N}$. We write B^d the unit open ball in $\mathbf{R}^d$ and put, for all $x \in B^d$,

$$W(x) := w_m (1 - \|x\|^2)^{(m-2)/2},$$

where

$$w_m := \left[\int_{B^d} (1 - \|x\|^2)^{(m-2)/2} dx\right]^{-1} = \frac{\Gamma((m+d)/2)}{\pi^{d/2}\Gamma(m/2)}$$

[11, (2.3)]. On $L^2(B^d, W(x)dx)$ we define the scalar product

$$(f|g)_w := \int_{B^d} f(x)\overline{g(x)}W(x)dx.$$

We write $\mathcal{V}_l(B^d)$ the vector space of all polynomials on $\mathbf{R}^d$ of degree $l \in \mathbf{N}_0$ which are orthogonal, with respect to $(\,.\,|\,.\,)_w$, to all polynomials of inferior degree; we write its dimension $r_l = r_l^d$. Let $(Q_1^l, \dots, Q_{r_l}^l)$ be an orthonormal basis of $\mathcal{V}_l(B^d)$. For $f \in L^2(B^d, W(x)dx)$, the series

$$\sum_{l=0}^{+\infty} \sum_{j=1}^{r_l} (f|Q_j^l)_w \, Q_j^l,$$

called *Fourier series of* f (with respect to W), converges to f in $L^2(B^d, W(x)dx)$. We write

$$\Xi_l(f) := \sum_{j=1}^{r_l} (f|Q_j^l)_w \, Q_j^l,$$

the orthogonal projection of f on $\mathcal{V}_l(B^d)$. For $x \in B^d$ we have

$$\Xi_l(f)(x) = \int_{B^d} \Xi_l(x, \xi) f(\xi) W(\xi) d\xi,$$

where

$$\Xi_l(x, \xi) := \sum_{j=1}^{r_l} Q_j^l(x)\overline{Q_j^l(\xi)}$$

is the *reproducing kernel of* $\mathcal{V}_l(B^d)$. We write $\mathcal{E}'(B^d)$ the set of distributions on $\mathbf{R}^d$ with support in B^d. For $\tau \in \mathcal{E}'(B^d)$, the *Fourier series of* τ (with respect to W) is

$$\sum_{l=0}^{+\infty} \sum_{j=1}^{r_l} \tau(\overline{Q_j^l}\, W) Q_j^l = \sum_{l=0}^{+\infty} \Xi_l(\tau),$$

with, for $x \in B^d$,

$$\Xi_l(\tau)(x) := \tau[\xi \mapsto \Xi_l(x, \xi) W(\xi)].$$

THEOREM 2. *Let* $\tau \in \mathcal{E}'(B^d)$ *be of order* $p \in \mathbf{N}_0$. i) *If there exist* $k \geq 0$ *and* U *an open subset of* B^d *on which*

$$\sum_{l=0}^{+\infty} \Xi_l(\tau)(x) = 0 \quad (C, k) \tag{5}$$

uniformly (in x), then τ is zero on U. ii) *Conversely, if $k > (d+m)/2 - 1 + p$, then (5) holds uniformly on every closed subset of $B^d \setminus \operatorname{supp} \tau$.*

Proof. From τ we construct a distribution T on $\mathbf{S}^{d+m-1}$ by

$$T(\varphi) := \tau[x \mapsto \frac{W(x)}{\omega_{m-1}} \int_{\mathbf{S}^{m-1}} \varphi(x, \sqrt{1 - \|x\|^2}\,\eta) d\sigma_{m-1}(\eta)]$$

for all $\varphi \in \mathcal{D}(\mathbf{S}^{d+m-1})$, with $x \in B^d(\mathbf{S}^0 = \{-1, 1\}$ and $\omega_0^{-1} \int_{\mathbf{S}^0} \psi(\eta) d\sigma_0(\eta)$ means $(\psi(-1)+\psi(1))/2$). Tedious but straightforward calculations show that T is indeed a distribution, that its order is equal to the order of τ and that

$$\operatorname{supp} T = (\operatorname{supp} \tau \times \mathbf{R}^m) \cap \mathbf{S}^{d+m-1}.$$

We will now use the link established by [11, theorem 2.6] between the zonal of degree l on $\mathbf{S}^{d+m-1}$ and the reproducing kernel of $\mathcal{V}_l(B^d)$:

$$\Xi_l(x, \xi) = \frac{\omega_{d+m-1}}{\omega_{m-1}} \int_{\mathbf{S}^{m-1}} Z_l(\zeta, (\xi, \sqrt{1 - \|\xi\|^2}\,\eta)) d\sigma_{m-1}(\eta)$$

with x, $\xi \in B^d$ and $\zeta := (x, y) \in \mathbf{S}^{d+m-1}$, where $y \in \mathbf{R}^m$ must only satisfy the condition $\|x\|^2 + \|y\|^2 = 1$. From this we immediately deduce:

$$\begin{aligned} & \Xi_l(\tau)(x) \\ = \;& \tau[\xi \mapsto \Xi_l(x, \xi) W(\xi)] \\ = \;& \tau[\xi \mapsto W(\xi) \frac{\omega_{d+m-1}}{\omega_{m-1}} \int_{\mathbf{S}^{m-1}} Z_l(\zeta, (\xi, \sqrt{1 - \|\xi\|^2}\,\eta) d\sigma_{m-1}(\eta)] \\ = \;& \omega_{d+m-1} T[z \mapsto Z_l(\zeta, z)] \\ = \;& \omega_{d+m-1} \Pi_l(T)(\zeta) \end{aligned}$$

where $\zeta := (x, y) \in \mathbf{S}^{d+m-1}$. Therefore the (C, k)-summability of the Fourier series of τ in x, $\sum_{l=0}^{+\infty} \Xi_l(\tau)(x)$, is equivalent to the (C, k)-summability of the Fourier-Laplace series of T in $\zeta = (x, y)$,

$$\sum_{l=0}^{+\infty} \Pi_l(T)(\zeta).$$

The conclusion follows from theorem 1 and remark 2. □

7. Fourier inversion on the real projective space

We write $\iota : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ ($n \geq 2$) the antipodal map $\iota(x) := -x$ and $\mathbf{P}^{n-1}(\mathbf{R})$ the real projective space of dimension $n-1$, that is, $\mathbf{S}^{n-1}/\langle \iota \rangle$. A function f on $\mathbf{S}^{n-1}$ is *even* if $f = f \circ \iota$ and *odd* if $f = -f \circ \iota$. Every function f on $\mathbf{S}^{n-1}$ is sum of its even and odd parts: $f_e := (f + f \circ \iota)/2$ and $f_o := (f - f \circ \iota)/2$. To every even function f on $\mathbf{S}^{n-1}$ corresponds one and exactly one function $\widetilde{f}$ on $\mathbf{P}^{n-1}(\mathbf{R})$ by $f(x) = \widetilde{f}(\pi(x))$, where $\pi : \mathbf{S}^{n-1} \to \mathbf{P}^{n-1}(\mathbf{R})$ is the canonical map. A distribution T on $\mathbf{S}^{n-1}$ is *even* if $T(\varphi) = T(\varphi \circ \iota)$ for all $\varphi \in \mathcal{D}(\mathbf{S}^{n-1})$; in that case $T(\varphi) = T(\varphi_e)$ and $T(\varphi_o) = 0$. Since $P_l^{(n-2)/2}$ is even for l even and odd for l odd (this follows from [2, (3.3.26) p.89] and $P_0^{(n-2)/2} = 1$, $P_1^{(n-2)/2}(t) = t$), the zonal $Z_l(\zeta, \eta)$ is, in each variable, even for l even and odd for l odd, by (2). Hence, given $T \in \mathcal{D}'(\mathbf{S}^{n-1})$ even, we have, for all $\zeta \in \mathbf{S}^{n-1}$,

$$\Pi_l(T)(\zeta) = T[\eta \mapsto Z_l(\zeta, \eta)] = 0$$

if l is odd; moreover,

$$\Pi_{2l}(T)(-\zeta) = T[\eta \mapsto Z_{2l}(-\zeta, \eta)] = T[\eta \mapsto Z_{2l}(\zeta, \eta)] = \Pi_{2l}(T)(\zeta),$$

that is, $\Pi_{2l}(T)$ is even. We now take a distribution $\widetilde{T}$ on $\mathbf{P}^{n-1}(\mathbf{R})$. From it we construct a distribution T on $\mathbf{S}^{n-1}$ by $T(\varphi) := \widetilde{T}(\widetilde{\varphi_e})$ for all $\varphi \in \mathcal{D}(\mathbf{S}^{n-1})$; T is even and therefore has the Fourier-Laplace series

$$\sum_{l=0}^{+\infty} \Pi_{2l}(T).$$

Since each of the functions in this series is even, we can go back to $\mathbf{P}^{n-1}(\mathbf{R})$ and obtain in this way the *Fourier-Laplace series of* $\widetilde{T}$:

$$\sum_{l=0}^{+\infty} \widetilde{\Pi_{2l}(T)}.$$

THEOREM 3. *Let $\widetilde{T} \in \mathcal{D}'(\mathbf{P}^{n-1}(\mathbf{R}))$ be of order $m \in \mathbf{N}_0$.* i) *If there exist $k \geq 0$ and U an open subset of $\mathbf{P}^{n-1}(\mathbf{R})$ on which*

$$\sum_{l=0}^{+\infty} \widetilde{\Pi_{2l}(T)}(\zeta) = 0 \quad (C, k) \tag{6}$$

uniformly (in ζ), then $\widetilde{T}$ is zero on U. ii) *Conversely, if $k > n/2 - 1 + m$, then* (6) *holds uniformly on every closed subset of $\mathbf{P}^{n-1}(\mathbf{R}) \setminus \operatorname{supp} \widetilde{T}$.*

Proof. This follows from the above discussion, theorem 1 and remark 2. □

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