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ALMOST FIXED POINT THEOREMS OF THE ZIMA TYPE

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ABSTRACT. In this paper, from the KKM theorem, we deduce almost fixed point theorems for convex-valued u.s.c. (or l.s.c.) multimaps satisfying certain conditions originated from Zima [19]. From these results, we obtain various fixed point theorems which extend a number of known results.

1. Introduction

The celebrated Brouwer fixed point theorem in 1912 was extended to normed vector spaces by Schauder in 1930, to locally convex Hausdorff topological vector spaces by Tychonoff in 1935, and to topological vector spaces E on which its topological dual E^* separates points by Ky Fan in 1964. On the other hand, the multimap versions of the above results are obtained by Kakutani, Bohnenblust-Karlin, Fan, Glicksberg, Himmelberg, and other authors. Moreover, some interesting related results have appeared on topological vector spaces not necessarily locally convex; see [15].

In 1977, Zima [19] obtained a generalization of the Schauder fixed point theorem with respect to para-normed spaces which are not locally convex.

Let E be a vector space over the real or complex number field. A real function $\|\cdot\|^* : E \to [0, \infty)$ is called a *para-norm* if and only if:

1) $\|x\|^* = 0 \iff x = 0$.
2) $\|-x\|^* = \|x\|^*$ for every $x \in E$.
3) $\|x+y\|^* \leq \|x\|^* + \|y\|^*$ for every $x, y \in E$.
4) If $\|x_n - x_0\|^* \to 0$ and $r_n \to r_0$, then $\|r_n x_n - r_0 x_0\|^* \to 0$.

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Then $(E, \| \|^*)$ is called a *para-normed space.* The para-normed space $(E, \| \|^*)$ is a topological vector space if a basis of neighborhoods of zero in E is given by the family $\{V_r\}_{r>0}$, where $V_r = \{x \in E : \|x\|^* < r\}$.

In fact, Zima has proved a generalization of Schauder fixed point theorem in a para-normed space for the map $f : K \to K$, where K is a closed convex and bounded subset of a para-normed space E, f is completely continuous map, and there exists $C > 0$ such that $\|tx\|^* \leq Ct\|x\|^*$ for every $t \in [0,1]$ and every $x \in f(K) - f(K)$.

Later Hadžić showed that the set K is of the so-called Zima type. Since then there have appeared a number of works on fixed point problems related to conditions of the Zima type; see Hadžić et al. [4–9, 10, 11]. The proofs of those works are based on various methods.

Motivated by the second author's previous works [16, 17], in the present paper, we show that most of those fixed point theorems are consequences of the celebrated Knaster-Kuratowski-Mazurkiewicz (simply, KKM) theorem [13]. In fact, we obtain new forms of almost fixed point theorems of the Zima type for upper [resp. lower] semicontinuous multimaps, and consequently, some fixed point theorems. Our results generalizes and unifies a number of known results with more transparent proofs.

2. Preliminary

Let E be a topological vector space (t.v.s) and $\mathcal{V}$ a basis of neighborhoods of the origin 0 of E. We say that a subset X of E is *of the Zima type* whenever for every $U \in \mathcal{V}$ there exists $V \in \mathcal{V}$ such that

$$\text{co}(V \cap (X - X)) \subset U;$$

for details, see [7] and references therein.

A *multimap* or a *map* $T : X \multimap Y$ is a function from a set X into the power set of a set Y with *nonempty values* $T(x)$ for $x \in X$ and *fibers* $T^-(y)$ for $y \in Y$. Note that $x \in T^-(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a map $T : X \multimap Y$ is said to be *closed* if its graph $\text{Gr}(T) = \{(x,y) : x \in X, y \in T(x)\}$ is closed in $X \times Y$, and *compact* if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y.

A map $T : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c) if for each closed set $B \subset Y$, the set

$$T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$$

is a closed subset of X; *lower semicontinuous* (l.s.c) if for each open set $B \subset Y$, the set $T^{-}(B)$ is open in X; and *continuous* if it is u.s.c. and l.s.c.

From the KKM theorem and its open version, we immediately have the following form as in Fan [2]:

THEOREM 1. *Let X be a subset of a topological vector space, D a nonempty subset of X such that $\operatorname{co} D \subset X$, and $F : D \multimap X$ a multimap with closed [resp. open] values. If*

$$\operatorname{co} A \subset F(A)$$

for every nonempty finite subset A of D, then the family $\{F(x)\}_{x \in D}$ has the finite intersection property.

It is well known that the closed and open versions of Theorem 1 can be derived from each other; see [14].

A nonempty subset Y of a topological vector space E is said to be *almost convex* [12] if for any $V \in \mathcal{V}$ and for any finite subset $\{y_1, y_2, \dots, y_n\}$ of Y, there exists a finite subset $\{z_1, z_2, \dots, z_n\}$ of Y, such that $z_i - y_i \in V$ for each $i = 1, \dots, n$ and $\operatorname{co}\{z_1, z_2, \dots, z_n\} \subset Y$.

3. Almost fixed point theorems

From Theorem 1, in this section, we deduce a very general almost fixed point theorem for convex-valued u.s.c. (or l.s.c.) multimaps defined on almost convex subsets having a certain form of the Zima type and some of its direct consequences.

The following almost fixed point theorem is our main result in this paper:

THEOREM 2. *Let X be a subset of a topological vector space E and Y an almost convex subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. Suppose that*

(Z_1) *for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that*

$$\operatorname{co}(V \cap (T(Y) - Y)) \subset U.$$

If there is a totally bounded subset K of $\overline{X}$ such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$ and $Y \cap K$ is dense in K, then for any $U \in \mathcal{V}$, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Proof. Let $U \in \mathcal{V}$, and V be a symmetric open neighborhood of 0 satisfying (Z_1). There exists a symmetric open neighborhood $W \in \mathcal{V}$ such that $\overline{W} + \overline{W} \subset V$. Since K is totally bounded in E, there exists a finite subset $\{x_1, x_2, \dots, x_n\} \subset K$ such that $K \subset \bigcup_{i=1}^n (x_i + W)$. Moreover, since Y is almost convex and $Y \cap K$ is dense in K, there exists a finite subset $D := \{y_1, y_2, \dots, y_n\}$ of Y such that $y_i - x_i \in W$ for each $i = 1, 2, \dots, n$, and $Z := \mathrm{co}\{y_1, y_2, \dots, y_n\} \subset Y$. [In fact, choose a neighborhood $W' \in \mathcal{V}$ such that $W' + W' \subset W$. Since $Y \cap K$ is dense in K, there is a subset $\{z_1, z_2, \dots, z_n\}$ of $Y \cap K$ such that $z_i - x_i \in W'$ for each i. Since Y is almost convex, there exists a subset $\{y_1, y_2, \dots, y_n\}$ of Y such that $y_i - z_i \in W'$ for each i and $\mathrm{co}\{y_1, y_2, \dots, y_n\} \subset Y$. Then we have $y_i - x_i = (y_i - z_i) + (z_i - x_i) \in W' + W' \subset W$ for each i.]

If T is lower semicontinuous, for each i, let

$$F(y_i) := \{z \in Z : T(z) \cap (x_i + W) = \emptyset\},$$

which is closed in Z. Moreover we have

$$\bigcap_{i=1}^n F(y_i) = \{z \in Z : T(z) \cap \bigcup_{i=1}^n (x_i + W) = \emptyset\} = \emptyset,$$

since

$$\emptyset \neq T(z) \cap K \subset T(z) \cap \bigcup_{i=1}^n (x_i + W)$$

for each $z \in Y$.

If T is upper semicontinuous, for each i, let

$$F(y_i) := \{z \in Z : T(z) \cap (x_i + \overline{W}) = \emptyset\},$$

which is open in Z. Moreover we have

$$\bigcap_{i=1}^n F(y_i) = \emptyset$$

as in the above.

Now we apply Theorem 1 replacing (X, D) by $(Z, \{y_i\}_{i=1}^n)$. Since the conclusion of Theorem 1 does not hold, in any case, there exist a finite subset $N := \{y_{i_1}, \dots, y_{i_k}\}$ of D and $x_U \in \mathrm{co}\, N \subset Y$ such that $x_U \notin F(N)$ or $T(x_U) \cap (x_{i_j} + \overline{W}) \neq \emptyset$ for all $j = 1, \dots, k$. Note that

$$x_{i_j} + \overline{W} = x_{i_j} - y_{i_j} + y_{i_j} + \overline{W} \subset y_{i_j} + W + \overline{W} \subset y_{i_j} + V$$

and hence

$$T(x_U) \cap (y_{i_j} + V) \neq \emptyset \quad \text{or} \quad V \cap (T(x_U) - y_{i_j}) \neq \emptyset.$$

Therefore, there exists a $z_j \in T(x_U)$ such that

$$z_j - y_{i_j} \in V \cap (T(x_U) - y_{i_j}) \subset (V \cap (T(Y) - Y)).$$

Let $x_U = \sum_{j=1}^{k} \alpha_j y_{i_j}$ and $y_U = \sum_{j=1}^{k} \alpha_j z_j$, where $0 \leq \alpha_j \leq 1$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$. Then $y_U \in \operatorname{co}\{z_j\}_{j=1}^{k} \subset \operatorname{co} T(x_U) = T(x_U)$ and

$$y_U - x_U = \sum_{j=1}^{k} \alpha_j (z_j - y_{i_j}) \in \operatorname{co}(V \cap (T(Y) - Y)) \subset U.$$

This completes our proof. □

The point x_U in the conclusion of Theorem 2 is called a *U-almost fixed point* of the multimap T.

Note that if $X = Y$ then the condition (Z_1) reduces to the following:

(Z_2) *for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that*

$$\operatorname{co}(V \cap (T(X) - X)) \subset U.$$

According to Hadžić [6], a subset K of E is said to be *of the Zima type* if for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that $\operatorname{co}(V \cap (K - K)) \subset U$.

COROLLARY 3. *Let E be a topological vector space and K a nonempty convex and totally bounded subset of E. Let $F : K \multimap E$ a lower* [resp. upper] *semicontinuous map with convex values such that for every $x \in K$, $F(x) \cap K \neq \emptyset$. If $K \cup F(K)$ is of Zima type, then F has a U-almost fixed point for every $U \in \mathcal{V}$.*

Proof. Put $X = Y = K$ in Theorem 2. Since $K \cup F(K)$ is of the Zima type, X and T can be replaced by K and F, respectively in the condition (Z_2). Therefore F has a U-almost fixed point. □

EXAMPLES 1. Hadžić [9, Theorem 2] is the lower semicontinuous case of Corollary 3.

2. A particular form of Corollary 3, for the case where E is completely metrizable and $X = Y$ is closed and convex, can be used to obtain [9, Theorem 1].

COROLLARY 4. *Let X be a subset of a topological vector space E and Y an almost convex subset of X. Let $T : X \multimap E$ be a multimap such that $T(y)$ is convex for each $y \in X$ and $T^-(z)$ is open for each $z \in E$. If the condition* (Z_1) *holds and if there is a totally bounded subset K of $\overline{X}$ such that $T(y) \cap K \neq \emptyset$* for each $y \in Y$ and $Y \cap K$ is *dense in K, then for each $U \in \mathcal{V}$, T has a U-almost fixed point.*

Proof. Simply T is lower semicontinuous. □

For a map $T : X \multimap X$ with $X = Y$ and $F(X) = K$, Corollary 4 reduces to the following:

COROLLARY 5. *Let X be an almost convex subset of a topological vector space E, and $T : X \multimap X$ a multimap such that*

(1) *$T(x)$ is nonempty and convex for each $x \in X$,*
(2) *$T^-(y)$ is open for each $y \in X$, and*
(3) *$T(X)$ is totally bounded.*

Then for any convex neighborhood $U \in \mathcal{V}$, T has a U-almost fixed point.

Note that Corollary 5 improves [16, Corollary 7].

Note that if U itself is convex, then the condition (Z_1) holds trivially, and Theorem 2 reduces to the following:

THEOREM 6. *Let X be a subset of a topological vector space E and Y an almost convex subset of X. Let $T : X \multimap E$ be a lower [resp. upper] semicontinuous multimap such that $T(y)$ is convex for all $y \in Y$. If there is a totally bounded subset K of $\overline{X}$ such that $T(y) \cap K \neq \emptyset$ for each $y \in Y$ and $Y \cap K$ is dense in K, then for any convex neighborhood $U \in \mathcal{V}$, there exists a point $x_U \in Y$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.*

EXAMPLES 1. Note that Theorem 6 slightly sharpens [16, Theorem 2].

2. In [16], Theorem 6 was shown to extend results of Ky Fan [3], Lassonde [14], Park and Tan [17], and Himmelberg [12].

For $X = Y$ and $T(X) = K$, Theorem 6 reduces to the following:

COROLLARY 7. *Let X be an almost convex subset of a topological vector space E and $T : X \multimap X$ a lower [resp. upper] semicontinuous multimap with convex values. If $T(X)$ is totally bounded, then for any convex neighborhood $U \in \mathcal{V}$, T has a U-almost fixed point.*

4. Fixed point theorems

In this section, we show that Theorem 2 is useful to deduce various forms of fixed point theorems including that of the Zima type.

For an upper semicontinuous map, we have the following fixed point theorem:

THEOREM 8. *Let X be a subset of a Hausdorff topological vector space E and Y an almost convex subset of X. Let $T : X \multimap X$ be a compact upper semicontinuous multimap with closed values such that $T(y)$ is convex for all $y \in Y$, and $Y \cap T(X)$ is dense in $T(X)$. If the condition (Z_1) holds, then T has a fixed point $x_0 \in X$, that is, $x_0 \in T(x_0)$.*

Proof. Let $K = T(X)$. By Theorem 2, for each $U \in \mathcal{V}$, there exist $x_U, y_U \in X$ such that $y_U \in T(x_U)$ and $y_U \in x_U + U$. Since $T(X)$ is relatively compact, we may assume that the net $\{y_U\}$ converges to some $x_0 \in X$. Since E is Hausdorff, the net $\{x_U\}$ also converges to x_0. Because T is upper semicontinuous with closed values, the graph of T is closed in $X \times X$ and hence we have $x_0 \in T(x_0)$. This completes our proof. □

EXAMPLE. If E is locally convex and Y is dense in X, then Theorem 8 reduces to Park and Tan [17, Theorem 1], which extends Himmelberg's theorems [12] and many others; see [17].

In particular, for $Y = X$, we obtain

THEOREM 9. *Let X be an almost convex subset of a Hausdorff topological vector space E. Then any compact upper semicontinuous multimap $T : X \multimap X$ with closed convex values has a fixed point in X whenever the condition (Z_2) holds.*

COROLLARY 10. *Let X be an almost convex subset of a locally convex Hausdorff topological vector space E. Then any compact upper semicontinuous multimap $T : X \multimap X$ with closed convex values has a fixed point in X.*

If X itself is convex, then Theorem 9 holds under a slightly more general condition than (Z_2):

THEOREM 11. *Let X be a convex subset of a Hausdorff topological vector space E. Then any compact upper semicontinuous multimap*

$T : X \multimap X$ with closed convex values has a fixed point in X whenever the following holds:

(Z_3) *for each $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ such that*

$$\mathrm{co}(V \cap (T(X) - T(X))) \subset U.$$

Proof. Let $K := T(X)$ and choose $D := \{x_1, x_2, \dots, x_n\} \subset K$ in the proof of Theorem 2. Then follow the proofs of Theorems 2 and 5. □

Note that (Z_2) $\Longrightarrow$ (Z_3), and (Z_3) simply tells that $T(X)$ is of the Zima type.

EXAMPLES 1. Since any subset of a locally convex t.v.s. is of the Zima type, Theorem 11 generalizes the well-known Himmelberg theorem [12].

2. Hadžić [4, Corollary 2], [6, Theorem 8] obtained Theorem 11 under the restriction that X is closed. A number of consequences and applications of her results were given in [4, 6], and Hadžić and Gajić [11]. Moreover, Arandelović [1] gave a simple proof of a particular form of Hadžić's theorem using the KKM–Fan theorem.

3. Hadžić [5, Theorem 2] obtained a particular form of Theorem 11 for a compact convex subset X of a metrizable t.v.s. E.

4. Hadžić [8, Theorem 3] obtained a particular form of Theorem 11 for a subset X of the Zima type in a complete t.v.s. E and applied her result to some economic problems.

The following is due to Hadžić and Gajić [10]:

COROLLARY 12. *Let K be a nonempty compact subset of a Hausdorff topological vector space E such that K is of the Zima type, and $T : K \multimap K$ an upper semicontinuous map such that $K \subset T(K)$, $T(K) = \mathrm{co}\, T(K)$, $T(x)$ is closed for every $x \in K$, and $\overline{\mathrm{co}}\, T^-(x) = T^-(x)$. Then T has a fixed point in K.*

Proof. Let $X = T(K)$. Then $T^- : X \multimap K \subset X$ is a compact upper semicontinuous map with closed convex values. Then (Z_3) holds for T^- instead of T. Therefore, by Theorem 11, the map T^- has a fixed point. □

Corollary 12 was used in [11] to obtain some minimax theorems as in Himmelberg [12].

Weber [18] defined that a subset K of E is said to be *strongly convexly totally bounded* (sctb) if every neighborhood U of 0 in E there is a convex subset C of U and a finite subset F of E such that $K \subset F + C$.

He showed that if K is totally bounded, then K is sctb if and only if K is of the Zima type. Therefore in Theorem 11, the condition (Z_3) can be replaced by the following:

(W) *$T(X)$ is sctb.*

Moreover, Weber also showed that if K is compact and convex, then

$$K \text{ is sctb} \iff K \text{ is the Zima type} \iff K \text{ is locally convex.}$$

From this, we can obtain some particular forms of Theorem 11. For example, we have the following:

COROLLARY 13. *Let X be a compact, convex and locally convex subset of a Hausdorff topological vector space E. Then any upper semi-continuous multimap $T : X \multimap X$ with nonempty closed convex values has a fixed point.*

Finally, it is well-known that the Brouwer fixed point theorem is equivalent to the KKM Theorem 1 and, since each of theorems and corollaries in this paper implies the Brouwer theorem and is deduced from Theorem 1, they are all equivalent to the Brouwer fixed point theorem.

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