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ON DISTANCE-PRESERVING MAPPINGS

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Abstract. We generalize a theorem of W. Benz by proving the following result: Let H_θ be a half space of a real Hilbert space with dimension ≥ 3 and let Y be a real normed space which is strictly convex. If a distance $\rho > 0$ is contractive and another distance $N\rho$ ($N \geq 2$) is extensive by a mapping $f : H_\theta \to Y$, then the restriction $f|_{H_{\theta+\rho/2}}$ is an isometry, where $H_{\theta+\rho/2}$ is also a half space which is a proper subset of H_θ. Applying the above result, we also generalize a classical theorem of Beckman and Quarles.

1. Introduction

Let X and Y be normed spaces. A mapping $f : X \to Y$ is called an isometry (or a congruence) if f satisfies

$$\|f(x) - f(y)\| = \|x - y\|$$

for all $x, y \in X$. A distance $\rho > 0$ is said to be contractive (or non-expanding) by $f : X \to Y$ if $\|x-y\| = \rho$ always implies $\|f(x)-f(y)\| \leq \rho$. Similarly, a distance ρ is said to be extensive (or non-shrinking) by f if the inequality $\|f(x)-f(y)\| \geq \rho$ is true for all $x, y \in X$ with $\|x-y\| = \rho$. We say that ρ is conservative (or preserved) by f if ρ is contractive and extensive by f simultaneously.

If f is an isometry, then every distance $\rho > 0$ is conservative by f, and conversely. At this point, we can raise a question:

Is a mapping that preserves certain distances an isometry?

In 1970, A. D. Aleksandrov [1] had raised a question whether a mapping $f : X \to X$ preserving a distance $\rho > 0$ is an isometry, which is now

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known to us as the Aleksandrov problem. Without loss of generality, we may assume $\rho = 1$ when X is a normed space (see [15]).

Indeed, earlier than Aleksandrov, F. S. Beckman and D. A. Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X = E^n$:

> *If a mapping $f : E^n \to E^n$ $(2 \le n < \infty)$ preserves distance 1, then f is a linear isometry up to translation.*

For $n = 1$, they suggested the mapping $f : E^1 \to E^1$ defined by

$$f(x) = \begin{cases} x + 1 & \text{for integral } x, \\ x & \text{otherwise} \end{cases}$$

as an example for a non-isometric mapping that preserves distance 1. For $X = E^\infty$, Beckman and Quarles also presented an example for a unit distance preserving mapping that is not an isometry (*cf.* [12]).

We may find a number of papers on a variety of subjects in the Aleksandrov problem (see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and also the references cited therein).

In 1985, W. Benz [3] introduced a sufficient condition under which a mapping, with a contractive distance and an extensive one, is an isometry (*cf.* [5]):

> *Let X and Y be real normed spaces such that* $\dim X \ge 2$ *and Y is strictly convex. Suppose $f : X \to Y$ is a mapping and $N \ge 2$ is a fixed integer. If a distance $\rho > 0$ is contractive and $N\rho$ is extensive by f, then f is a linear isometry up to translation.*

In this paper, we prove a theorem which generalizes a theorem of W. Benz (see [3]); more precisely, let H_θ be a half space of a real Hilbert space X with dimension larger than 2 and let Y be a real normed space which is strictly convex. If a distance $\rho > 0$ is contractive and another distance $N\rho$, $N \ge 2$, is extensive by a mapping $f : H_\theta \to Y$, then the restriction $f|_{H_{\theta+\rho/2}}$ is an isometry, where $H_{\theta+\rho/2}$ is a half space and also a proper subset of H_θ.

Moreover, applying this result, we generalize a classical theorem of Beckman and Quarles by proving that if a mapping, from a half space of X into Y, preserves a distance ρ, then the restriction of f to a subset of the half space is an isometry.

2. On a theorem of Benz

Let X be a real Hilbert space with $\dim X \geq 3$ for which there exists a unit vector $w \in X$ and a subspace X_s of X with $X = X_s \oplus Sp(w)$ and $X_s \perp Sp(w)$, where $Sp(w)$ denotes the subspace spanned by w. We now define half spaces,

$$H_\theta = \{x + \lambda w : x \in X_s\,;\, \lambda > \theta\}$$

for a fixed real number θ. Assume that Y is a real normed space which is strictly convex.

Throughout this section, let a real number $\rho > 0$ and an integer $N \geq 2$ be fixed. Furthermore, assume that a mapping $f : H_\theta \to Y$ satisfies both the following properties:
$(P1)$ ρ is contractive by f;
$(P2)$ $N\rho$ is extensive by f.

Following the steps presented in the paper [3], we prove in the following two lemmas that if a mapping $f : H_\theta \to Y$ satisfies both $(P1)$ and $(P2)$, then f preserves the distances ρ and 2ρ.

LEMMA 1. *For all $x, y \in H_\theta$, $\|x - y\| = \rho$ implies $\|f(x) - f(y)\| = \rho$.*

Proof. Assume that x and y of H_θ satisfy $\|x-y\| = \rho$ and $x-y \in \overline{H}_0$, where we set $\overline{H}_0 = \{x + \lambda w : x \in X_s\,;\, \lambda \geq 0\}$. Define $p_n = y + n(x-y)$ for $n = 0, 1, \ldots, N$. Then, $p_n \in H_\theta$, $\|p_N - y\| = N\rho$ and $\|p_n - p_{n-1}\| = \rho$ for $n = 1, \ldots, N$. Using $(P1)$ and $(P2)$, we have

$$N\rho \leq \|f(p_N) - f(y)\| \leq \sum_{n=1}^{N} \|f(p_n) - f(p_{n-1})\| \leq N\rho.$$

Hence, we conclude that $\|f(x) - f(y)\| = \|f(p_1) - f(p_0)\| = \rho$.

If $x-y \notin \overline{H}_0$ then $y-x \in \overline{H}_0$. In this case, we define $p_n = x+n(y-x)$ and we get the same result by following a similar process as before. □

LEMMA 2. *For all $x, y \in H_\theta$, $\|x-y\| = 2\rho$ implies $\|f(x)-f(y)\| = 2\rho$.*

Proof. Assume that x and y in H_θ satisfy $\|x-y\| = 2\rho$ and $x-y \in \overline{H}_0$, where we may refer to the proof of Lemma 1 for the definition of $\overline{H}_0$. Let us define

$$p_n = y + (n/2)(x-y)$$

for $n = 0, 1, \ldots, N$. Then, $p_n \in H_\theta$, $\|p_N - y\| = N\rho$ and $\|p_n - p_{n-1}\| = \rho$ for $n = 1, \ldots, N$. Now, we make use of $(P1)$ and $(P2)$ to get

$$N\rho \leq \|f(p_N) - f(y)\| \leq \sum_{n=1}^{N} \|f(p_n) - f(p_{n-1})\| \leq N\rho,$$

i.e.,

$$\|f(p_N) - f(y)\| = \sum_{n=1}^{N} \|f(p_n) - f(p_{n-1})\|. \tag{1}$$

If we assume $\|f(p_2) - f(p_0)\| < \|f(p_2) - f(p_1)\| + \|f(p_1) - f(p_0)\|$, then it should be $N \geq 3$ in view of (1) and further

$$\begin{aligned} \|f(p_N) - f(y)\| &\leq \sum_{n=3}^{N} \|f(p_n) - f(p_{n-1})\| + \|f(p_2) - f(p_0)\| \\ &< \sum_{n=1}^{N} \|f(p_n) - f(p_{n-1})\|, \end{aligned}$$

which is contrary to (1). Therefore, we conclude by Lemma 1 that

$$\|f(x) - f(y)\| = \|f(p_2) - f(p_0)\| = \|f(p_2) - f(p_1)\| + \|f(p_1) - f(p_0)\| = 2\rho.$$

For the case of $x - y \notin \overline{H}_0$, we define $p_n = x + (n/2)(y - x)$ and follow the same process as before to prove our assertion. □

Because of the strict convexity of Y, the following lemma is obvious (or see [3]). Hence, we omit the proof.

LEMMA 3. *For all $a, b, c \in Y$ and for any $\alpha > 0$, $\|b - a\| = \alpha = \|c - b\|$ and $\|c - a\| = 2\alpha$ imply $c = 2b - a$.*

We use the mathematical induction to prove the following lemma which turns out to be essential for treating the cases when x and y have the same X_s-components.

From now on, we denote by x_s, y_s and z_s the X_s-component of x, y and z, respectively, if there is no specification.

LEMMA 4. *For any given $n \in \mathbb{N}$, let $x = x_s + \lambda w$ and $y = y_s + \mu w$ be any points of H_θ with $x_s = y_s$ and $\lambda, \mu > \theta + (2^{-2} + 2^{-3} + \cdots + 2^{-(n+1)})\rho$. Then, $\|x - y\| = 2^{-n}\rho$ implies $\|f(x) - f(y)\| = 2^{-n}\rho$.*

Proof. Assume that $x = x_s + \lambda w$ and $y = y_s + \mu w$ are points of H_θ such that $x_s = y_s$, $\lambda, \mu > \theta + \rho/4$ and $\|x - y\| = |\lambda - \mu| = \rho/2$. Choose a $z = z_s + (\lambda + \mu)w/2 \in H_\theta$ with $\|x - z\| = \|y - z\| = \rho$.

Furthermore, select x' and y' on the rays $\overline{zx}$ and $\overline{zy}$, respectively, such that $\|x' - z\| = \|y' - z\| = 2\rho$. Then, $\|x' - y'\| = \rho$.

If we set $x' = x'_s + \lambda' w$ and $y' = y'_s + \mu' w$, then

$$\lambda' = \lambda + (\lambda - \mu)/2 > \theta + \rho/4 + (-\rho/2)/2 = \theta$$

and

$$\mu' = \mu + (\mu - \lambda)/2 > \theta + \rho/4 + (-\rho/2)/2 = \theta.$$

So, we know that both x' and y' are in H_θ.

According to Lemmas 1 and 2, we have

$$\|f(x) - f(z)\| = \|f(y) - f(z)\| = \|f(x') - f(y')\| = \rho,$$
$$\|f(x') - f(x)\| = \|f(y') - f(y)\| = \rho,$$
$$\|f(x') - f(z)\| = \|f(y') - f(z)\| = 2\rho.$$

In view of Lemma 3, $f(x)$ is a midpoint of $f(x')$ and $f(z)$, and likewise for $f(y)$. Hence, the triangles $f(x)f(z)f(y)$ and $f(x')f(z)f(y')$ are similar and we conclude that $\|f(x) - f(y)\| = \rho/2$.

Now, we assume that our assertion is true for some $n \in \mathbb{N}$ and suppose that $x = x_s + \lambda w$ and $y = y_s + \mu w$ satisfy $x_s = y_s$, $\lambda, \mu > \theta + (2^{-2} + 2^{-3} + \cdots + 2^{-(n+2)})\rho$ and $\|x - y\| = 2^{-(n+1)}\rho$. Choose a $z = z_s + (\lambda + \mu)w/2$ with $\|x - z\| = \|y - z\| = \rho$. Moreover, select x' and y' on the rays $\overline{zx}$ and $\overline{zy}$ respectively such that $\|x' - z\| = \|y' - z\| = 2\rho$. Then, $\|x' - y'\| = 2^{-n}\rho$. Similarly as in the first part, we know that both the x' and y' lie in H_θ.

By Lemmas 1 and 2, we get

$$\|f(x) - f(z)\| = \|f(y) - f(z)\| = \rho,$$
$$\|f(x') - f(x)\| = \|f(y') - f(y)\| = \rho,$$
$$\|f(x') - f(z)\| = \|f(y') - f(z)\| = 2\rho.$$

By Lemma 3, $f(x)$ is a midpoint of $f(x')$ and $f(z)$, and likewise for $f(y)$. Furthermore, we know that $x' = x'_s + \lambda' w$ and $y' = y'_s + \mu' w$ satisfy $x'_s = y'_s$, $\lambda', \mu' > \theta + (2^{-2} + 2^{-3} + \cdots + 2^{-(n+1)})\rho$ and $\|x' - y'\| = 2^{-n}\rho$. By the assumption of the induction, we see that $\|f(x') - f(y')\| = 2^{-n}\rho$.

Since the triangles $f(x)f(z)f(y)$ and $f(x')f(z)f(y')$ are similar, we may conclude that $\|f(x) - f(y)\| = 2^{-(n+1)}\rho$. □

In the following lemma, we prove that if x and y are separated from each other by a specific distance, then some equidistant points on the line through x and y are mapped by f onto some equidistant points of the line through $f(x)$ and $f(y)$.

LEMMA 5. *(a) If x and y are any points of H_θ with $\|x-y\| = \rho$, then $f(x+m(y-x)) = f(x)+m(f(y)-f(x))$ holds for all $m \in \mathbb{N} \cup \{0\}$ with $x+m(y-x) \in H_\theta$.*

(b) For any $n \in \mathbb{N}$, let x, y be points of $H_{\theta+\rho/2}$ with $x_s = y_s$ and $\|x-y\| = 2^{-n}\rho$. If $x+m(y-x) \in H_{\theta+\rho/2}$ for $m \in \mathbb{N}$, then $f(x+m(y-x)) = f(x)+m(f(y)-f(x))$.

Proof. (a) Assume that $x, y \in H_\theta$ satisfy $\|x-y\| = \rho$. We use induction to show that $f(x+m(y-x)) = f(x)+m(f(y)-f(x))$ holds for all $m \in \mathbb{N} \cup \{0\}$ with $x+m(y-x) \in H_\theta$. There is nothing to prove for $m = 0$ or 1. We now assume that our assertion is true for $m = 0, 1, \ldots, k$, where $k \geq 1$ is some integer. Put $p_i = x+i(y-x)$ for $i \in \mathbb{N}$ and let $p_{k+1} \in H_\theta$. Then, we get

$$\|p_k - p_{k-1}\| = \rho = \|p_{k+1} - p_k\| \text{ and } \|p_{k+1} - p_{k-1}\| = 2\rho.$$

According to Lemmas 1 and 2, we have

$$\|f(p_k)-f(p_{k-1})\| = \rho = \|f(p_{k+1})-f(p_k)\| \text{ and } \|f(p_{k+1})-f(p_{k-1})\| = 2\rho.$$

Hence, it follows from Lemma 3 that

$$f(p_{k+1}) = 2f(p_k) - f(p_{k-1}) = f(x) + (k+1)(f(y) - f(x)),$$

as we desired.

(b) Let $x = x_s + \lambda w$ and $y = y_s + \mu w$ be any points of $H_{\theta+\rho/2}$. Assume that $x_s = y_s$ and $\|x-y\| = 2^{-n}\rho$ for some $n \in \mathbb{N}$.

We also use induction to prove our assertion. There is nothing to prove for $m = 1$. We assume that our assertion holds for $m = 1, \ldots, k$, where $k \geq 1$ is some integer.

Set $p_i = x+i(y-x)$ for $i \in \mathbb{N}$ and let $p_{k+1} \in H_{\theta+\rho/2}$. Then, we have

$$\|p_k - p_{k-1}\| = 2^{-n}\rho = \|p_{k+1} - p_k\| \text{ and } \|p_{k+1} - p_{k-1}\| = 2^{-(n-1)}\rho.$$

Since the X_s-component of p_i is equal to x_s and $p_i \in H_{\theta+\rho/2}$ for $i = 1, \ldots, k+1$, we can make use of Lemma 4 to show

$$\|f(p_k) - f(p_{k-1})\| = 2^{-n}\rho = \|f(p_{k+1}) - f(p_k)\|,$$
$$\|f(p_{k+1}) - f(p_{k-1})\| = 2^{-(n-1)}\rho.$$

Hence, it follows from Lemma 3 that

$$f(p_{k+1}) = 2f(p_k) - f(p_{k-1}) = f(x) + (k+1)(f(y) - f(x)),$$

which completes the proof of (b). □

LEMMA 6. *Let n be a fixed positive integer. If $x, y \in H_\theta$ satisfy $\|x-y\| = n\rho$, then $\|f(x) - f(y)\| = n\rho$.*

Proof. Assume that x and y are points of H_θ and are separated from each other by a distance $n\rho$. Choose a point z on the segment between x and y such that $x = y + n(z - y)$. Then, we have $\|z - y\| = \rho$. From Lemma 5 (a), it follows that $f(x) = f(y) + n(f(z) - f(y))$. Hence, by Lemma 1, we get

$$\|f(x) - f(y)\| = n\,\|f(z) - f(y)\| = n\rho,$$

which completes our proof. □

Using Lemmas 4, 5 and 6, we can prove the following lemma which is indispensable for the proof of Theorem 9 below.

LEMMA 7. *Let $x = x_s + \lambda w$ and $y = y_s + \mu w$ be any points of H_θ. Assume that $m, n \in \mathbb{N}$ are given.*

(a) If $x_s \neq y_s$ and $\|x - y\| = n\rho/m$, then $\|f(x) - f(y)\| = n\rho/m$.

(b) If $x, y \in H_{\theta+\rho/2}$, $x_s = y_s$, and if $\|x - y\| = 2^{-n}m\rho$, then $\|f(x) - f(y)\| = 2^{-n}m\rho$.

Proof. (a) Assume that x and y are points of H_θ with $\|x-y\| = n\rho/m$ which are represented by $x = x_s + \lambda w$ and $y = y_s + \mu w$, where $x_s \neq y_s$, $\lambda \geq \mu > \theta$, and where $m \geq 2$ and n are positive integers.

Set $z = z_s + \mu w$ and examine whether there exists a $z_s \in X_s$ which is a solution of the following parametric equations

$$\begin{aligned}
\|z - x\|^2 &= \|z_s - x_s\|^2 + (\mu - \lambda)^2 &&= k^2\rho^2, \\
\|z - y\|^2 &= \|z_s - y_s\|^2 &&= k^2\rho^2, \\
\|x - y\|^2 &= \|x_s - y_s\|^2 + (\mu - \lambda)^2 &&= (n\rho/m)^2,
\end{aligned}$$

where k is a parameter whose value is integral. It follows from these equations that

$$\begin{aligned}
\|z_s - x_s\| &= \sqrt{k^2\rho^2 - (\mu - \lambda)^2}, \\
\|z_s - y_s\| &= k\rho, \\
\|x_s - y_s\| &= \sqrt{(n\rho/m)^2 - (\mu - \lambda)^2}.
\end{aligned} \tag{2}$$

The sphere in X_s of radius $\sqrt{k^2\rho^2 - (\mu - \lambda)^2}$ and with center at x_s is expressed by the first equation of (2). Let us use the notation S_1 for this sphere. The second one of (2) is an equation for the sphere S_2 in X_s of radius $k\rho$ and with center at y_s. If k is so large that the inequality

$$k\rho \leq \sqrt{k^2\rho^2 - (\mu - \lambda)^2} + \sqrt{(n\rho/m)^2 - (\mu - \lambda)^2}$$

holds, then $S_1 \cap S_2 \neq \emptyset$. Hence, we can select a z_s from $S_1 \cap S_2$, i.e., the parametric equations (2) are solvable in z_s. With such a z_s, $z = z_s + \mu w$ is separated from x resp. from y by a same distance $k\rho$.

Choose $x', y' \in H_\theta$ on the ray $\overline{zx}$ resp. $\overline{zy}$ such that $\|x' - z\| = \|y' - z\| = km\rho$. We then have $\|x' - y'\| = n\rho$. By Lemma 6, we get

$$\begin{aligned}
&\|f(x) - f(z)\| = \|f(y) - f(z)\| = k\rho, \\
&\|f(x') - f(z)\| = \|f(y') - f(z)\| = km\rho, \\
&\|f(x') - f(y')\| = n\rho.
\end{aligned}$$

Furthermore, by a slight modification of Lemma 5 (a), we conclude that $f(x)$ lies on the segment between $f(z)$ and $f(x')$ and also that $f(y)$ lies on the segment between $f(z)$ and $f(y')$.

Hence, the triangles $f(x)f(z)f(y)$ and $f(x')f(z)f(y')$ are similar. Therefore, we obtain $\|f(x) - f(y)\| = n\rho/m$.

(b) Assume that $x = x_s + \lambda w$ and $y = y_s + \mu w$ are points of $H_{\theta+\rho/2}$ with $x_s = y_s$ and $\|x - y\| = 2^{-n}m\rho$. Choose a z on the segment between x and y with $\|z - y\| = 2^{-n}\rho$. Then, by Lemma 4, $\|f(z) - f(y)\| = 2^{-n}\rho$. Further, in view of Lemma 5 (b), we get

$$f(x) = f(y + m(z - y)) = f(y) + m(f(z) - f(y)),$$

i.e.,

$$\|f(x) - f(y)\| = m\,\|f(z) - f(y)\| = 2^{-n}m\rho,$$

which completes the proof. □

LEMMA 8. *Assume that α and β are real numbers with $2\beta \geq \alpha > 0$. Then, for all $x, y \in H_\theta$ with $\|x - y\| = \alpha$, there exists a $z \in H_\theta$ satisfying $\|z - x\| = \beta = \|z - y\|$. In particular, if $x_s \neq y_s$, then $z_s \notin \{x_s, y_s\}$.*

Proof. Assume that $x = x_s + \lambda w$ and $y = y_s + \mu w$ are points of H_θ with $\|x - y\| = \alpha$, where $\lambda, \mu > \theta$. It is to find a $z = z_s + \delta w \in H_\theta$ which is a solution of the following equations:

$$\begin{aligned}
&\|z - x\|^2 = \|z_s - x_s\|^2 + (\delta - \lambda)^2 = \beta^2, \\
&\|z - y\|^2 = \|z_s - y_s\|^2 + (\delta - \mu)^2 = \beta^2, \qquad (3)\\
&\|x - y\|^2 = \|x_s - y_s\|^2 + (\lambda - \mu)^2 = \alpha^2.
\end{aligned}$$

Put $\delta = (\lambda + \mu)/2\,(> \theta)$. It then follows from (3) that

$$\begin{aligned}
&\|z_s - x_s\|^2 = \beta^2 - (\mu - \lambda)^2/4, \\
&\|z_s - y_s\|^2 = \beta^2 - (\mu - \lambda)^2/4, \\
&\|x_s - y_s\|^2 = \alpha^2 - (\mu - \lambda)^2.
\end{aligned}$$

Since $\dim X_s \geq 2$ and since

$$\begin{aligned}\|z_s - x_s\| + \|z_s - y_s\| &= 2\,\|z_s - x_s\| \\ &= \sqrt{(2\beta)^2 - (\mu - \lambda)^2} \\ &\geq \sqrt{\alpha^2 - (\mu - \lambda)^2} \\ &= \|x_s - y_s\|\end{aligned}$$

(where $\|x_s - y_s\| > 0$ for $x_s \neq y_s$, and hence $z_s \neq x_s$ and $z_s \neq y_s$), there exists at least one $z_s \in X_s$ which is a solution of the above equations. With such a z_s, $z = z_s + (\lambda + \mu)w/2 \in H_\theta$ satisfies our requirement. Hence, the proof is complete. □

So far, we have proved all preliminary lemmas to the main theorem of this section. In the following theorem, we generalize a theorem of Benz:

THEOREM 9. *Let a real number $\rho > 0$ and an integer $N \geq 2$ be given. If ρ is contractive and $N\rho$ is extensive by a mapping $f : H_\theta \to Y$, then $f|_{H_{\theta+\rho/2}}$ is an isometry. In particular, it holds that*

$$\|f(x) - f(y)\| = \|x - y\|$$

for any points x and y of H_θ with $x_s \neq y_s$.

Proof. Assume that $x, y \in H_{\theta+\rho/2}$ are distinct. For those x and y, choose the sequences, (k_i), (m_i) and (n_i), of non-negative integers with the following properties:

(K) $2^{-n_i} k_i \rho \leq \|x - y\| < 2^{-n_i}(k_i + 1)\rho$ for all sufficiently large integers i;

(M) $2^{-n_i}(m_i - 1)\rho < \|x - y\| \leq 2^{-n_i} m_i \rho$ for all sufficiently large integers i;

(N) (n_i) increases strictly to infinity.

Since $H_{\theta+\rho/2}$ is open, we can select a z_i on the segment $\overline{xy}$ and a $w_i \in H_{\theta+\rho/2}$ such that

$$\|x - z_i\| = 2^{-n_i} k_i \rho \text{ and } \|z_i - w_i\| = \|w_i - y\| = 2^{-n_i}\rho$$

for any sufficiently large i. It then follows from Lemma 7 (a) and (b) that

$$\|f(x) - f(z_i)\| = 2^{-n_i} k_i \rho \text{ and } \|f(z_i) - f(w_i)\| = \|f(w_i) - f(y)\| = 2^{-n_i}\rho$$

for any sufficiently large integer i. Thus, it follows from (K) that

$$\begin{aligned}\|f(x) - f(y)\| &\leq \|f(x) - f(z_i)\| + \|f(z_i) - f(w_i)\| + \|f(w_i) - f(y)\| \\ &\leq \|x - y\| + 2^{1-n_i}\rho\end{aligned}$$

for any sufficiently large integer i, i.e., we get $\|f(x) - f(y)\| \leq \|x - y\|$.

On the other hand, since $H_{\theta+\rho/2}$ is open, we can choose a $v_i \in H_{\theta+\rho/2}$ such that

$$\|x - v_i\| = 2^{-n_i} m_i \rho \text{ and } \|y - v_i\| = 2^{-n_i} \rho$$

for all sufficiently large integers i. From Lemma 7 (a) and (b) we get

$$\|f(x) - f(v_i)\| = 2^{-n_i} m_i \rho \text{ and } \|f(y) - f(v_i)\| = 2^{-n_i} \rho.$$

Hence, it follows from (M) that

$$\|f(x) - f(y)\| \geq \|f(x) - f(v_i)\| - \|f(y) - f(v_i)\| \geq \|x - y\| - 2^{-n_i} \rho$$

for all sufficiently large integers i, i.e., we get $\|f(x) - f(y)\| \geq \|x - y\|$, which completes the proof of the first part.

For the second part of this theorem, let $x, y \in H_\theta$ satisfy $x_s \neq y_s$ and $r_1 \rho < \|x - y\| < r_2 \rho$, where $r_1, r_2 > 0$ are given rational numbers. We prove that $r_1 \rho \leq \|f(x) - f(y)\| \leq r_2 \rho$: According to Lemma 8, there exists a $z \in H_\theta$ with $\|z - x\| = r_2 \rho / 2 = \|z - y\|$, $x_s \neq z_s$ and $y_s \neq z_s$. Due to Lemma 7 (a), we get

$$\|f(z) - f(x)\| = r_2 \rho / 2 = \|f(z) - f(y)\|.$$

Hence,

$$\|f(x) - f(y)\| \leq \|f(x) - f(z)\| + \|f(z) - f(y)\| = r_2 \rho.$$

On the other hand, assume that there existed $x, y \in H_\theta$ with $x_s \neq y_s$, $r_1 \rho < \|x - y\| < r_2 \rho$ and $\|f(x) - f(y)\| < r_1 \rho$. Then,

$$r_2 \rho - \|x - y\| < r_2 \rho - r_1 \rho < r_2 \rho - \|f(x) - f(y)\|. \tag{4}$$

Define $z = x + \lambda(y - x)$ for the case $y - x \in \overline{H}_0$ with $\lambda = r_2 \rho \, \|x - y\|^{-1} > 1$. (Otherwise, i.e., if $y - x \notin \overline{H}_0$, we replace the definition of z by $y + \lambda(x - y)$ and repeat the following process similarly.) It then follows that $x_s \neq z_s$, $y_s \neq z_s$ and $\|z - x\| = r_2 \rho$. Furthermore, (4) implies that $\|z - y\| = (\lambda - 1)\|x - y\| < (r_2 - r_1)\rho$. Due to Lemma 7 (a), we have $\|f(z) - f(x)\| = r_2 \rho$ and by considering the argument in the last paragraph, we see that $\|f(z) - f(y)\| \leq (r_2 - r_1)\rho$. Subsequently, we have

$$\begin{aligned} r_2 \rho = \|f(z) - f(x)\| &\leq \|f(z) - f(y)\| + \|f(y) - f(x)\| \\ &< (r_2 - r_1)\rho + r_1 \rho = r_2 \rho, \end{aligned}$$

which is a contradiction. Therefore, it should be $r_1 \rho \leq \|f(x) - f(y)\| \leq r_2 \rho$.

Since the set of all rational numbers is dense in $\mathbb{R}$, we conclude that the second assertion is true. □

3. On a theorem of Beckman and Quarles

Throughout this section, let X and Y denote n-dimensional Euclidean spaces, where $n \geq 3$ is a fixed integer, for which there exists a unit vector $w \in X$ and a subspace X_s of X such that $X = X_s \oplus Sp(w)$ and X_s is orthogonal to $Sp(w)$, where $Sp(w)$ is the subspace of X which is spanned by w.

Let us define

$$r_0 = \theta,\ r_1 = \theta + \rho,\ r_2 = \theta + \rho + \rho_1,\ r_3 = \theta + (1 + 1/n)\rho + \rho_1,$$

where θ is a real number, ρ is a positive real number and

$$\rho_1 = \sqrt{2(n+1)/n}\,\rho.$$

Using these r_k's we define

$$E_k = \{x + \lambda w : x \in X_s\,;\ \lambda > r_k\}$$

for $k = 0, 1, 2, 3$. We remark that $E_3 \subset E_2 \subset E_1 \subset E_0 \subset X$.

Let E be a subset of an n-dimensional Euclidean space X. Following W. Benz, we will call a set of n distinct points of E a β-set in E if the points are pairwise of distance $\beta > 0$. If there are two distinct points of X, which have distance α from each point of a β-set P in E, the two points will be called the α-associated points of P.

For the proofs of the following two lemmas, we may refer the reader to 2) and 3) in section 2 of [4].

LEMMA 10. *Assume that α and β are positive real numbers with*

$$\gamma(\alpha, \beta) := 4\alpha^2 - 2\beta^2(1 - 1/n) > 0$$

and that P is a β-set in E. The α-associated points of P are uniquely determined and the distance between them is $\sqrt{\gamma(\alpha, \beta)}$.

LEMMA 11. *Assume that α and β are positive real numbers with $\gamma(\alpha, \beta) > 0$. If x and y are points of X (or of Y) with $\|x - y\| = \sqrt{\gamma(\alpha, \beta)}$, then there exists a β-set P in X (or in Y) such that x and y are the α-associated points of P.*

LEMMA 12. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the distance $\rho_1 = \sqrt{\gamma(\rho, \rho)}$ is preserved by $f|_{E_1}$.*

Proof. Assume that x and y are points of E_1 satisfying $\|x - y\| = \rho_1$. According to Lemma 11 and the definition of E_k, there exists a ρ-set P in E_0 such that x and y are the ρ-associated points of P. Since f preserves ρ, $P' = f(P)$ is also a ρ-set in Y.

Due to Lemma 10, there are exactly two distinct ρ-associated points x' and y' of P' and they satisfy $\|x' - y'\| = \sqrt{\gamma(\rho, \rho)} = \rho_1$. Since there exist only two ρ-associated points of P', we have $\{f(x), f(y)\} \subset \{x', y'\}$, i.e., $\|f(x) - f(y)\| = 0$ or ρ_1.

Assume that $f(x) = f(y)$. Choose a $z \in E_0$ with $\|x - z\| = \rho_1$ and $\|y - z\| = \rho$. In view of Lemma 11, there exists a ρ-set Q in E_0 such that x and z are the ρ-associated points of Q (Because $x \in E_1$ and $\|x - q\| = \rho$ for each $q \in Q$, Q is a subset of E_0). Similarly, $Q' = f(Q)$ is a ρ-set in Y.

Due to Lemma 10, there exist exactly two distinct ρ-associated points x'' and z'' of Q' which satisfy $\|x'' - z''\| = \sqrt{\gamma(\rho, \rho)} = \rho_1$. Hence, $\{f(x), f(z)\} \subset \{x'', z''\}$, i.e., $\|f(x) - f(z)\| = 0$ or ρ_1, i.e., $\|f(y) - f(z)\| = 0$ or ρ_1 because we assumed $f(x) = f(y)$.

On the other hand, we get $\rho = \|y - z\| = \|f(y) - f(z)\| = 0$ or ρ_1, which is a contradiction. Altogether, we conclude that $\|f(x) - f(y)\| = \rho_1$. □

LEMMA 13. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the distance $\rho_2 = \sqrt{\gamma(\rho_1, \rho_1)} = (n + 1)(2\rho/n)$ is preserved by $f|_{E_2}$.*

Proof. Assume that x and y are points of E_2 with $\|x - y\| = \rho_2$. According to Lemma 11, there exists a ρ_1-set P in E_1 such that x and y are the ρ_1-associated points of P (see also the definition of E_k). Since $f|_{E_1}$ preserves ρ_1 (see Lemma 12), $P' = f(P)$ is also a ρ_1-set in Y.

By Lemma 10, there exist only two distinct ρ_1-associated points x' and y' of P' whose distance is $\|x' - y'\| = \rho_2$. Thus, we get $\{f(x), f(y)\} \subset \{x', y'\}$, i.e., $\|f(x) - f(y)\| = 0$ or ρ_2.

Assume $f(x) = f(y)$. Choose a $z \in E_1$ with $\|x - z\| = \rho_2$ and $\|y - z\| = \rho_1$ (Because of $y \in E_2$ and $\|y - z\| = \rho_1$, we conclude that $z \in E_1$). In view of Lemma 11, there exists a ρ_1-set Q in E_1 such that x and z are the ρ_1-associated points of Q (Because $x \in E_2$ and $\|x - q\| = \rho_1$ for all $q \in Q$, Q is a subset of E_1). Hence, $Q' = f(Q)$ is a ρ_1-set in Y (see Lemma 12).

By Lemma 10, there exist exactly two distinct ρ_1-associated points x'' and z'' of Q' and $\|x'' - z''\| = \rho_2$. Therefore, we have $\|f(x) - f(z)\| = 0$ or ρ_2, i.e., $\|f(y) - f(z)\| = 0$ or ρ_2 because we assumed $f(x) = f(y)$.

Since $y, z \in E_1$, by Lemma 12, we get $\rho_1 = \|y - z\| = \|f(y) - f(z)\| = 0$ or ρ_2, a contradiction. Altogether, we conclude that $\|f(x) - f(y)\| = \rho_2$. □

LEMMA 14. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the distance $\rho_3 = \sqrt{\gamma(\rho, \rho_1)} = 2\rho/n$ is contractive by $f|_{E_2}$.*

Proof. Assume that x and y are points of E_2 with $\|x-y\|=\rho_3$. By Lemma 11, there exists a ρ_1-set P in E_1 such that x and y are the ρ-associated points of P ($x \in E_2$ and $\|x-p\|=\rho$ for all $p \in P$. Hence, P is a subset of E_1). By Lemma 12, $P'=f(P)$ is also a ρ_1-set in Y.

According to Lemma 10, there exist only two distinct ρ-associated points x' and y' of P' with $\|x'-y'\|=\rho_3$. Hence, it follows that $\|f(x)-f(y)\|=0$ or ρ_3. Consequently, we have $\|f(x)-f(y)\| \le \rho_3$. □

We are now ready to generalize a classical theorem of Beckman and Quarles by proving that if a mapping, from a half space E_0 of X into Y, preserves a distance ρ, then the restriction of f to a half space E_3 is an isometry.

THEOREM 15. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the restriction $f|_{E_3}$ is an isometry. In particular, if any x, y of E_2 satisfy $x_s \neq y_s$, where x_s and y_s are the X_s-components of x and y, then it holds that $\|f(x)-f(y)\|=\|x-y\|$.*

Proof. According to Lemmas 13 and 14, the distance $2\rho/n$ is contractive and the distance $(n+1)(2\rho/n)$ is extensive (preserved) by $f|_{E_2}$. Hence, by Theorem 9, the restriction $f|_{E_3}$ is an isometry.

In view of the second part of Theorem 9, the second part of this theorem is obviously true. □

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