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THE EXISTENCE OF SEMIALGEBRAIC SLICES AND ITS APPLICATIONS

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ABSTRACT. Let G be a compact semialgebraic group and M a semialgebraic G-set. We prove that there exists a semialgebraic slice at every point of M. Moreover M can be covered by finitely many semialgebraic G-tubes. As an application we give a different proof that every semialgebraic G-set admits a semialgebraic G-embedding into some semialgebraic orthogonal representation space of G, which has been proved in [15].

1. Introduction

Let X be a topological space with an action of a topological group G. For a point $x \in X$ a *slice* of x is a subset S of X containing x such that

1. $G_x S = S$, where $G_x = \{g \in G \mid gx = x\}$, and
2. the map $\varphi\colon G \times_{G_x} S \to M$ defined by $[g, s] \mapsto gs$ is a G-embedding onto an open neighborhood GS of the orbit $G(x)$ in X.

When G is a compact Lie group and X is a completely regular G-space, the existence of a slice was studied by Gleason [6], Montgomery and Yang [9], and finally proved in the most general form by Mostow [10].

In this paper we prove the existence of a slice in the semialgebraic category. Namely we have the following theorem.

THEOREM 1.1. *Let G be a compact semialgebraic group and M a semialgebraic G-set. Then*

1. *for each $x \in M$, there is a semialgebraic G_x-slice S at x, and*
2. *M can be covered by a finite number of semialgebraic G-tubes.*

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The existence of a slice plays an important role on further developments in the theory of topological and smooth transformation groups. For instance, Mostow [10] used the existence of a slice to prove that any finite dimensional separable metric G-space with finitely many orbit types can be topologically embedded in some orthogonal representation space of G when G is a compact Lie group.

Mostow [10] and Palais [12] also proved the smooth version of the above embedding theorem independently. In the semialgebraic category Park and Suh [15] proved the semialgebraic version of the above embedding theorem, without using the existence of a semialgebraic slice, see Section 2 for the definition of semialgebraic group actions. Since Theorem 1.1 is available, now we can follow Mostow's method to give a different proof of the semialgebraic embedding theorem. Namely we reprove the following theorem.

THEOREM 1.2 ([15]). *Let G be a compact semialgebraic linear group. Then every semialgebraic G-set can be equivalently and semialgebraically embedded in some semialgebraic orthogonal representation space of G.*

Note that in Theorem 1.2 we do not assume that the G-space has finitely many orbit types. However this can be intrinsically assumed because Theorem 2.6 shows that there are only finitely many orbit types in a semialgebraic space with a semialgebraic action of a compact semialgebraic group.

A *semialgebraic space* is an object obtained by pasting finitely many semialgebraic sets together along open semialgebraic subsets. R. Robson [16, Theorem 1] proved that every semialgebraic space which is 'regular', equivalent to the old fashioned terminology if the base filed is $\mathbb{R}$, admits a semialgebraic embedding into $\mathbb{R}^n$ for some n. In other words, every regular semialgebraic space is semialgebraically homeomorphic to a semialgebraic set in $\mathbb{R}^n$ for some n. A semialgebraic space M is called *locally complete* if every $x \in M$ has a semialgebraic neighborhood which is a complete semialgebraic set. If the base field is $\mathbb{R}$, locally complete is equivalent to locally compact. Furthermore every locally complete space is regular. Thus every locally complete semialgebraic space is semialgebraically homeomorphic to a semialgebraic set in some $\mathbb{R}^n$. Moreover, H. Delf and M. Knebusch proved the following result which is stated for an arbitrary real closed field. Since we are working in the real field $\mathbb{R}$ we give the statement of the result for the real field case.

PROPOSITION 1.3 ([4, 5]). *Any locally compact semialgebraic space can be semialgebraically embedded in some* $\mathbb{R}^n$ *as a closed semialgebraic subset.*

Let G be a semialgebraic group. A *semialgebraic G-space* is a semialgebraic space M with a semialgebraic action $\theta\colon G \times M \to M$ of G.

As a corollary of Theorem 1.2, we have the following equivariant version of Proposition 1.3.

COROLLARY 1.4. *Let G be a compact semialgebraic linear group. Then every locally compact semialgebraic G-space can be equivariantly and semialgebraically embedded in some semialgebraic orthogonal representation space Ω of G as a closed semialgebraic G-subset of Ω.*

Throughout this paper the base field is the real numbers $\mathbb{R}$ and all semialgebraic maps are assumed to be continuous. In this paper we consider the semialgebraic sets in $\mathbb{R}^n$ equipped with the subspace topology induced by the usual topology.

2. Background materials

In this section we review some background materials on semialgebraic geometry which some readers may not be familiar with. For more properties in semialgebraic geometry we refer the reader to [1, 3, 4, 5, 13, 14, 15].

The class of *semialgebraic sets* in $\mathbb{R}^n$ is the smallest collection of subsets containing all $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ for each polynomial $p\colon \mathbb{R}^n \to \mathbb{R}$ which is stable under finite union, finite intersection and complement.

It follows from the definition of a semialgebraic set that a subset M of $\mathbb{R}^n$ is semialgebraic if and only if there exist polynomials f_{ij} and g_{ij} for $i = 1, \ldots, k$ and $j = 1, \ldots, l$, such that

$$M = \bigcup_{i=1}^{k} \{x \in \mathbb{R}^n \mid f_{ij}(x) > 0,\ g_{ij}(x) = 0 \text{ for all } j\}.$$

It is easy to see that the union and the intersection of a finite number of semialgebraic sets are semialgebraic and that the complement of a semialgebraic set is semialgebraic. Furthermore, the closure, and hence the interior, of a semialgebraic set is semialgebraic. In particular, the cartesian product of two semialgebraic sets is also semialgebraic.

A continuous map $f\colon M \to N$ between semialgebraic sets $M(\subset \mathbb{R}^m)$ and $N(\subset \mathbb{R}^n)$ is called a *semialgebraic map* if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. It is easy to show that the composition of two semialgebraic maps and the inverse of a semialgebraic homeomorphism are semialgebraic maps.

The following are elementary properties in semialgebraic geometry.

PROPOSITION 2.1 ([1, Proposition 2.2.7]). *Let $f\colon M \to N$ be a semialgebraic map.*

(1) *If $A(\subset M)$ is semialgebraic, then its image $f(A)$ is semialgebraic.*
(2) *If $B(\subset N)$ is semialgebraic, then its preimage $f^{-1}(B)$ is semialgebraic.*

PROPOSITION 2.2 ([13, Lemma 2.4]). *Let A, B, and C be semialgebraic sets, and let $f\colon A \to B$ and $g\colon A \to C$ be semialgebraic maps. Assume f is surjective. If $h\colon B \to C$ is a continuous map such that $h \circ f = g$, then h is a semialgebraic map.*

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g"] & \\ B \arrow[r, "h"'] & C \end{tikzcd}$$

REMARK 2.3. Let $A \subset \mathbb{R}^n$ be a non-empty semialgebraic set. For every $x \in \mathbb{R}^n$, the distance between x and A

$$\operatorname{dist}(x, A) = \inf\{\|x - y\| \mid y \in A\}$$

is well-defined. Moreover, the map $\operatorname{dist}_A\colon \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \operatorname{dist}(x, A)$, is semialgebraic and vanishes on the closure $\overline{A}$ of A (see, [1, Proposition 2.2.8]).

We now deal with the semialgebraic triangulations of semialgebraic sets. Let $a_0, \ldots, a_n$ be generically independent points of $\mathbb{R}^m$. The *n-simplex* $\langle a_0, \ldots, a_n\rangle$ spanned by $a_0, \ldots, a_n$ is defined by

$$\langle a_0, \ldots, a_n\rangle = \{\sum_{i=0}^{n} t_i a_i \in \mathbb{R}^m \mid \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0\}.$$

The *open n-simplex* $(a_0, \ldots, a_n)$ spanned by $a_0, \ldots, a_n$ is defined by

$$(a_0, \ldots, a_n) = \{\sum_{i=0}^{n} t_i a_i \in \mathbb{R}^m \mid \sum_{i=0}^{n} t_i = 1,\ t_i > 0\}.$$

Note that the open 0-simplex (a) is equal to $\langle a\rangle$ from the definition. Clearly $\langle a_0, \ldots, a_n\rangle$ and $(a_0, \ldots, a_n)$ are semialgebraic sets in $\mathbb{R}^m$.

A *finite open simplicial complex* $(K, (\sigma_i \mid i \in I))$ is defined as a subset of some $\mathbb{R}^m$ equipped with a partition $(\sigma_i \mid i \in I)$ composed of a finite number of open simplices σ_i in $\mathbb{R}^m$, such that the intersection $\bar{\sigma}_i \cap \bar{\sigma}_j$ of the closures of any two open simplices σ_i and σ_j is either empty or a common face of $\bar{\sigma}_i$ and $\bar{\sigma}_j$. Thus a finite open simplicial complex $(K, (\sigma_i))$ is obtained from some finite, hence compact, simplicial complex L by deleting some open simplices σ of L.

PROPOSITION 2.4 ([7, 8]). *Let M be a semialgebraic set and let $M_1, \ldots, M_k$ be semialgebraic subsets of M. Then there exist a finite open simplicial complex K and a semialgebraic homeomorphism $\tau \colon |K| \to M$ such that each M_i is a finite union of some of the $\tau(\sigma)$, where σ is an open simplex of K.*

The above pair (K, τ) is called a *semialgebraic open triangulation* of M *compatible with* $\{M_i\}$. Proposition 2.4 implies that every semialgebraic set has a finite number of connected components because an open simplex is connected. Moreover, every connected component of a semialgebraic set is also semialgebraic.

The definition of a semialgebraic group is similar to that of a Lie group. A semialgebraic set $G \subset \mathbb{R}^n$ is called a *semialgebraic group* if it is a topological group such that the group multiplication and the inversion

$$\mu \colon G \times G \to G, \quad (g, h) \mapsto gh$$
$$i \colon G \to G, \quad g \mapsto g^{-1}$$

are semialgebraic. A semialgebraic subgroup is obviously defined. It is easy to see that the identity component G_0, and the center $Z(G)$ of a semialgebraic group G are semialgebraic subgroups. Moreover, the normalizer $N(H)$ of a semialgebraic subgroup H of G is also a semialgebraic subgroup. For a semialgebraic homomorphism $f \colon G_1 \to G_2$ between two semialgebraic groups the image $f(H)$ (resp. the preimage $f^{-1}(K)$) is a semialgebraic subgroup of G_2 (resp. G_1) when H (resp. K) is a semialgebraic subgroup of G_1 (resp. G_2). In particular, $\ker(f)$ is a semialgebraic subgroup of G_1.

A semialgebraic set M is called a *semialgebraic G-set* if the action map $\theta \colon G \times M \to M$ of G on M is a semialgebraic map.

PROPOSITION 2.5 ([3]). *Let G be a compact semialgebraic group and M a semialgebraic G-set. Then the orbit space M/G exists as a semialgebraic set such that the orbit map $\pi \colon M \to M/G$ is semialgebraic.*

As an immediate consequence of Propositions 2.5 with 2.2, if G is a semialgebraic group and H a compact semialgebraic subgroup of G, the homogeneous space G/H is a semialgebraic G-set. On the other hand, for a semialgebraic G-set M, the orbit $G(x)$ of $x \in M$ is clearly a semialgebraic G-set. Moreover, the isotropy subgroup $G_x = \{g \in G \mid gx = x\}$ is also a closed semialgebraic subgroup of G for all $x \in M$. As in the theory of Lie group actions, the natural map

$$\alpha_x \colon G/G_x \to G(x), \quad gG_x \mapsto gx$$

is a semialgebraic G-homeomorphism (see, [15, Proposition 2.6]).

Unlike topological or smooth transformation group theory we have the following theorem in the semialgebraic category.

THEOREM 2.6. *Let G be a compact semialgebraic group. Then every semialgebraic G-set has only finitely many orbit types.*

Proof. Let M be a semialgebraic G-set. Note that the orbit space M/G and the orbit map $\pi \colon M \to M/G$ are semialgebraic by Proposition 2.5. Apply the nonequivariant local triviality to π (see [1, Theorem 9.3.2]), then we can find semialgebraic subsets $B_1, \ldots, B_k$ of M/G with $M/G = \bigcup_{i=1}^k B_i$ and semialgebraic homeomorphisms $\pi^{-1}(b_i) \times B_i \cong \pi^{-1}(B_i)$ preserving fibers for some $b_i \in B_i$. Since a semialgebraic set has a finite number of connected components, we can assume B_i are all connected without loss of generality. Now the claim is that $\pi^{-1}(B_i)$ has one orbit type for each i. We fix $B = B_i$, $b = b_i$ and first prove this claim locally.

For a given $b \in B$, choose a neighborhood W of b in B such that there is a continuous (which may not be semialgebraic) G-map $f \colon \pi^{-1}(W) \to \pi^{-1}(b)$. (The existence of such W follows from the existence of slices in the topological category [2, Theorem II.4.2].) Moreover we assume W is connected. For another point $b' \in W$ we assert that the type of b' is equal to that of b. On the contrary, suppose $\mathrm{type}(\pi^{-1}(b)) < \mathrm{type}(\pi^{-1}(b'))$. Let b_t be a path in W connecting b to b'. Let $\Psi_t \colon \pi^{-1}(b) \to \pi^{-1}(b_t)$ be the parameterized homeomorphisms occurring from the restriction of the homeomorphism $\pi^{-1}(B) \cong \pi^{-1}(b) \times B$. Let $\phi_t \colon \pi^{-1}(b) \to \pi^{-1}(b)$ denote the composition $f \circ \Psi_t$. Then ϕ_0 is the identity map and ϕ_1 is a covering map with nontrivial fibers. These two maps could not be homotopic, which is a contradiction. Thus we proved that the orbit type is constant over W.

Now for any two points of B we can connect them by a curve. Then by the compactness of the path and the consecutive application of the above argument, we can show that the two points have the same orbit type and the claim is proved over B. Since the number of B_i's is finite, the proof is complete. □

For any subgroup H of G, let $M_{(H)}$ denote the set of points having orbit type G/H, i.e.,

$$M_{(H)} = \{x \in M \mid G_x = gHg^{-1} \text{ for some } g \in G\},$$

then $M_{(H)}$ is a semialgebraic G-subset of M. In particular the set of fixed points of G on M,

$$M^G = \{x \in M \mid gx = x \text{ for all } g \in G\} = M_{(G)}$$

is a closed semialgebraic subset of M.

3. The existence of semialgebraic slices

This section is devoted to the proof of the existence of semialgebraic slices of semialgebraic G-sets. A semialgebraic slice is a semialgebraic analogue of a topological slice defined in Introduction. More precisely, it is defined as follows.

Let M be a semialgebraic G-set. For $x \in M$, the isotropy subgroup G_x of G at x is a semialgebraic subgroup of G. A *semialgebraic slice* S of x is a semialgebraic subset of M containing x such that

1. $G_x S = S$
2. the map $\varphi\colon G \times_{G_x} S \to M$ defined by $[g, s] \mapsto gs$ is a semialgebraic G-embedding onto an open semialgebraic neighborhood GS of the orbit $G(x)$ in M.

We call GS a *semialgebraic G-tubular neighborhood* of $G(x)$. The map φ is called a *semialgebraic G-tube* about $G(x)$.

LEMMA 3.1. *Let M be a semialgebraic G-set and let S be a semialgebraic subset of M containing x. Then the following statements are equivalent.*

(1) *S is a semialgebraic slice at x.*
(2) *GS is a semialgebraic G-invariant open neighborhood of $G(x)$ and there is a semialgebraic G-retraction $f\colon GS \to G(x)$ such that $f^{-1}(x) = S$.*

Proof. To show that (1) implies (2), let S be a semialgebraic slice. Define a map $h\colon G \times_{G_x} S \to G/G_x$ by $h([g, s]) = gG_x$. Then we have the following commutative diagram

$$\begin{tikzcd} G \times S \arrow[r, "\pi_1"] \arrow[d, "p_1"'] & G \times_{G_x} S \arrow[d, "h"] \\ G \arrow[r, "\pi_2"'] & G/G_x \end{tikzcd}$$

where π_1 and π_2 are semialgebraic orbit maps and p_1 is the natural projection. By Proposition 2.2, h is a surjective semialgebraic map.

Define $f\colon GS \to G(x)$ by $f(gs) = gx$. Then we have the following commutative diagram

$$\begin{tikzcd} G \times_{G_x} S \arrow[r, "\varphi", "\cong"'] \arrow[d, "h"'] & GS \arrow[d, "f"] \\ G/G_x \arrow[r, "\cong", "\alpha_x"'] & G(x). \end{tikzcd}$$

Therefore f has the desired property by Proposition 2.2.

That (2) implies (1) is clear since S is a topological slice at x (see, [2, Theorem II.4.2]). □

PROPOSITION 3.2. *If S is a semialgebraic slice at x in a semialgebraic G-set M, then the natural map*

$$\kappa\colon S/G_x \to M/G, \quad ([s] \mapsto [s])$$

is a semialgebraic homeomorphism onto the open subset GS/G.

Proof. By the corresponding fact in topological group theory (see, [2, Proposition II.4.7]), κ is a homeomorphism. So it is enough to show that κ is semialgebraic. This is clear by Proposition 2.2 with the following commutative diagram

$$\begin{tikzcd} G \times S \arrow[d, "p_2"'] \arrow[dr, "\theta"] & \\ S \arrow[r, "i"] \arrow[d, "\pi_S"'] & GS \arrow[d, "\pi"] \\ G/G_x \arrow[r, "\kappa"'] & GS/G, \end{tikzcd}$$

where π_S and π are semialgebraic orbit maps, p_2 is the natural projection, i is the inclusion and θ is the action map. Note that π and κ are surjective. □

To construct a semialgebraic slice, we need two technical lemmas concerning nonequivariant semialgebraic sets. Recall that a function $f\colon M \to \mathbb{R}$ is lower (resp. upper) semi-continuous if the set $\{x \in M \mid f(x) > a\}$ (resp. $\{x \in M \mid f(x) < a\}$) is open for all $a \in \mathbb{R}$.

LEMMA 3.3. *Let M be a semialgebraic set, and let $f\colon M \to \mathbb{R}$ (resp. $g\colon M \to \mathbb{R}$) be a lower (resp. upper) semi-continuous semialgebraic function such that $g(x) < f(x)$ for all $x \in M$. Then there exists a continuous semialgebraic function $h\colon M \to \mathbb{R}$ such that $g(x) < h(x) < f(x)$ for all $x \in M$.*

Proof. We can choose a semialgebraic open triangulation (K, τ) of M such that both f and g are continuous over the interior of each simplex of K. Indeed, we can see this as follows. By the local triviality theorem for semialgebraic maps ([1, Theorem 9.3.2]), there exists a finite cover $\{M_i^f\}$ of semialgebraic subsets of M such that for the projection $p\colon G(f) = \{(x, y) \in M \times \mathbb{R} \mid y = f(x)\} \to M$, $(x, y) \mapsto x$, the inverse image $p^{-1}(M_i^f)$ is semialgebraically homeomorphic to $M_i^f \times p^{-1}(b_i)$ for some $b_i \in M_i^f$. (Here, we applied the local triviality theorem to the map $p\colon G(f) \to M$ instead of f to achieve the continuity of the map p which is needed in the theorem.) Let $p^{-1}(b_i) = \{y_i\}$. Then clearly f is continuous on each M_i^f since $f(x) = \pi_2 \circ \Psi^{-1}(x, y_i)$ where $\Psi\colon p^{-1}(M_i^f) \to M_i^f \times p^{-1}(b_i)$ is the semialgebraic homeomorphism obtained by the local triviality theorem and $\pi_2\colon G(f) \to \mathbb{R}$ is the projection. Similarly we can find such $\{M_j^g\}$ corresponding to the function g. If we choose a semialgebraic open triangulation (K, τ) of M compatible with the finite collection $\{M_i^f\} \cup \{M_j^g\}$, then we are done.

We construct the desired function $h\colon M \to \mathbb{R}$ by the induction on the skeleta of K. Assume that a continuous function $h_1\colon K^{(n-1)} \to \mathbb{R}$ is defined such that $g(x) < h_1(x) < f(x)$ for all $x \in |K^{(n-1)}|$. Let δ be an n-dimensional simplex of K which is closed in K. (Since we are working with an open simplicial complex, δ may not be compact.) Then the proof is reduced to the construction of $h\colon \delta \to \mathbb{R}$ satisfying the inequality condition and extending $h_1\colon \partial\delta \to \mathbb{R}$ where $\partial\delta$ means the boundary of δ.

First, let us construct a lower semi-continuous semialgebraic function $f'\colon \delta \to \mathbb{R}$ and an upper semi-continuous semialgebraic function $g'\colon \delta \to \mathbb{R}$ so that $g(x) < g'(x) < f'(x) < f(x)$ for all $x \in \mathring{\delta}$ where $\mathring{\delta} = \delta - \partial\delta$ is the interior of δ. For this aim, let $\alpha\colon \mathring{\delta} \to (0, \infty)$ be a continuous semialgebraic function such that it vanishes when it approaches to the boundary of δ, e.g., $\alpha(x) = \mathrm{dist}(x, \partial\delta)$ for $x \in \mathring{\delta}$. Define $\alpha' \colon \mathring{\delta} \to [0, \infty)$ by $\alpha'(x) = \min(\alpha(x), (f-g)(x)/3)$ for $x \in \mathring{\delta}$, then α' is continuous in the interior of δ and vanishes when it approaches the boundary of δ. Thus it extends continuously over δ. Define $f', g' \colon \delta \to \mathbb{R}$ by $f'(x) = f(x) - \alpha'(x)$ and $g'(x) = g(x) + \alpha'(x)$ for $x \in \delta$.

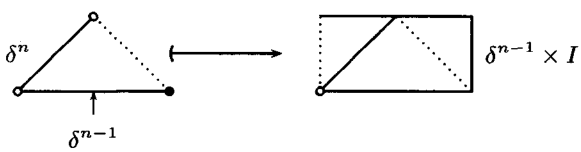
Finally let us construct $h\colon \delta \to \mathbb{R}$ such that $g'(x) \le h(x) \le f'(x)$ for $x \in \mathring{\delta}$ and that h extends h_1; then, it follows that $g(x) < g'(x) \le h(x) \le h'(x) < f(x)$ for $x \in \mathring{\delta}$ and $g(x) < h(x) = h_1(x) < f(x)$ for $x \in \partial\delta$, and hence the proof will be complete.

By the semialgebraic version of the Tietze extension theorem ([4, Theorem 3]), the continuous semialgebraic function $h_1 \colon \partial\delta \to \mathbb{R}$ has a continuous semialgebraic extension $h_2 \colon \delta \to \mathbb{R}$. We now modify h_2 so that it satisfies the inequality over all δ. Let $\tilde{f} = \min(f', h_2)$. Since both f' and h_2 are continuous on $\mathring{\delta}$, automatically $\tilde{f}$ is continuous on $\mathring{\delta}$. On the other hand $\tilde{f}$ is continuous on $\partial\delta$: since f' is lower semi-continuous and $h_2(x) < f'(x)$ on $\partial\delta$, it follows that $\tilde{f}(x) = h_2(x)$ on a neighborhood of $\partial\delta$. So both facts imply that $\tilde{f}$ is continuous all over δ.

Now let $h = \max(g', \tilde{f})$, then h is continuous on δ by the same reason for $\tilde{f} = \min(f', h_2)$. It is obvious that $g'(x) \le h(x) \le f'(x)$ for all $x \in \delta$ from its definition, which completes the proof. □

LEMMA 3.4. *Let (δ^n, δ^{n-1}) be a pair of simplices in an open simplicial complex K such that δ^{n-1} is a face of δ^n. Let U be a given semialgebraic open neighborhood of δ^{n-1} in δ^n. Then there is a semialgebraic closed neighborhood $N \subset U$ of δ^{n-1} and a semialgebraic retraction $\gamma\colon \delta^n \to N$ such that $\gamma(\delta^n - \delta^{n-1}) \subset N - \delta^{n-1}$.*

Proof. If δ^n and δ^{n-1} are closed simplices, then the proof might be easier, but note that they need not be closed simplices in our case. Embed (δ^n, δ^{n-1}) in $(\delta^{n-1} \times I, \delta^{n-1})$ as in the following figure.



Let V be a semialgebraic open neighborhood of δ^{n-1} in $\delta^{n-1} \times I$ such that $V \cap \delta^n = U$. Let $f\colon \delta^{n-1} \to \mathbb{R}$ be a semialgebraic function defined by

$$f(x) = \sup\{r \in \mathbb{R} \mid x \times [0, r) \subset V\}.$$

Then f is lower semi-continuous. (For any $x \in \delta^{n-1}$ such that $f(x) > a$, let $b = f(x) - a$. Then $x \times [0, a + b/2] \subset V$. Since the interval $[0, a + b/2]$ is compact, we can find a neighborhood $A \subset \delta^{n-1}$ such that $A \times [0, a + b/2] \subset V$. Thus $f(y) \geq a + b/2 > a$ whenever $y \in A$.)

We now apply Lemma 3.3 with the above f and with $g = 0$, to get a continuous semialgebraic function $h\colon \delta^{n-1} \to \mathbb{R}$ such that $0 < h(x) < f(x)$ for all $x \in \delta^{n-1}$. Let $N' = \{(x, s) \mid 0 \leq s \leq h(x),\ x \in \delta^{n-1}\}$. Let $\gamma'\colon \delta^{n-1} \times I \to N'$ defined by

$$\gamma'(x,t) = \begin{cases} (x, h(x)), & t \geq h(x) \\ (x, t), & t \leq h(x). \end{cases}$$

Now let $N = N' \cap \delta^n$ and let $\gamma = \gamma'|_{\delta^n}$. Then it is easy to see that γ and N satisfy the desired properties. $\square$

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let x_0 be a given point in M. By Theorem 2.6 M has finitely many orbit types, say $\operatorname{type}(G/H_1), \ldots, \operatorname{type}(G/H_k)$. Consider the $N(H_i)$-subspace $M^{H_i} \subset M$ for each $i = 1, \ldots, k$. The orbit map $\pi\colon M \to M/G$ restricts to

$$\pi_{H_i}\colon M^{H_i} \to M^{H_i}/N(H_i).$$

We apply semialgebraic trivialization to each π_i, we get B_{j_i} such that $\cup B_{j_i} = M^{H_i}/N(H_i)$ and semialgebraic cross sections $s_{j_i}\colon B_{j_i} \to M^{H_i}$. We simply write the index j_i just as i. Now we choose a semialgebraic triangulation (K, τ) of $B = M/G$ compatible with B_i. We replace K by its barycentric subdivision to contain $\pi(x_0)$ as a 0-simplex. Then

(A) every $\delta \in K$ contains a 0-simplex, and

(B) the interior $\overset{\circ}{\delta}$ of δ has a continuous semialgebraic cross section $s\colon \overset{\circ}{\delta} \to M$ of $\pi\colon M \to M/G$ such that $s(\overset{\circ}{\delta})$ has a constant stabilizer.

Let $\{v_0 = \pi(x_0), v_1, \ldots, v_l\}$ be the set of all vertices of K. By the property (A), the set $\{\operatorname{St}(v_i) \mid i = 0, \ldots, l\}$ of the (open) star neighborhoods covers $|K| = M/G$, and thus $\pi^{-1}(\operatorname{St}(v_0)), \ldots, \pi^{-1}(\operatorname{St}(v_l))$ cover M. We claim that $\pi^{-1}(\operatorname{St}(v))$ is a semialgebraic G-tube of the orbit $\pi^{-1}(v)$ for $v = v_0, \ldots, v_l$.

By Lemma 3.1, showing that $\pi^{-1}(\operatorname{St}(v))$ is a semialgebraic G-tube is equivalent to constructing a semialgebraic G-retraction $f\colon \pi^{-1}(\operatorname{St}(v)) \to$

$\pi^{-1}(v)$. We shall construct a semialgebraic G-retraction

$$f_{n-1}\colon \pi^{-1}(\mathrm{St}(v)^{(n-1)}) \to \pi^{-1}(\mathrm{St}(v)^{(n-2)}),$$

for each n by using the induction on n. Then the composition $f = f_0 \circ f_1 \circ \cdots$ will be the desired G-retraction.

Let δ be an n-simplex (closed in K) of K containing v as a vertex and let $\delta^n = \delta \cap \mathrm{St}(v)^{(n)}$. Since each n-simplex of $\mathrm{St}(v)$ is of the form δ^n we restrict our attention to constructing a semialgebraic G-retraction

$$f_n\colon X = \pi^{-1}(\delta^n) \to \pi^{-1}(\partial\delta^n) = \partial X$$

where $\partial\delta^n = \delta \cap \mathrm{St}(v)^{(n-1)}$. Here, the symbol ∂ does not mean the boundary but we use it just for notational convenience.

Since $\delta^n - \partial\delta^n = \mathring{\delta}$, by the property (B), there is a semialgebraic cross section $s\colon \delta^n - \partial\delta^n \to X^H \subset X$ where H is an $H = H_i$ for some i occurring in the orbit types. Let Y be the closure of $s(\delta^n - \partial\delta^n)$ in X, and we simply denote $\partial Y = Y \cap \partial X$.

We now claim that there exists a semialgebraic retraction

$$\tilde{r}\colon Y \to \partial Y.$$

Let $\widetilde{U}$ be a semialgebraic regular neighborhood of ∂Y in Y. Since G is compact and so π is a closed map, the set $U = \pi(\widetilde{U})$ is again a semialgebraic neighborhood of $\partial\delta^n$. By a suitable semialgebraic homeomorphism, the pair $(\delta^n, \partial\delta^n)$ is homeomorphic to a pair composed of a simplex and one of its faces (δ, δ^{n-1}) with a neighborhood which is homeomorphic to U. By Lemma 3.4 there is a semialgebraic closed neighborhood $N \subset U$ and a retraction $r\colon \delta \to N$ such that $r(\delta - \delta^{n-1}) \subset N - \delta^{n-1}$. We lift this retraction to a map $r'\colon Y \to \widetilde{U}$, more precisely define by

$$r'(x) = \begin{cases} s \circ r \circ \pi(x), & x \in Y - \partial Y \\ x, & x \in \partial Y. \end{cases}$$

Since the regular neighborhood $\widetilde{U}$ has a retraction to ∂Y, the composition of this map followed by r' gives a retraction $\tilde{r}\colon Y \to \partial Y$. This proves the claim.

Now the rest is routine. From the retraction $\tilde{r}\colon Y \to \partial Y$ we define a semialgebraic G-retraction $f\colon X = GY \to G(\partial Y) = \partial X$ by $f(gx) = g\tilde{r}(x)$ which was needed to complete the proof. $\square$

4. Semialgebraic G-representation spaces

In this section we discuss some properties of semialgebraic representations of semialgebraic groups. A semialgebraic group is called a *semialgebraic linear group* if it is semialgebraically isomorphic to a semialgebraic subgroup of some general linear group $\mathrm{GL}_n(\mathbb{R})$. Note that $\mathrm{GL}_n(\mathbb{R})$ is a semialgebraic set in $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, where $\mathrm{M}_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices over $\mathbb{R}$. A *semialgebraic representation* of G is, by definition, a semialgebraic homomorphism $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ for some n. By a semialgebraic orthogonal representation of G we mean a semialgebraic homomorphism $\rho\colon G \to \mathrm{O}(n)$ where $\mathrm{O}(n)$ is the orthogonal group. In this case $\mathbb{R}^n$ equipped with the linear action of G via ρ is denoted by $\mathbb{R}^n(\rho)$ and called a *semialgebraic orthogonal representation space* of G.

PROPOSITION 4.1 ([15, Lemma 2.2]). *Every compact subgroup H of a semialgebraic linear group G is a semialgebraic subgroup of G.*

PROPOSITION 4.2 ([15, Corollary 2.3]). *Let $f\colon G \to H$ be a topological group homomorphism between two semialgebraic linear groups G and H. If G is compact, then f is semialgebraic.*

It is well known that, for a closed subgroup H of a compact Lie group G, there exist a representation $\rho\colon G \to \mathrm{O}(n)$ and a point $v \in \mathbb{R}^n(\rho)$ such that $G_v = H$. We have the semialgebraic analogue of this fact as follows.

PROPOSITION 4.3 ([15, Proposition 2.4]). *Let G be a compact semialgebraic linear group and H a closed (semialgebraic) subgroup of G. Then there exist a semialgebraic faithful representation $\rho\colon G \to \mathrm{O}(n)$ for some n, and a point $u(\neq 0)$ of $\mathbb{R}^n(\rho)$ such that $G_u = H$.*

PROPOSITION 4.4. *Let G be a compact semialgebraic linear group and H a closed (semialgebraic) subgroup of G. If Ω is a semialgebraic orthogonal H-representation space then there is a semialgebraic orthogonal G-representation space Ξ which, considered as an H-space by restriction, has Ω as an H-invariant linear subspace.*

Proof. It is a consequence of Proposition 4.2 together with the corresponding facts in topological group theory (see, [12, Proposition 1.4.2]). $\square$

Let X be a semialgebraic G-set and H a closed semialgebraic subgroup of G. A semialgebraic subset S of X is called a *semialgebraic H-kernel* if there exists a semialgebraic G-map $f\colon GS \to G/H$ such that $f^{-1}(eH) = S$. Note that a kernel is a more generalized notion of slice, in particular, every slice at x is a G_x-kernel.

COROLLARY 4.5. *Let G be a compact semialgebraic linear group and H a closed (semialgebraic) subgroup of G. If Ω is a semialgebraic orthogonal H-representation space then there exists a semialgebraic H-embedding of Ω onto a semialgebraic H-kernel in some orthogonal semialgebraic G-representation space Ξ.*

Proof. By Proposition 4.3, there exist a semialgebraic orthogonal G representation space Ξ' and a point $u_0(\neq 0)$ of Ξ' such that $G_{u_0} = H$. By Proposition 4.4, there is a semialgebraic orthogonal G-representation space Ω' which includes Ω as an H-invariant linear subspace. Set $\Xi = \Xi' \oplus \Omega'$. Then Ξ is a semialgebraic orthogonal G-representation space. Clearly $\varphi\colon \Omega \hookrightarrow \Xi = \Xi' \oplus \Omega'$, $v \mapsto (u_0, v)$, is a semialgebraic H-embedding. The image $S = \varphi(\Omega)$ is an H-invariant closed semialgebraic subset of Ξ. Moreover if $g \notin H$ and $(u_0, v) \in S$ then $g(u_0, v) = (gu_0, gv) \notin S$ because $g \notin H = G_{u_0}$. Consider the semialgebraic G-map $f\colon GS \to G/H$, $gs \mapsto gH$. Then $f^{-1}(eH) = S$. Hence S is a semialgebraic H-kernel. □

LEMMA 4.6. *Let Ω be a semialgebraic orthogonal representation space of G. Then there exists a semialgebraic embedding $\varphi\colon \Omega \to \Omega \oplus \mathbb{R}$ such that $\|\varphi(x)\| = 1$ for all $x \in \Omega$ where $\mathbb{R}$ denotes the real one dimensional trivial representation space of G.*

Proof. Let v be a non-zero real number. Clearly the map $\psi\colon \Omega \to \Omega \oplus \mathbb{R}$ defined by $\psi(x) = (x, v)$ is a semialgebraic G-embedding. Define $\varphi\colon \Omega \to \Omega \oplus \mathbb{R}$ by $\varphi(x) = \psi(x)/\|\psi(x)\|$, then φ is the desired semialgebraic G-embedding. □

5. Equivariant semialgebraic embeddings

In this section we prove the semialgebraic version of the equivariant embedding theorem. Since many things of the proofs in this section are similar to that of [12], here we just sketch them and refer the reader to the cited paper for more detail.

LEMMA 5.1. *Let G be a compact semialgebraic group and M a semialgebraic G-set. If $M - M^G$ admits an equivariant semialgebraic embedding in some orthogonal semialgebraic representation space of G then so does M.*

Sketch of the proof. Let M be a semialgebraic subset of $\mathbb{R}^n$ and M/G a semialgebraic subset of $\mathbb{R}^k$. We define a semialgebraic map $h\colon M/G \to \mathbb{R}$ by $h(z) = \operatorname{dist}(z, M^G/G) = \inf\{\|z - y\| \mid y \in M^G/G\}$. Then the composition map $\tilde{h} = h \circ \pi \colon M \to \mathbb{R}$ is semialgebraically G-invariant. Let $f \colon M - M^G \to \Omega$ be a semialgebraic G-embedding. By Lemma 4.6 we can assume $\|f(x)\| = 1$ for all $x \in M - M^G$. Define $\tilde{f} \colon M \to \Omega$ by

$$\tilde{f}(x) = \begin{cases} \tilde{h}(x)f(x) & \text{if } x \in M - M^G \\ 0 & \text{if } x \in M^G. \end{cases}$$

Then $\tilde{f}$ is clearly a semialgebraic G-map. Now we define $\phi \colon M \to \mathbb{R}^k \oplus \Omega$ by $\phi(x) = (\pi(x), \tilde{f}(x))$ where $\mathbb{R}^k$ denotes k-dimensional trivial real G-representation space. Then ϕ can be shown to be continuous (see [12, p.22]). Hence ϕ is a semialgebraic G-embedding. □

PROPOSITION 5.2. *Let G be a compact semialgebraic group and M a semialgebraic G-set. Let $U_1, \dots, U_k$ be a covering of M by open semialgebraic G-subsets of M. If each U_i admits a semialgebraic G-embedding in a semialgebraic orthogonal G-representation space Ω_i then so does M.*

Sketch of the proof. Let $\pi \colon M \to M/G$ be the semialgebraic orbit map. Let $U_i^* = \pi(U_i)$ and let $h_1^*, \dots, h_k^* \colon M/G \to [0,1]$ be a semialgebraic partition of unity subordinate to $U_1^*, \dots, U_k^*$ (see [4, Theorem 1.6]). Define a semialgebraic G-invariant map $h_i \colon M \to [0,1]$ by $h_i = h_i^* \circ \pi$.

Let $\phi_i \colon U_i \to \Omega_i$ be a semialgebraic G-embedding. Now we define a semialgebraic G-map $\varphi_i \colon M \to \Omega_i$ by

$$\varphi_i(x) = \begin{cases} h_i(x)\phi_i(x) & \text{if } x \in U_i \\ 0 & \text{if } x \notin U_i. \end{cases}$$

Let $\mathbb{R}^k$ denote k-dimensional trivial real G-representation space. Then the map $\varphi_0 \colon M \to \mathbb{R}^k$ defined by $\varphi_0(x) = (h_1(x), \dots, h_k(x))$ is a semialgebraic G-invariant map. The map

$$\varphi \colon M \to \mathbb{R}^k \oplus \Omega_1 \cdots \oplus \Omega_k, \; x \mapsto (\varphi_0(x), \varphi_1(x), \dots, \varphi_k(x))$$

is a G-embedding (see [12] for the detail). Hence φ is a desired semialgebraic G-embedding. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. By the induction argument, we can assume that the theorem is true for all proper closed (semialgebraic) subgroups of G. By Lemma 5.1 it suffices to show that the semialgebraic G-set $M - M^G$ admits a semialgebraic G-embedding in a semialgebraic orthogonal G-representation space. By Theorem 1.1 there are finite number of semialgebraic H_i-slices $S_1, \dots, S_k$ of $M - M^G$ such that $GS_1, \dots, GS_k$

cover $M - M^G$. Since each H_i is a proper subgroup of G, by the induction hypothesis, there is a semialgebraic H_i-embedding $\varphi_i \colon S_i \to \Omega_i$ in a semialgebraic orthogonal H_i-representation space Ω_i. By Corollary 4.5, there exists a semialgebraic H_i-embedding ψ_i of Ω_i onto a semialgebraic H_i-kernel in some semialgebraic orthogonal G-representation space Ξ_i. Then the map $f_i \colon GS_i \to \Xi_i$, defined by $f_i(gs) = g\psi_i(\varphi_i(s))$, is a semialgebraic G-embedding. Since each GS_i is an open semialgebraic subset in M, by Proposition 5.2, $M - M^G$ admits a semialgebraic G-embedding in a semialgebraic orthogonal G-representation space. $\square$

REMARK 5.3. The linearity condition of the semialgebraic group is necessary as well as sufficient. Let G be a compact semialgebraic group and let M be equal to G viewed as a semialgebraic G-set with left multiplication. If M has a semialgebraic embedding $f \colon M \to \mathbb{R}^n(\rho)$ for some semialgebraic representation space $\mathbb{R}^n(\rho)$ of G, it follows that G acts effectively on $\mathbb{R}^n(\rho)$, i.e. ρ is faithful, so that G is a semialgebraic linear group. Moreover there is a compact semialgebraic group which is not linear (see, [15]).

REMARK 5.4. Let G be a compact semialgebraic linear group. Let M be a regular semialgebraic G-space. By Theorem 1 of [16], there exist a semialgebraic set $N \subset \mathbb{R}^k$ and a semialgebraic homeomorphism $f \colon M \to N$. Then f induces a semialgebraic action of G on N, so that $f \colon M \to N$ is a semialgebraic G-homeomorphism. Since N is a semialgebraic G-set, there exists a semialgebraic G-embedding φ into some semialgebraic orthogonal representation space of G by Theorem 1.2. Hence every regular semialgebraic G-space can be equivariantly and semialgebraically embedded in some semialgebraic orthogonal representation space of G by $\varphi \circ f$.

We now prove Corollary 1.4.

Proof of Corollary 1.4. Let M be a locally compact semialgebraic G-space. By Remark 5.4, we can view M as a semialgebraic G-subset of a semialgebraic orthogonal G-representation space Ω.

Set $A = \overline{M} - M$ where $\overline{M}$ is the closure of M in Ω. Since M is locally compact, A is a closed semialgebraic subset of Ω (see [4, Proposition 3.3]). We may assume $A \neq \varnothing$ unless M is already closed. The map $f \colon \Omega \to \mathbb{R}$, defined by, $f(x) = \operatorname{dist}(x, A)$, is semialgebraic, and G-invariant. Define a semialgebraic embedding $\varphi \colon M \to \Omega \oplus \mathbb{R}$ by $\varphi(x) = (x, 1/f(x))$. Since Ω is an orthogonal G-representation, φ is a G-map. Clearly, the image of φ is the closed semialgebraic set defined by

$$\{(x,y) \in \Omega \oplus \mathbb{R} \mid x \in \overline{M},\ yf(x) = 1\}.$$

□

COROLLARY 5.5. *Let G be a compact semialgebraic linear group. Then every semialgebraic G-manifold can be equivariantly and semialgebraically embedded in some semialgebraic orthogonal representation space Ω of G as a closed semialgebraic G-subset of Ω.*

Proof. Immediate from Corollary 1.4 since every semialgebraic manifold is locally compact. □

COROLLARY 5.6. *Let G be a compact semialgebraic linear group. If M is a locally compact semialgebraic G-set, then there exists a semialgebraic one point G-compactification of M.*

Proof. By Corollary 1.4, we may assume that M is a closed semialgebraic G-subset of some orthogonal semialgebraic representation space Ω of G. We may assume that $0 \notin M$ because otherwise we can replace M by $M \times \{1\} \subset \Omega \oplus \mathbb{R}$. Let $\vartheta\colon \Omega - \{0\} \to \Omega - \{0\}$ be the inversion through the unit sphere, $\vartheta(x) = x/\|x\|^2$. Clearly ϑ is a semialgebraic homeomorphism, and thus $\vartheta(M) \cup \{0\}$ is a semialgebraic set in Ω. Form this we can see that $\vartheta(M) \cup \{0\}$ is the desired compact semialgebraic G-set. □

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