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ON OPERATORS WITH AN ABSOLUTE VALUE CONDITION

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ABSTRACT. Let $\mathfrak{A}$ denote the class of bounded linear Hilbert space operators with the property that $|A^2| \geq |A|^2$. In this paper we show that $\mathfrak{A}$-operators are finitely ascensive and that, for non-zero operators A and B, $A \otimes B$ is in $\mathfrak{A}$ if and only if A and B are in $\mathfrak{A}$. Also, it is shown that if A is an operator such that $p(A)$ is in $\mathfrak{A}$ for a non-trivial polynomial p, then Weyl's theorem holds for $f(A)$, where f is a function analytic on an open neighborhood of the spectrum of A.

1. Introduction

Let H be a Hilbert space, and let $\mathcal{B}(H)$ denote the algebra of bounded linear operators on H. Recall ([1]) that an operator A is *p-hyponormal*, $0 < p \leq 1$, if $|A^*|^{2p} \leq |A|^{2p}$. Evidently, 1-hyponormality is hyponormality. Let $\mathbf{H}(p)$ denote the class of p-hyponormal operators. $\mathbf{H}(\frac{1}{2})$ operators were first introduced by Xia (see [29]). The class of $\mathbf{H}$(p) operators, though strictly larger than the class of hyponormal operators ([5], [9], [29]), shares a large number of properties with hyponormal operators (see [1], [5], [7], [8]). We say that an operator $A \in \mathcal{B}(H)$ is *paranormal* if A satisfies the norm condition $||A^2x|| ||x|| \geq ||Ax||^2$ for all $x \in H$. An operator $A \in \mathcal{B}(H)$ is said to be *normaloid* if $||A|| = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. It is well known that a p-hyponormal operator A is paranormal and that a paranormal operator is normaloid.

Recently, Furuta-Ito-Yamazaki ([10]) have defined the following very interesting class of Hilbert space operators.

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DEFINITION 1-1. The operator $A \in \mathcal{B}(H)$ is said to belong to *Class* A if A satisfies an absolute value condition $|A^2| \geq |A|^2$.

In the following we denote "Class A" by simply $\mathfrak{A}$. In [10], it is shown that $\mathfrak{A}$ stands in the middle of classes of p-hyponormal and paranormal operators. More explicitly, we have the following inclusions:

$$\begin{aligned}\{\text{hyponormal operators}\} &\subsetneq \{\text{p-hyponormal operators}\}\\ &\subsetneq \{\ \mathfrak{A} - \text{operators}\}\\ &\subsetneq \{\text{paranormal operators}\}\\ &\subsetneq \{\text{normaloid operators}\}.\end{aligned}$$

It is well known that all of these inclusions may be proper (for details, see [9]). Ito ([16]) has shown that there are some parallelisms between absolute value conditions of $\mathfrak{A}$-operators and norm conditions of paranormal operators. Uchiyama ([26]) proved basic properties of $\mathfrak{A}$-operators and that Weyl's theorem holds for $\mathfrak{A}$-operators.

Recall ([17], [18]) that the operator $A \in \mathcal{B}(H)$ is said to be *finitely ascensive* if for every $\lambda \in \mathbb{C}$ there is a $p \in \mathbb{N}$ such that

$$\ker(A-\lambda)^p = \ker(A-\lambda)^{p+1}.$$

The class of finitely ascensive operators is considerably large and plays a significant role in the study of local spectral theory (see [18], [20]). In section 2 we study basic properties of $\mathfrak{A}$-operators, which would make more explicit the relationship between the theory of $\mathfrak{A}$-operators and of paranormal operators. In particular, we prove that $\mathfrak{A}$-operators are finitely ascensive.

Given non-zero $A, B \in \mathcal{B}(H)$, let $A \otimes B$ denote the tensor product on the product space $H \otimes H$. The operation of taking tensor products $A \otimes B$ preserves many properties of $A, B \in \mathcal{B}(H)$, but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products, the spectraloid property is not (see [24, pp. 623 and 631]); again, whereas $A \otimes B$ is normal if and only if A and B are ([14], [25]), there exist paranormal operators A and B such that $A \otimes B$ is not paranormal ([2]). In section 3, for non-zero $A, B \in \mathcal{B}(H)$ it is shown that $A \otimes B \in \mathfrak{A}$ if and only if $A, B \in \mathfrak{A}$, which extends an analogous result on p-hyponormal operators in [7].

Recall ([12]) that an operator $A \in \mathcal{B}(H)$ is called *Fredholm* if it has closed range and finite dimensional null space and its range is of finite co-dimension. The *index* of a Fredholm operator $A \in \mathcal{B}(H)$ is given by

$$\operatorname{ind}(A) = \dim \ker(A) - \dim \ker(A^*).$$

An operator $A \in \mathcal{B}(H)$ is called *Weyl* if it is Fredholm of index zero. The *Weyl spectrum* $\omega(A)$ of A is defined by

$$\omega(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}.$$

We write $\pi_0(A)$ for the set of eigenvalues of A and $\pi_{00}(A)$ for the isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity. We say that *Weyl's theorem holds for* $A \in \mathcal{B}(H)$ if there is the equality

$$\sigma(A) \setminus \omega(A) = \pi_{00}(A).$$

DEFINITION 1-2. An operator $A \in \mathcal{B}(H)$ is said to be a *polynomially* $\mathfrak{A}$*-operator* if $p(A)$ is in $\mathfrak{A}$ with a non-trivial polynomial p.

In section 4, we show that Weyl's theorem holds for $f(A)$ whenever A is a polynomially $\mathfrak{A}$-operator and f is an analytic function on an open neighborhood of $\sigma(A)$, which completely extends earlier results in [8] and [11] through slightly different approaches.

2. Basic properties of $\mathfrak{A}$-operators

First, we recall that a paranormal operator is normaloid ([15]), that a compact paranormal operator is normal ([15, Theorem 2] or [23]), and that scalar perturbations of paranormal operators are not paranormal as noted in [1, pp.174–175]. But as the case of hyponormal operator, if $A \in \mathcal{B}(H)$ is paranormal and $A - \lambda$ for any $\lambda \in \mathbb{C}$ is quasinilpotent, then $A = \lambda I$. Also, if $A \in \mathcal{B}(H)$ is paranormal, $\lambda \in \operatorname{iso}\sigma(A)$ and E_λ is the Riesz projection corresponding to λ, then $\operatorname{ran} E_\lambda = \ker(A - \lambda)$ ([6] or [27]), which implies A is isoloid (i.e., $\operatorname{iso}\sigma(A) \subseteq \pi_0(A)$). Furthermore, if $\lambda \neq 0$ then E_λ is self-adjoint and $\ker(A - \lambda) = \ker(A - \lambda)^*$ ([27]).

$\mathfrak{A}$-operators share these properties with paranormal operators and have the following result.

LEMMA 2-1. ([26]) *The following holds:*

(i) *If* $A \in \mathfrak{A}$, *then the restriction* $A|_{\mathcal{M}}$ *to its invariant subspace* $\mathcal{M}$ *is also in* $\mathfrak{A}$.
(ii) *If* $A \in \mathfrak{A}$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$, *then* $(A - \lambda)x = 0$ *implies that* $(A - \lambda)^* x = 0$.

REMARK 2-2. In the case of paranormal operators, the corresponding result of Lemma 2-1(i) follows immediately by its definition but that of Lemma 2-1(ii) does not. In fact, there is a counterexample given by A. Uchiyama ([28]). It looks very interesting and valuable.

The following result says that $\mathfrak{A}$-operators are finitely ascensive.

THEOREM 2-3. *Let $A \in \mathfrak{A}$. Then*

$$\ker(A-\lambda) = \ker(A-\lambda)^2 \text{ for all } \lambda \in \mathbb{C}. \tag{2.3.1}$$

Proof. First, let $\lambda = 0$; if $x \neq 0 \in \ker A^2$, then we have

$$\begin{aligned} 0 = ||A^2x||||x|| &= |||A^2|x||||x|| \\ &\geq \langle |A^2|x, x\rangle \geq \langle |A|^2x, x\rangle \\ &= |||A|x||^2 = ||Ax||^2. \end{aligned} \tag{2.3.2}$$

Second, let $\lambda \neq 0 \in \mathbb{C}$; if $x \neq 0 \in \ker(A-\lambda)^2$, then by Lemma 2-1(ii) we have $(A-\lambda)x \in \ker(A-\lambda)^*$. Thus

$$\begin{aligned} 0 &= ||(A-\lambda)^*(A-\lambda)x||||x|| \\ &\geq \langle (A-\lambda)^*(A-\lambda)x, x\rangle \\ &= ||(A-\lambda)x||^2. \end{aligned} \tag{2.3.3}$$

Since (2.3.2) and (2.3.3) imply $\ker(A-\lambda)^2 \subseteq \ker(A-\lambda)$ for all $\lambda \in \mathbb{C}$ and $\ker(A-\lambda) \subseteq \ker(A-\lambda)^2$ in general, this completes the proof. □

If $A \in \mathcal{B}(H)$ and F is a closed set in $\mathbb{C}$, we define

$$H_A(F) = \{x \in H : \text{there exists an analytic } H\text{-valued function } f : \mathbb{C} \setminus F \longrightarrow H \text{ such that } (A-\lambda)f(\lambda) = x\}.$$

$H_A(F)$ is said to be a *spectral manifold* of A. If A has the single valued extension property, then the above definition is identical with $H_A(F) = \{x \in H : \sigma_A(x) \subseteq F\}$, where $\sigma_A(x)$ is the local spectrum of A at x (see [20] for details).

COROLLARY 2-4. *Let $A \in \mathfrak{A}$ and $\lambda \in iso\sigma(A)$. Then A has the single valued extension property and*

$$\operatorname{ran} E_\lambda = \ker(A-\lambda) = H_A(\{\lambda\}), \tag{2.3.4}$$

where E_λ is the Riesz projection corresponding to λ.

Proof. Since A is finitely ascensive, [18, Proposition 1.8] implies that A has the single valued extension property. Combining [18, Corollary 2.4] and [27, Theorem 3.7] we easily have (2.3.4), and hence the proof is complete. □

REMARK 2-5. Proofs of Theorem 2-3 and Corollary 2-4 are thoroughly dependent on Lemma 2-1(ii). So we may notice it is impossible to get analogous results for paranormal operators. Actually, A. Uchiyama's example ([28]) shows that (2.3.1) generally is not true for paranormal operators.

3. Tensor products of $\mathfrak{A}$-operators

In this section we completely extend earlier results on tensor products of p-paranormal operators in [7]. We start with

LEMMA 3-1. ([25, Proposition 2.2]) *Let $A_i, B_i \in \mathcal{B}(H)$ $(i = 1, 2)$ be non-zero positive operators. Then the following conditions are equivalent:*

(i) $A_1 \otimes B_1 \leq A_2 \otimes B_2$.

(ii) *There exists $c > 0$ such that $A_1 \leq c\,A_2$ and $B_1 \leq c^{-1}\,B_2$.*

THEOREM 3-2. *For non-zero $A, B \in \mathcal{B}(H)$ $A \otimes B \in \mathfrak{A}$ if and only if A and $B \in \mathfrak{A}$.*

Proof. Suppose $A \otimes B \in \mathfrak{A}$. Then

$$|A|^2 \otimes |B|^2 = |A \otimes B|^2 \leq |(A \otimes B)^2| = |A^2 \otimes B^2| = |A^2| \otimes |B^2|.$$

Hence, by lemma 3.1, there exists a scalar $c > 0$ such that

$$|A|^2 \leq c|A^2| \text{ and } |B|^2 \leq c^{-1}|B^2|.$$

This implies that

$$\begin{aligned} ||A||^2 &= \sup_{||x||=1}\langle |A|^2 x, x\rangle \\ &\leq \sup_{||x||=1}\langle c|A^2|x, x\rangle \\ &\leq c|||A^2||| = c||A^2|| \leq c||A||^2 \end{aligned}$$

and

$$\begin{aligned}||B||^2 &= \sup_{||x||=1}\langle |B|^2 x, x\rangle \\ &\le \sup_{||x||=1}\langle c^{-1}|B^2|x, x\rangle \\ &\le c^{-1}|||B^2||| = c^{-1}||B^2|| \le c^{-1}||B||^2.\end{aligned}$$

Clearly, we must have $c = 1$, and then $A, B \in \mathfrak{A}$. Conversely, if $A, B \in \mathfrak{A}$, then $(|A^2| - |B|^2) \otimes (|B^2| - |B|^2) \ge 0$ implies

$$\begin{aligned}&(|A^2| \otimes |B^2|) - (|A|^2 \otimes |B|^2) \\ &\ge |A^2| \otimes |B|^2 + |A|^2 \otimes |B^2| - 2|A|^2 \otimes |B|^2 \\ &= (|A^2| - |A|^2) \otimes |B|^2 + |A|^2 \otimes (|B^2| - |B|^2) \ge 0.\end{aligned}$$

Hence $A \otimes B \in \mathfrak{A}$. □

For any $X \in \mathcal{B}(H)$ let $\tau_{AB^*} : \mathcal{B}(H) \to \mathcal{B}(H)$ be defined by $\tau_{AB^*}(X) = AXB^*$ and $C_2(H)$ denote the class of Hilbert-Schmidt operators on H. Then we have

COROLLARY 3-3. *For non-zero $A, B \in \mathcal{B}(H)$, $A, B \in \mathfrak{A}$ if and only if $\tau_{AB^*}|_{C_2(H)} \in \mathfrak{A}$.*

Proof. It is well known that the tensor product $A \otimes B$ can be identified with the mapping $\tau_{AB^*}|_{C_2(H)}$ (cf., [3, Lemma 2]). This completes the proof. □

4. Polynomially $\mathfrak{A}$-operators

Let $H(K)$ be the set of all analytic functions on an open neighborhood of compact subset $K \subset \mathbb{C}$. In this section we prove that if A is a polynomially $\mathfrak{A}$-operator, then Weyl's theorem holds for $f(A)$ for $f \in H(\sigma(A))$. This extends well-known results of [8] and [11]: our proof however employs slightly different techniques.

THEOREM 4-1. *If $A \in \mathcal{B}(H)$ is a polynomially $\mathfrak{A}$-operator and $f \in H(\sigma(A))$, then Weyl's theorem holds for $f(A)$.*

The proof will be given by following several lemmas. We begin by elementary properties of polynomially $\mathfrak{A}$-operators.

LEMMA 4-2. *Let A be a polynomially $\mathfrak{A}$-operator. Then the following holds.*

(i) *If A is quasinilpotent, then A is nilpotent.*
(ii) *A is isoloid.*
(iii) *A is finitely ascensive.*

Proof. Towards (i), suppose $p(A)$ is an $\mathfrak{A}$-operator for a non-trivial polynomial p. We may write

$$p(\lambda) - p(0) = a_0 \lambda^m \prod_{i=1}^{n} (\lambda - \lambda_i)$$

for some scalars $a_0, \lambda_1, \dots, \lambda_n$ and integers m, n. If A is quasinilpotent, then

$$\sigma(p(A)) = p(\sigma(A)) = p(0),$$

so that $p(A) - p(0)$ is also quasinilpotent. Thus it follows that

$$p(A) - p(0) = a_0 A^m \prod_{i=1}^{n} (A - \lambda_i) = 0.$$

Since $A - \lambda_i$ is invertible for every $1 \leq i \leq n$, we have that $A^m = 0$.

Towards (ii), suppose $p(A)$ is an $\mathfrak{A}$-operator for a non-trivial polynomial p. Let $\lambda \in \operatorname{iso}\sigma(A)$. Then using the spectral decomposition, we can represent A as the direct sum $A = A_1 \oplus A_2$, where $\sigma(A_1) = \{\lambda\}$ and $\sigma(A_2) = \sigma(A) \setminus \{\lambda\}$. Since $p(A_1)$ is also $\mathfrak{A}$-operator by Lemma 2-1(i), the quasinilpotency of $p(A_1) - p(\lambda)$ implies the nilpotency of $A_1 - \lambda$ from similar arguments of proof of (i). Therefore $\lambda \in \pi_0(A_1)$ and hence $\lambda \in \pi_0(A)$. This shows that A is isoloid.

Towards (iii), suppose $p(A)$ is an $\mathfrak{A}$-operator for a non-trivial polynomial p. If $\lambda \in \sigma(A)$, then we may assume that for some scalars $a_0, \lambda_1, \dots, \lambda_n$ and integers m, n

$$p(A) - p(\lambda) = a_0 (A - \lambda)^m \prod_{i=1}^{n} (A - \lambda_i). \tag{4.2.1}$$

Let $x (\neq 0) \in \ker(A - \lambda)^{m+1}$. Then

$$(p(A) - p(\lambda))x = b(A - \lambda)^m x \text{ for some scalar } b. \tag{4.2.2}$$

Let $p(\lambda) = 0$;

$$\begin{aligned} 0 &= ||(A-\lambda)^{2m}x||||x|| \\ &= |||b^{-2}p(A)^2|x||||x|| \\ &\geq \langle |b^{-2}p(A)^2|x, x\rangle \\ &\geq \langle |b^{-1}p(A)|^2 x, x\rangle \\ &= ||b^{-1}p(A)x||^2 \\ &= ||(A-\lambda)^m x||^2. \end{aligned} \tag{4.2.3}$$

Let $p(\lambda) \neq 0$; since by Lemma 2-1(ii)

$$(p(A)-p(\lambda))(A-\lambda)^m x = 0 \;\Rightarrow\; (p(A)-p(\lambda))^*(A-\lambda)^m x = 0,$$

we have

$$\begin{aligned} ||(A-\lambda)^m x||^2 &= \langle (A-\lambda)^m x, (A-\lambda)^m x\rangle \\ &= \langle b^{-1}(p(A)-p(\lambda))x, (A-\lambda)^m x\rangle \\ &= \langle x, b^{*-1}(p(A)-p(\lambda))^*(A-\lambda)^m x\rangle \\ &= 0. \end{aligned} \tag{4.2.4}$$

Thus (4.2.3) and (4.2.4) implies that $x \in \ker(A-\lambda)^m$. Therefore $\ker(A-\lambda)^{m+1} \subseteq \ker(A-\lambda)^m$ and the reverse inclusion is evident. This completes the proof. □

In view of Remark 2-5, it also seems to be impossible to get Lemma 4-2(iii) in the context of (polynomially) paranormal operators.

LEMMA 4-3. ([17, Theorem 2]) *Let $A \in \mathcal{B}(H)$ be finitely ascensive. Then Weyl's theorem holds for A if and only if* $\operatorname{ran}(A-\lambda)$ *is closed for every* $\lambda \in \pi_{00}(A)$.

PROPOSITION 4-4. *Weyl's theorem holds for every polynomially $\mathfrak{A}$-operators.*

Proof. Let A be a polynomially $\mathfrak{A}$-operator. Then by Lemma 4-2(iii) A is finitely ascensive. So it suffices to show that $\operatorname{ran}(A-\lambda)$ is closed for every $\lambda \in \pi_{00}(A)$ by Lemma 4-3. Suppose $\lambda \in \pi_{00}(A)$ and let E_λ be the Riesz projection with corresponding to λ. Then $\operatorname{ran}(E_\lambda)$ is finite dimensional because $(A-\lambda)|_{\mathrm{ran}E_\lambda}$ is nilpotent as shown in the proof of Lemma 4-2(ii), and

$$0<\dim\ker(A-\lambda)|_{\mathrm{ran}E_\lambda}=\dim\ker(A-\lambda)<\infty.$$

From [18, Corollary 2.4] we have $\mathrm{ran}E_\lambda=H_A(\{\lambda\})$, and so [19, Lemma 2] implies that $A-\lambda$ is Fredholm. Hence $\mathrm{ran}(A-\lambda)$ is closed for $\lambda\in\pi_{00}(A)$. □

We show that the Weyl spectrum obeys the spectral mapping theorem for polynomially $\mathfrak{A}$-operators.

LEMMA 4-5. *If $A\in\mathcal{B}(H)$ is a polynomially $\mathfrak{A}$-operator, then*

(4.5.1) $$\omega(f(A))=f(\omega(A))\quad \textit{for every } f\in H(\sigma(A)).$$

Proof. First, let f be a polynomial. Since it is well known ([4, Theorem 3.2]) that

$$\omega(f(A))\subseteq f(\omega(A)),$$

in view of [13, Theorem 5], it suffices to show that

(4.5.2) $$\mathrm{ind}\,(A-\lambda I)\,\mathrm{ind}\,(A-\mu I)\geq 0\quad\text{for each pair } \lambda,\mu\in\mathbb{C}\setminus\sigma_e(A).$$

By Lemma 4-2(iii), $A-\lambda I$ has finite ascent for every $\lambda\in\mathbb{C}$. Observe that if $A-\lambda$ is Fredholm of finite ascent, then $\mathrm{ind}(A-\lambda)\leq 0$ by the same arguments in the proof of [13, Theorem 3]. Thus we can see that (4.5.2) holds for every polynomially $\mathfrak{A}$-operators T. This proves that the equality (4.5.1) holds for every polynomial f. Now the equality (4.5.1) for $f\in H(\sigma(A))$ follows at once from an argument of Oberai ([22, Theorem 2]). □

Now, we conclude this paper with the proof of Theorem 4-1.

Proof of Theorem 4-1. Remembering [21, Lemma] that if A is isoloid, then

$$f\big(\sigma(A)\setminus\pi_{00}(A)\big)=\sigma(f(A))\setminus\pi_{00}(f(A))\quad\text{for every } f\in H(\sigma(A));$$

it follows from Lemma 4-2(ii), Proposition 4-4 and Lemma 4-5 that

$$\sigma(f(A))\setminus\pi_{00}(f(A))=f\big(\sigma(A)\setminus\pi_{00}(A)\big)=f(\omega(A))=\omega(f(A)),$$

which implies that Weyl's theorem holds for $f(A)$. □

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