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FOUNDATIONS OF THE THEORY OF ℓ_1 HOMOLOGY

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ABSTRACT. In this paper, we provide the algebraic foundations to the theory of relative ℓ_1 homology. In particular, we prove that ℓ_1 homology of topological spaces, both for the absolute case and for the relative case, depends only on their fundamental groups. We also provide a proof of Gromov's Equivalence theorem for ℓ_1 homology, stated by Gromov without proof [4].

1. Introduction

The simplicial ℓ_1 was introduced by M. Gromov and W. Thurston in Thurston's 1979 lecture notes: Geometry and topology of three-manifolds. Then, on the basis of simplicial ℓ_1 norm, Gromov [4] introduced ℓ_1 homology of topological spaces. He [4] also defined bounded cohomology of topological spaces by taking the dual of the simplicial ℓ_1 norm.

Furthermore, Gromov [4] demonstrated the importance of both the theory of ℓ_1 homology and the theory of bounded cohomology by applying them to Riemannian geometry. He also proved a number of profound theorems about them [4]. However, Gromov's proofs in [4] are based on a specific technique developed by him, which he called the theory of simplicial multicomplexes, rather than on standard ideas of algebraic topology.

R. Brooks [1] made a first step in understanding the theory of bounded cohomology from the point of view of homological algebra. However, Brooks's approach did not let one precisely reconstruct the natural seminorm on bounded cohomology groups. In [5] N. Ivanov improved Brooks's approach using a suitable version of relative homological algebra, modified so that it takes into account a natural seminorm in

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the bounded cohomology. Afterwards, H. Park [8] extended Ivanov's approach to the theory of relative bounded cohomology.

The main purpose of this paper is to provide the algebraic foundation and appropriate definitions for the relative ℓ_1 homology, so that this theory is well understood from a more conventional point of view.

We first approach the theory of absolute ℓ_1 homology based on the ideas of the relative homological algebra by modifying Ivanov's approach [5] to the theory of bounded cohomology. Then we extend this approach to the relative case.

As in the case of the relative bounded cohomology in [8], we extend the theory of the relative ℓ_1 homology from the usual case of a pair of spaces (X, Y) with $Y \subset X$ to the more general case of any continuous map $Y \to X$ of spaces X and Y (similarly, from a pair of groups (G, A) to any homomorphism $A \to G$). In this general case, the pairs (X, Y) and (G, A) correspond to inclusions $Y \hookrightarrow X$ and $A \hookrightarrow G$ respectively. This more general framework with continuous maps and homomorphisms turns out to be necessary for comparing the relative ℓ_1 homology of spaces with the relative ℓ_1 homology of groups.

Let X be a topological space. For every $n \geq 0$, we denote by $C_n(X)$ the real n-dimensional chain group of X: a chain $c \in C_n(X)$ is a finite combination $\sum_i r_i\sigma_i$ of singular n-simplices σ_i in X with real coefficients r_i. We define the *simplicial ℓ_1 norm* on $C_n(X)$ by setting $\|c\|_1 = \sum_i |r_i|$.

Let $C_n^{\ell_1}(X)$ be the completion of $C_n(X)$ with respect to this norm, that is,

$$C_n^{\ell_1}(X) = \left\{ \sum_{i=1}^{\infty} r_i\sigma_i \,\middle|\, \sum_{i=1}^{\infty} |r_i| < \infty \right\}.$$

We have the chain complex

$$\cdots \to C_3^{\ell_1}(X) \xrightarrow{\partial_3} C_2^{\ell_1}(X) \xrightarrow{\partial_2} C_1^{\ell_1}(X) \xrightarrow{\partial_1} C_0^{\ell_1}(X) \to 0$$

of Banach spaces and bounded operators, where the boundary operator ∂_n is defined by extending linearly the boundary operator on the ordinary chain complex $C_*(X)$ and has the norm $\|\partial_n\| \leq n+1$. The homology of this complex is called *the ℓ_1 homology of X* and is denoted by $H_*^{\ell_1}(X)$.

Taking the dual Banach space of $\{C_*^{\ell_1}(X), \partial_*\}$, we obtain a cochain complex $\{B^*(X), d_*\}$. However, the complex $\{B^n(X)\}$ has its own independent description as the space of bounded real valued functions on the set of singular n-simplices in X (see [4], [5]). The cohomology of the complex $\{B^*(X)\}$ is called the bounded cohomology of X and is denoted by $\widehat{H}^*(X)$.

On $H_*^{\ell_1}(X)$ there is a natural seminorm $\|\cdot\|_1$ defined by $\|[x]\|_1 = \inf \|c\|_1$ for a homology class $[x] \in H_n^{\ell_1}(X)$, where the infimum is taken over all cycles $c \in C_n^{\ell_1}(X)$ representing the homology class $[x]$. Notice that the inclusions $C_*(X) \hookrightarrow C_*^{\ell_1}(X)$ induce a canonical map $H_*(X) \to H_*^{\ell_1}(X)$, which is in general neither injective nor surjective.

Let us consider the relative case. For a continuous map of spaces $\varphi : Y \to X$, there is an induced chain map $\varphi_* \colon \{C_*^{\ell_1}(Y), \partial'_*\} \to \{C_*^{\ell_1}(X), \partial_*\}$. Then the mapping cone $\{C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y), d_n\}$ is a complex, where the boundary operators d_* are defined by

$$d_n(x_n, a_{n-1}) = (\partial_n x_n + \varphi_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1}).$$

The n-th homology of this complex is called *the n-th relative ℓ_1 homology of X modulo Y* and is denoted by $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$. We define the norm $\|\cdot\|_1$ on $\{C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y), d_n\}$ by setting

$$\|(x,\, a)\|_1 = \|x\|_1 + \|a\|_1.$$

This norm induces a seminorm $\|\cdot\|_1$ on $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$.

There is a group-theoretic analogue of ℓ_1 homology. We use the standard bar resolution (see [2]). For a discrete group G, let $C_n(G)$ be a free $\mathbf{R}$-module generated by the n-tuples $[g_1|\cdots|g_n]$, where $g_i \in G$. We define the ℓ_1 norm $\|\cdot\|_1$ on $C_n(G)$ by putting

$$\|\sum r_i[g_{i_1}|\cdots|g_{i_n}]\|_1 = \sum |r_i|.$$

Let $C_n^{\ell_1}(G)$ be the norm completion of $C_n(G)$, that is,

$$C_n^{\ell_1}(G) = \left\{ \sum_{i=1}^{\infty} r_i[g_{i_1}|\cdots|g_{i_n}] \mid \sum_{i=1}^{\infty} |r_i| < \infty \right\}.$$

We have the chain complex

$$\cdots \to C_3^{\ell_1}(G) \xrightarrow{\partial_3} C_2^{\ell_1}(G) \xrightarrow{\partial_2} C_1^{\ell_1}(G) \xrightarrow{\partial_1 = 0} \mathbf{R} \to 0$$

of Banach spaces and bounded operators, where the boundary operator ∂_n for every $n \geq 2$ is defined by

$$\partial_n[g_1|\cdots|g_n] = [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}],$$

and has the norm $\|\partial_n\| \le n+1$. The homology of this complex is called *the ℓ_1 homology of G* and is denoted by $H_*^{\ell_1}(G)$.

For the relative case, we consider a homomorphism of groups $\varphi\colon A \to G$. Using the standard bar resolutions, we define $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ as the homology of the complex of mapping cone $\{C_n^{\ell_1}(G) \bigoplus C_{n-1}^{\ell_1}(A), d_n\}$ [Definition 3.3].

If we consider an inclusion map $\varphi\colon Y \hookrightarrow X$ for $Y \subset X$, there is an exact sequence

$$0 \to C_*^{\ell_1}(Y) \hookrightarrow C_*^{\ell_1}(X) \to C_*^{\ell_1}(X)/C_*^{\ell_1}(Y) \to 0.$$

It is clear that $\{C_*^{\ell_1}(X)/C_*^{\ell_1}(Y)\}$ is a complex. As in the ordinary case, we define the relative ℓ_1 homology groups $H_*^{\ell_1}(X,Y)$ as the homology of the complex of $\{C_*^{\ell_1}(X)/C_*^{\ell_1}(Y)\}$. While our definition is different from this ordinary case, the groups $H_*^{\ell_1}(X,Y)$ and $H_*^{\ell_1}(Y \hookrightarrow X)$ are canonically isomorphic as vector spaces [Theorem 4.6]. Similarly, for a pair of groups $A \subset G$, we define $H_*^{\ell_1}(G,A)$. Notice that for a pair of spaces $Y \subset X$ we can define $H_*^{\ell_1}(\pi_1 X, \pi_1 Y)$ only when the inclusion map $Y \hookrightarrow X$ induces an injective homomorphism $\pi_1 Y \hookrightarrow \pi_1 X$.

Using our definition, for any continuous map $Y \to X$ and the induced homomorphism $\pi_1 Y \to \pi_1 X$, we can construct a homomorphism between $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ and $H_n^{\ell_1}(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$. Moreover, it turns out that these two groups are in fact isometrically isomorphic [Theorem 4.4]. This supports the idea that our definition of the relative ℓ_1 homology is more natural.

Now we describe the content of this paper. In Section 2, we construct a theory of ℓ_1 homology of discrete groups. Amenable groups [Definition 2.7] play a special role on the theory of ℓ_1 homology. As a main result, we prove that ℓ_1 homology of amenable groups [Corollary 2.9] is zero. In Section 3, for a group homomorphism $A \xrightarrow{\varphi} G$ we define the relative ℓ_1 homology of a group G modulo A and denote it by $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$[Definition 3.3]. We prove Gromov's equivalence theorem to the effect that the groups $H_*^{\ell_1}(G)$ and $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ are isometrically isomorphic if A is an amenable subgroup of G [Theorem 3.6]. In Section 4, for a continuous map $\varphi\colon Y \to X$ of spaces we define the relative ℓ_1 homology of X modulo Y and denote it by $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ [Definition 4.2]. The main result of this section is that the relative ℓ_1 homology of a continuous map $\varphi\colon Y \to X$ is isometrically isomorphic with the relative ℓ_1 homology of the induced homomorphism $\varphi_*\colon \pi_1 Y \to \pi_1 X$ [Theorem 4.4].

2. ℓ_1 homology of groups

Throughout this section, G denotes a discrete group. We dualize the notion of *relative injectivity* in [5] to define ℓ_1 homology of groups.

By a *bounded left G module* we mean a real Banach space V together with a left action of G on V such that $\|g \cdot v\| \leq \|v\|$ for all $g \in G$ and $v \in V$. We define a bounded right G-module similarly. We call a bounded left G-module simply a G-module. The simplest important example of G-module is $\mathbf{R}$ with the trivial action of G.

We introduce another important example of G-module for ℓ_1 homology. Using the bar notation [2], we consider $C_n(G)$ the free $\mathbf{R}$-module generated by the n-tuples $[g_1|g_2|\cdots|g_n]$ with the G-action. Since the operation on a basis with an element of $g \in G$ yields an element $g[g_1|\cdots|g_n]$ in $C_n(G)$, we may describe $C_n(G)$ as the free $\mathbf{R}$-module generated by all $g[g_1|\cdots|g_n]$ so that an element of $C_n(G)$ can be written as a finite sum of the form $\sum r_i g_i[g_{i_1}|\cdots|g_{i_n}]$ where $r_i \in \mathbf{R}$, $g_i \in G$.

In particular, $C_0(G)$ has one generator, denoted by $[\,]$, so its element is a finite sum of the form $\sum r_i g_i[\,]$. We define the ℓ_1 norm $\|\cdot\|_1$ on $C_n(G)$ by

$$\|\sum r_i g_i[g_{i_1}|\cdots|g_{i_n}]\|_1 = \sum |r_i|.$$

Let $C_n^{\ell_1}(G)$ be the norm completion of $C_n(G)$. Thus

$$C_n^{\ell_1}(G) = \left\{ \sum_{i=1}^{\infty} r_i g_i[g_{i_1}|\cdots|g_{i_n}] \;\middle|\; \sum_{i=1}^{\infty} |r_i| < \infty \right\}$$

is a Banach space with the G-action such that $\|g \cdot c\|_1 \leq \|c\|_1$ for every $g \in G$, and $c \in C_n^{\ell_1}(G)$. Hence $C_n^{\ell_1}(G)$ is a G-module.

DEFINITION 2.1. A surjective G-morphism of G-modules $\pi\colon V \to W$ is said to be *strongly projective* if there exists a bounded linear operator $\sigma\colon W \to V$ such that $\pi \circ \sigma = id$ and $\|\sigma\| \leq 1$. Also a G-module U is said to be *relatively projective*, if for any strongly projective G-morphism of G-modules $\pi\colon V \to W$ and any G-morphism of G-modules $\alpha\colon U \to W$ there exists a G-morphism $\beta\colon U \to V$ such that $\pi \circ \beta = \alpha$ and $\|\beta\| \leq \|\alpha\|$. The definition is illustrated by the following diagram (2.1.1) :

$$\begin{array}{ccc} U & = & U \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ V & \xrightarrow{\pi} & W \end{array}$$

LEMMA 2.1. *Let U be a free $\mathbf{R}$-module generated by $\{u_i\}$ with the G-action. Suppose that the ℓ_1 norm on U is defined by $\| \sum_{i=1}^{n} r_i g_i u_i \|_1 = \sum_{i=1}^{n} |r_i|$, where $r_i \in \mathbf{R}$ and $g_i \in G$. Then the norm completion U^{ℓ_1} of U is a relatively projective G-module. In particular, the G-modules $C_n^{\ell_1}(G)$ are relatively projective for all $n \geq 0$.*

Proof. Let $\pi\colon V \to W$ be strongly projective G-morphism of G-modules. We consider the situation pictured in diagram (2.1.1), in which $U = U^{\ell_1}$ and all the rest are given.

For $x = \sum_{i=1}^{\infty} r_i g_i u_i \in U^{\ell_1}$, we define β by the formula $\beta(\sum_{i=1}^{\infty} r_i g_i u_i) = \sum_{i=1}^{\infty} r_i g_i \sigma\alpha(u_i)$. It is easy to check that $\pi\beta = \alpha$ and β commutes with the action of G. Also we have

$$\begin{aligned} \|\beta(x)\| &= \|\beta(\sum_{i=1}^{\infty} r_i g_i u_i)\|_1 = \| \sum_{i=1}^{\infty} r_i g_i \sigma\alpha(u_i) \|_1 \\ &\leq \sum_{i=1}^{\infty} | r_i | \, \|g_i\|_1 \|\sigma\| \|\alpha\| \leq \sum_{i=1}^{\infty} | r_i | \, \|\alpha\| = \|x\|_1 \|\alpha\|, \end{aligned}$$

so that $\|\beta\| \leq \|\alpha\|$. □

DEFINITION 2.2. A G-resolution of a G-module V

$$\cdots \to V_2 \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \xrightarrow{d_0} V \to 0 \tag{2.2.1}$$

is said to be *strong* if it is provided with a contracting homotopy, that is, a sequence of linear operators

$$\cdots \leftarrow V_3 \xleftarrow{k_2} V_2 \xleftarrow{k_1} V_1 \xleftarrow{k_0} V_0 \xleftarrow{k_{-1}} V$$

such that $d_0 k_{-1} = id$, $d_{n+1} k_n + k_{n-1} d_n = id$ for $n \geq 0$, and such that $\|k_n\| \leq 1$. The resolution in (2.2.1) is said to be *relatively projective* if all G-modules V_n are relatively projective.

We consider the sequence of G-modules and G-morphisms

$$\cdots \to C_3^{\ell_1}(G) \to C_2^{\ell_1}(G) \to C_1^{\ell_1}(G) \to C_0^{\ell_1}(G) \to \mathbf{R} \to 0, \tag{2.1}$$

where the boundary operator $\partial_n\colon C_n^{\ell_1}(G) \to C_{n-1}^{\ell_1}(G)$ for every $n \geq 1$ is defined by

$$\begin{aligned} \partial_n[g_1|\cdots|g_n] =& (-1)^n g_1[g_2|\cdots|g_n] \\ &+ \sum_{i=1}^{n-1} (-1)^{n-i}[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] + [g_1|\cdots|g_{n-1}], \end{aligned}$$

while $\varepsilon[\,] = 1$ is a G-morphism $\varepsilon\colon C_0^{\ell_1}(G) \to \mathbf{R}$.

Also we define $s_{-1}\colon \mathbf{R} \to C_0^{\ell_1}(G)$ and $s_n\colon C_n^{\ell_1}(G) \to C_{n+1}^{\ell_1}(G)$ by the formulas respectively:

$$s_{-1}1 = [\,] \quad \text{and} \quad s_n(g[g_1|\cdots|g_n]) = (-1)^{n+1}[g|g_1|\cdots|g_n].$$

It is clear that the sequence in (2.1) is a strong relatively projective G-resolution of the trivial G-module $\mathbf{R}$.

DEFINITION 2.3. The sequence in (2.1) is called the *bar resolution* of G.

DEFINITION 2.4. For any G-module V *the space of co-invariants of* V, denoted by V_G, is defined to be the quotient of V by the additive submodule generated by the elements of the form $gv - v$ for all $g \in G$ and $v \in V$.

For a strong relatively projective G-resolution

$$\cdots \to V_2 \to V_1 \to V_0 \to \mathbf{R} \to 0$$

of the trivial G-module $\mathbf{R}$, it is easy to see that the induced sequence

$$\cdots \to (V_2)_G \to (V_1)_G \to (V_0)_G \to 0 \tag{2.2}$$

is a complex. Notice that the homology of this complex depends only on G.

DEFINITION 2.5. The n-th homology group of the complex in (2.2) is called the *n-th ℓ_1 homology* group of G and is denoted by $H_n^{\ell_1}(G)$.

REMARK 2.1. It is proved that $H_1^{\ell_1}(G) = 0$ for any group G (see [3], [6]).

Remark that the homology of the complex in (2.2) has a natural seminorm which induces a topological vector space structure. Also remark that this seminorm depends on the choice of a resolution.

DEFINITION 2.6. We define the *canonical seminorm* on $H_*^{\ell_1}(G)$ as the supremum of the seminorms that arise from all strong relatively projective G-resolutions of the trivial G-module $\mathbf{R}$.

THEOREM 2.2. *Let*

$$\cdots \underset{t_2}{\overset{\partial_3'}{\rightleftarrows}} V_2 \underset{t_1}{\overset{\partial_2'}{\rightleftarrows}} V_1 \underset{t_0}{\overset{\partial_1'}{\rightleftarrows}} V_0 \underset{t_{-1}}{\overset{\varepsilon'}{\rightleftarrows}} \mathbf{R} \to 0$$

be a strong relatively projective G-resolution of trivial G-module $\mathbf{R}$. *Then there exists a G-morphism of the bar resolution of G to this resolution*

$$\begin{array}{ccccccccccc} \longrightarrow & C_2^{\ell_1}(G) & \xrightarrow{\partial_2} & C_1^{\ell_1}(G) & \xrightarrow{\partial_1} & C_0^{\ell_1}(G) & \xrightarrow{\varepsilon} & \mathbf{R} & \longrightarrow & 0 \\ \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & \downarrow id_{\mathbf{R}} & & \downarrow \\ \longrightarrow & V_2 & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & \mathbf{R} & \longrightarrow & 0 \end{array}$$

extending $id_{\mathbf{R}}$ *and such that* $\|f_n\| \leq 1$ *for every* $n \geq 0$.

Proof. We define f_n by the formula

$$f_n(g[g_1|\cdots|g_n]) = (-1)^n g t_{n-1}(g_1 t_{n-2}(g_2 \cdots (g_{n-1} t_0(g_n t_{-1}(1))\cdots).$$

It is clear that f_n commutes with the action of G. Since $\|t_*\| \leq 1$ and $\|g \cdot x\|_1 \leq \|x\|_1$ for all $g \in G, x \in V_*$, we have $\|f_n\| \leq 1$.

Notice that $\varepsilon' f_0[\,] = 1 = \varepsilon[\,]$. It remains for us to verify that $f_n \partial_{n+1} = \partial'_{n+1} f_{n+1}$ for every $n \geq 0$. We prove this by induction on n. First notice that we have $f_{n+1}([g_1|\cdots|g_{n+1}]) = (-1)^{n+1} t_n(g_1 f_n([g_2|\cdots|g_{n+1}]))$ and also $\partial'_{n+1} t_n + t_{n-1}\partial'_n = id$. Now we assume $f_{n-1}\partial_n = \partial'_n f_n$. Then

$$\begin{aligned} &\partial'_{n+1} f_{n+1}([g_1|\cdots|g_{n+1}]) = \partial'_{n+1}(-1)^{n+1}(t_n(g_1 f_n([g_2|\cdots|g_{n+1}])) \\ &= (-1)^{n+1}(id - t_{n-1}\partial'_n)(g_1 f_n([g_2|\cdots|g_{n+1}])) \\ &= (-1)^{n+1} g_1 f_n([g_2|\cdots|g_{n+1}]) - (-1)^{n+1} t_{n-1}\partial'_n(g_1 f_n([g_2|\cdots|g_{n+1}])) \\ &= (-1)^{n+1} f_n(g_1[g_2|\ldots|g_{n+1}]) + (-1)^n t_{n-1}(g_1 \partial'_n f_n([g_2|\cdots|g_{n+1}])) \\ &= (-1)^{n+1} f_n(g_1[g_2|\ldots|g_{n+1}]) + (-1)^n t_{n-1}(g_1 f_{n-1}\partial_n([g_2|\cdots|g_{n+1}])) \\ &= (-1)^{n+1} f_n(g_1|g_2|\cdots|g_{n+1}]) + (-1)^n t_{n-1}\{g_1 f_{n-1}((-1)^n g_2[g_3|\cdots|g_{n+1}] \\ &\quad + \sum_{i=2}^{n}(-1)^{n+1-i}[g_2|\cdots|g_i g_{i+1}|\cdots|g_{n+1}] + [g_2|\cdots|g_n])\} \\ &= \cdots \\ &= (-1)^{n+1} f_n(g_1[g_2|\cdots|g_{n+1}]) \\ &\quad + \sum_{i=1}^{n}(-1)^{n-i} f_n([g_1|g_2|\cdots|g_i g_{i+1}|\cdots|g_{n+1}]) + f_n([g_1|\cdots|g_n]) \\ &= f_n\partial_{n+1}([g_1|\cdots|g_{n+1}]). \end{aligned}$$

Thus we have $f_n\partial_{n+1} = \partial'_{n+1} f_{n+1}$. □

COROLLARY 2.3. *On* $H_*^{\ell_1}(G)$ *the seminorm induced by the bar resolution of G coincides with the canonical seminorm.*

Notice that, as in the ordinary homology of groups (see [2]), it is easy to check that $H_*^{\ell_1}(G)$ is a covariant functor of G: given a group homomorphism $\varphi\colon G \to H$ there is an induced homomorphism $H_*(\varphi)\colon H_*^{\ell_1}(G) \to H_*^{\ell_1}(H)$ which depends only on φ. Also notice that $\|H_*(\varphi)\| \le 1$.

Now we see the relationship between amenable groups and ℓ_1 homology. First we recall the definition of amenable groups. Let S be a set. The space $B(S)$ of all bounded functions on S is a Banach space with the norm $\|f\| = \sup\{\,|f(x)| \mid x \in S\,\}$. A linear functional $m\colon B(S) \to \mathbf{R}$ is called a *mean* if

$$\inf\{f(x) \mid x \in S\} \le m(f) \le \sup\{f(x) \mid x \in S\} \quad \text{for all } f \in B(S).$$

Let the group G act on S on the right. Then G acts on $B(S)$ on the left by the formula $g \cdot f(s) = f(s \cdot g)$, where $g \in G$, $f \in B(S)$, and $s \in S$. A mean m on $B(S)$ is called right-invariant if $m(g \cdot f) = m(f)$ for all $g \in G, f \in B(S)$.

DEFINITION 2.7. Let the group G act on itself by the right translation. If there is a right-invariant mean on $B(G)$, then the group G is called *amenable.*

As it is well known, finite groups, abelian groups, subgroups and the homomorphic images of amenable groups are amenable.

Let A be an amenable subgroup of G. We consider G/A, the set of (right) cosets Ag of A in G. Since the set of cosets Ag has the G-action by right translation, we can define $C_*^{\ell_1}(G/A)$ as the same manner with $C_*^{\ell_1}(G)$. Namely, we can take $C_n^{\ell_1}(G/A)$ as the free $\mathbf{R}$-module generated by the n-tuples of the form $[Ag_1|\cdots|Ag_n]$. The action of a G-module is given by the formula $g'[Ag_1|\cdots|Ag_n] = Ag'[Ag_1|\cdots|Ag_n]$. Notice that the canonical map $p_n\colon C_n^{\ell_1}(G) \to C_n^{\ell_1}(G/A)$ is a G-morphism and has the norm $\|p_n\| \le 1$.

LEMMA 2.4. *Let A be an amenable subgroup of G. Then there exists a G-morphism $q_1\colon C_1^{\ell_1}(G/A) \to C_1^{\ell_1}(G)$ such that $p_1q_1 = id$ and $\|q_1\| = 1$.*

Proof. Since A is amenable, there is a right-invariant mean $m\colon B(A) \to \mathbf{R}$. On any coset Ag, as shown at the point (2.1) in [5], m defines a mean $m_g\colon B(Ag) \to \mathbf{R}$ by $m_g(\varphi) = m(f)$, where $f(a) = \varphi(ag)$. For each $x \in G$, we consider the function $\delta_x\colon G \to \mathbf{R}$ defined by $\delta_x(y) = 1$ if $y = x$, and $\delta_x(y) = 0$ otherwise. We define q_1 by the formula

$$q_1(Ag'[Ag]) = \sum_{x \in G} m_g(\delta_x|_{Ag})g'[x].$$

Since $0 \le \delta_x(ag) \le 1$, for $a \in A$, we have $0 \le m_g(\delta_x|_{Ag}) \le 1$ and also
(2.4.1)

$$\sum_{x\in G} \left| m_g(\delta_x|_{Ag}) \right| = \sum_{x\in G} m_g(\delta_x|_{Ag}) = m_g(\sum_{x\in G} \delta_x|_{Ag}) = m_g(\bar{1}_{Ag}) = 1,$$

where $\bar{1}_{Ag}$ is a constant function on Ag with value 1. Thus q_1 is well defined and has the norm $\|q_1\| = 1$. It is easy to check that q_1 commutes with the action of G. Finally, notice that

$$\begin{aligned} p_1q_1([Ag]) &= p_1\big(\sum_{x\in G} m_g(\delta_x|_{Ag})[x]\big) = \sum_{x\in G} m_g(\delta_x|_{Ag})p_1([x]) \\ &= \sum_{x\in Ag} m_g(\delta_x|_{Ag})p_1([x]) = \sum_{x\in Ag} m_g(\delta_x|_{Ag})[Ag] \\ &= \big(\sum_{x\in G} m_g(\delta_x|_{Ag})\big)[Ag] = [Ag], \end{aligned}$$

where the last equality follows from (2.4.1). This shows $p_1q_1 = id$. □

Corollary 2.5. *Let A be an amenable subgroup of G. Then for every $n \ge 0$ there exists a G-morphism $q_n\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ such that $p_nq_n = 1$ and $\|q_n\| = 1$, where $p_n\colon C_n^{\ell_1}(G) \to C_n^{\ell_1}(G/A)$ is the canonical map.*

Proof. Since the spaces $C_0^{\ell_1}(G)$ and $C_0^{\ell_1}(G/A)$ have only one basis element denoted by $[\,]$, we define q_0 by the formula $q_0([\,]) = [\,]$.

Notice that $(G/A)^n \cong G^n/A^n$ and A^n is an amenable subgroup of G^n. We may consider $C_n^{\ell_1}(G/A)$ as $C_1^{\ell_1}(G^n/A^n)$ by setting up each basis $[Ag_1|\cdots|Ag_n]$ of $C_n^{\ell_1}(G/A)$ by

$$[Ag_1|\cdots|Ag_n] = A^n[(g_1,\ldots,g_n)].$$

Then Lemma 2.4 provides a G^n-morphism $q_n\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ such that $p_nq_n = 1$ and $\|q_n\| = 1$. Especially, the G-module structure on $C_n^{\ell_1}(G)$ (and similarly on $C_n^{\ell_1}(G/A)$) is the restriction of the canonical G^n-module structure: $g[g_1|\cdots|g_n] = (1,\ldots,1,g)[g_1|\cdots|g_n]$. Hence p_n is a G-morphism. □

Lemma 2.6. *Let A be an amenable subgroup of G. Then $C_n^{\ell_1}(G/A)$ is relatively projective G-module for every $n \ge 0$.*

Proof. We consider the diagram

$$\begin{array}{ccc} C_n^{\ell_1}(G) & \xrightarrow{\ p\ } & C_n^{\ell_1}(G/A) \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ V & \xrightarrow{\ \pi\ } & W \end{array}$$

where a G-morphism α and a strongly projective G-morphism π are given. We need to construct a G-morphism $\beta\colon C_n^{\ell_1}(G/A) \to V$ such that $\pi\beta = \alpha$ and $\|\beta\| \leq \|\alpha\|$. Since $C_n^{\ell_1}(G)$ is a relatively projective G-module, there exists a G-morphism $\beta'\colon C_n^{\ell_1}(G) \to V$ such that $\alpha p = \pi\beta'$ and $\|\beta'\| \leq \|\alpha p\| \leq \|\alpha\|$. Moreover, there is a G-morphism $q\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ constructed in Corollary 2.5. We define $\beta = \beta' q$. Then $\pi\beta = \pi\beta' q = \alpha p q = \alpha$ and also $\|\beta\| = \|\beta' q\| \leq \|\beta'\|\|q\| \leq \|\alpha\|\|q\| \leq \|\alpha\|$. □

Now we introduce another important strong relatively projective G-resolution. Let A be an amenable subgroup of G. From Lemma 2.6, every $C_*^{\ell_1}(G/A)$ is a relatively projective G-module and so the sequence (2.3)

$$\to C_3^{\ell_1}(G/A) \xrightarrow{\partial_3} C_2^{\ell_1}(G/A) \xrightarrow{\partial_2} C_1^{\ell_1}(G/A) \xrightarrow{\partial_1} C_0^{\ell_1}(G/A) \xrightarrow{\varepsilon} \mathbf{R} \to 0$$

is a strong relatively projective G-resolution of the trivial G-module $\mathbf{R}$, where the boundary and contracting operators are defined by the same formulas in the sequence (2.1). Notice that it induces the complex (2.4)

$$\to C_3^{\ell_1}(G/A)_G \xrightarrow{\partial_3} C_2^{\ell_1}(G/A)_G \xrightarrow{\partial_2} C_1^{\ell_1}(G/A)_G \xrightarrow{\partial_1} C_0^{\ell_1}(G/A)_G \xrightarrow{\partial_0} 0,$$

and the homology of which is $H_*^{\ell_1}(G)$.

PROPOSITION 2.7. *Let A be an amenable subgroup of G. Then the seminorm on $H_*^{\ell_1}(G)$ induced by the resolution in* (2.3) *coincides with the canonical seminorm.*

Proof. Let $\|\cdot\|_1$ denote the canonical seminorm on $H_*^{\ell_1}(G)$ and $\|\cdot\|_1^s$ the seminorm on $H_*^{\ell_1}(G)$ induced by the resolution (2.3). By definition of the canonical seminorm on $H_*^{\ell_1}(G)$, we have $\|\cdot\|_1^s \leq \|\cdot\|_1$.

From Corollary 2.5, there exists a G-morphism $q_n\colon C_n^{\ell_1}(G/A) \to C_n^{\ell_1}(G)$ such that $\|q_n\| = 1$. Thus the seminorm induced by the resolution (2.3) is not less than the canonical seminorm and so $\|\cdot\|_1 \leq \|\cdot\|_1^s$. Hence $\|\cdot\|_1 = \|\cdot\|_1^s$. □

THEOREM 2.8. *Let A be an amenable normal subgroup of G. Then the map $H_*(\varphi)\colon H_*^{\ell_1}(G) \to H_*^{\ell_1}(G/A)$ induced by the canonical map $\varphi\colon G \to G/A$ is an isometric isomorphism, that is, the isomorphism preserves the canonical seminorm.*

Proof. Notice that the sequence (2.3) is the bar resolution of G/A. So it induces the complex $\{C_*^{\ell_1}(G/A)_{G/A}\}$, and the homology of which is $H_*^{\ell_1}(G/A)$. Also the induced norm on $H_*^{\ell_1}(G/A)$ by the resolution (2.3) is the canonical one. Remark that $C_*^{\ell_1}(G/A)_G = C_*^{\ell_1}(G/A)_{G/A}$. Thus $H_*(\varphi)$ is an isomorphism. It follows from Proposition 2.7 that $H_*(\varphi)$ is an isometry. □

COROLLARY 2.9. *If G is amenable, then the group $H_*^{\ell_1}(G)$ is zero.*

We denote the coset A in A by $\{A\}$. Let A be an amenable group. By setting $A = G$ in sequence (2.4), we have a complex

(2.5)
$$\to C_3^{\ell_1}(\{A\})_A \xrightarrow{\partial_3'} C_2^{\ell_1}(\{A\})_A \xrightarrow{\partial_2'} C_1^{\ell_1}(\{A\})_A \xrightarrow{\partial_1'} C_0^{\ell_1}(\{A\})_A \xrightarrow{\partial_0'} 0.$$

The homology of the complex in (2.5) is $H_*^{\ell_1}(A)$ and the induced seminorm is the canonical one. Remark that the boundary operators in the complex (2.5) are in fact given by

$$\partial_n' = \begin{cases} id & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2.6}$$

This also proves that ℓ_1 homology of an amenable group is zero.

3. Relative ℓ_1 homology of groups

Let $\varphi\colon A \to G$ be a group homomorphism. Recall that there is an induced homomorphism $H_*(\varphi)\colon H_*^{\ell_1}(A) \to H_*^{\ell_1}(G)$ which depends only on φ. Also $\|H_*(\varphi)\| \le 1$.

DEFINITION 3.1. Let $\varphi\colon A \to G$ be a group homomorphism. A strong relatively projective G-resolution of a G-module U

$$\cdots \underset{k_2}{\overset{\partial_3}{\rightleftarrows}} U_2 \underset{k_1}{\overset{\partial_2}{\rightleftarrows}} U_1 \underset{k_0}{\overset{\partial_1}{\rightleftarrows}} U_0 \underset{k_{-1}}{\overset{\partial_0}{\rightleftarrows}} U \to 0$$

and a strong relatively projective A-resolution of an A-module U

$$\cdots \underset{t_2}{\overset{\partial_3'}{\rightleftarrows}} V_2 \underset{t_1}{\overset{\partial_2'}{\rightleftarrows}} V_1 \underset{t_0}{\overset{\partial_1'}{\rightleftarrows}} V_0 \underset{t_{-1}}{\overset{\partial_0'}{\rightleftarrows}} U \to 0$$

are called a co-allowable pair of resolutions for $(G, A; U)$ if id_U can be extended to an A-morphism of resolutions $\varphi_n\colon V_n \to U_n$ such that φ_n commutes with the contracting homotopies k_n and t_n for every $n \ge 0$.

PROPOSITION 3.1. *Let $\varphi\colon A \to G$ be a group homomorphism. The bar resolutions of G and A are a co-allowable pair of resolutions for $(G, A; \mathbf{R})$.*

Proof. Recall that the bar resolutions of G and A are strong relatively projective. We define a map $\varphi_n\colon C_n^{\ell_1}(A) \to C_n^{\ell_1}(G)$ by the formula

$$\varphi_n([a_1|\cdots|a_n]) = [\varphi(a_1)|\cdots|\varphi(a_n)].$$

It is easy to check φ_* is an A-morphism commuting with contracting homotopies. □

DEFINITION 3.2. Let $\varphi\colon A \to G$ be a group homomorphism. Let

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be the G- and A-resolutions respectively such that they are a co-allowable pair for $(G, A; \mathbf{R})$. The mapping cone and mapping cylinder of chain complexes induced by φ, respectively, are defined as follows:

$$C_n(A \xrightarrow{\varphi} G) = (U_n)_G \bigoplus (V_{n-1})_A$$
$$EC_n(A \xrightarrow{\varphi} G) = (V_n)_A \bigoplus (U_n)_G \bigoplus (V_{n-1})_A,$$

where the boundary operators on $C_n(A \xrightarrow{\varphi} G)$ and on $EC_n(A \xrightarrow{\varphi} G)$ are defined by the following formulas respectively

$$d_n(x_n, a_{n-1}) = (\partial_n x_n + \varphi_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1})$$
$$d_n(a_n, x_n, a_{n-1}) = (\partial'_n a_n - a_{n-1}, \partial_n x_n + \varphi_{n-1} a_{n-1}, -\partial'_{n-1} a_{n-1}).$$

It is easy to check that the mapping cone $\{C_*(A \xrightarrow{\varphi} G), d_*\}$ and the mapping cylinder $\{EC_*(A \xrightarrow{\varphi} G), d_*\}$ are complexes.

DEFINITION 3.3. The n-th homology of the complex $\{C_*(A \xrightarrow{\varphi} G), d_*\}$ is called *the n-th relative ℓ_1 homology of G modulo A* and is denoted by $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$. The n-th homology of the complex $\{EC_*(A \xrightarrow{\varphi} G), d_*\}$ is denoted by $H_n^{\ell_1}(EC(A \xrightarrow{\varphi} G))$.

We define the norm $\|\cdot\|_1$ on $C_n(A \xrightarrow{\varphi} G)$ (similarly on $EC_n(A \xrightarrow{\varphi} G)$) by

$$\|(x_n,\, a_{n-1})\|_1 = \|x_n\|_1 + \|a_{n-1}\|_1.$$

Notice that these norms define the seminorms $\|\cdot\|_1$ on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ and on $H_*^{\ell_1}(EC(A \xrightarrow{\varphi} G))$, respectively. Furthermore, for every $\omega \geq 0$ we define a norm $\|\cdot\|_1(\omega)$ on $C_n(A \xrightarrow{\varphi} G)$ by

$$\|(x_n,\, a_{n-1})\|_1(\omega) = \|x_n\|_1 + (1+\omega)\|a_{n-1}\|_1.$$

Then we have the corresponding seminorm $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$. Finally, we define these norms $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ for all ω in the closed interval $[0, \infty]$ by passing ω to the limits. Notice that, for $0 \le \omega_1 \le \omega_2$, we have

$$\|\cdot\|_1 = \|\cdot\|_1(0) \le \|\cdot\|_1(\omega_1) \le \|\cdot\|_1(\omega_2).$$

THEOREM 3.2. *Let $\varphi\colon A \to G$ be a group homomorphism. Then the inclusion map $\rho_n\colon C_n^{\ell_1}(G)_G \to C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ induces an isometric isomorphism $H_n(\rho)\colon H_n^{\ell_1}(G) \to H_n^{\ell_1}(EC(A \xrightarrow{\varphi} G))$ with respect to the norm $\|\cdot\|_1$.*

Proof. We consider the exact sequence

$$\begin{aligned} 0 &\to C_n^{\ell_1}(G)_G \\ &\xrightarrow{\rho_n} EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A \\ &\to C_n^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(A)_A \to 0. \end{aligned}$$

It is easy to check that $C_n^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(A)_A$ is a complex, and the homology of which is zero. Thus $H_*^{\ell_1}(G)$ and $H_*^{\ell_1}(EC(A \xrightarrow{\varphi} G))$ are isomorphic. For simplicity, we denote every boundary operator by the same notation d. We consider the diagram

$$\begin{array}{ccccc} C_n^{\ell_1}(G)_G & \xrightarrow{\rho_n} & C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A & \xrightarrow{\widetilde{\rho}_n} & C_n^{\ell_1}(G)_G \\ {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\ C_{n-1}^{\ell_1}(G)_G & \xrightarrow{\rho_{n-1}} & C_{n-1}^{\ell_1}(A)_A \bigoplus C_{n-1}^{\ell_1}(G)_G \bigoplus C_{n-2}^{\ell_1}(A)_A & \xrightarrow{\widetilde{\rho}_{n-1}} & C_{n-1}^{\ell_1}(G)_G, \end{array}$$

where $\widetilde{\rho}_n(a, x, b) = x + \varphi_n a$ and $\varphi_n\colon C_n^{\ell_1}(A)_A \to C_n^{\ell_1}(G)_G$ is an induced homomorphism by φ. It is clear that $\widetilde{\rho}_n \rho_n = id$ and that the first square is commutative. Also it is easy to check that $\widetilde{\rho}_n$ is a chain map. Since we have $\|\rho_n(x)\|_1 = \|(0, x, 0)\|_1 = \|x\|_1$, the map $H_n(\rho)$ has the norm $\|H_n(\rho)\| \le 1$. Also notice that

$$\begin{aligned} \|\widetilde{\rho}_n(a, x, b)\|_1 &= \|x + \lambda_n a\|_1 \le \|x\|_1 + \|a\|_1 \\ &\le \|a\|_1 + \|x\|_1 + \|b\|_1 = \|(a, x, b)\|_1, \end{aligned}$$

so that $\widetilde{\rho}_n$ has the norm $\|\widetilde{\rho}_n\| \le 1$. This shows that $(H_n(\rho))^{-1}$ has the norm $\|(H_n(\rho))^{-1}\| \le 1$. Hence the isomorphism $H_n(\rho)$ is also an isometry. □

Let $\varphi\colon A \to G$ be a group homomorphism. Let the sequences

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair of resolutions for $(G, A; \mathbf{R})$. Then there is an exact sequence

$$0 \to (V_n)_A \xrightarrow{i_n} EC_n(A \xrightarrow{\varphi} G) \xrightarrow{p_n} C_n(A \xrightarrow{\varphi} G) \to 0,$$

where i_n and p_n are natural injective and projective maps respectively. Also this sequence induces a long exact sequence

$$\cdots \to H_n^{\ell_1}(A) \to H_n^{\ell_1}(G) \to H_n^{\ell_1}(A \xrightarrow{\varphi} G) \to H_{n-1}^{\ell_1}(A) \to \cdots .$$

Remark that, by Theorem 3.2, the seminorm on $H_*^{\ell_1}(EC(A \xrightarrow{\varphi} G))$ induced by the bar resolutions coincides with the canonical one. Also remark that a seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ depends on the choice of a co-allowable pair of resolutions for $(G, A; \mathbf{R})$. As on $H_*^{\ell_1}(G)$, we define the canonical seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ by the supremum of the seminorms which arise from all co-allowable pairs of resolutions for $(G, A; \mathbf{R})$.

THEOREM 3.3. *The seminorm $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ induced by the bar resolutions of G and A coincides with the canonical seminorm for every $\omega \in [0, \infty]$.*

Proof. Let

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair for $(G, A; \mathbf{R})$ with an A-morphism $\varphi_n \colon V_n \to U_n$ as in Definition 3.2. Let $\alpha_n \colon C_n^{\ell_1}(G)_G \to (U_n)_G$ and $\gamma_{n-1} \colon C_{n-1}^{\ell_1}(A)_A \to (V_{n-1})_A$ be defined by the same formula in Theorem 2.2. We define a map

$$\beta_n \colon C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A \to (U_n)_G \bigoplus (V_{n-1})_A$$

by the formula $\beta_n(x, a) = (\alpha_n x,\ \gamma_{n-1} a)$. It is easy to check that β_n is a chain map. Also, for every $\omega \geq 0$

$$\begin{aligned}\|\beta_n(x,\ a)\|_1(\omega) &= \|(\alpha_n x,\ \gamma_{n-1} a)\|_1(\omega) = \|\alpha_n x\|_1 + (1+\omega)\|\gamma_{n-1} a\|_1 \\ &\leq \|x\|_1 + (1+\omega)\|a\|_1 = \|(x,\ a)\|_1(\omega).\end{aligned}$$

So β_* has the norm $\|\beta_*\| \leq 1$ with respect to the norm $\|\cdot\|_1(\omega)$. □

LEMMA 3.4. *Let A be an amenable subgroup of G. Then the sequences*

(3.4.1)

$$\to C_3^{\ell_1}(G/A) \xrightarrow{\partial_3} C_2^{\ell_1}(G/A) \xrightarrow{\partial_2} C_1^{\ell_1}(G/A) \xrightarrow{\partial_1} C_0^{\ell_1}(G/A) \xrightarrow{\varepsilon} \mathbf{R} \to 0$$

(3.4.2)

$$\to C_3^{\ell_1}(\{A\}) \xrightarrow{\partial_3'} C_2^{\ell_1}(\{A\}) \xrightarrow{\partial_2'} C_1^{\ell_1}(\{A\}) \xrightarrow{\partial_1'} C_0^{\ell_1}(\{A\}) \xrightarrow{\varepsilon} \mathbf{R} \to 0$$

are a co-allowable pair of resolutions for $(G, A; \mathbf{R})$.

Proof. We define $\varphi_n \colon C_n^{\ell_1}(\{A\}) \to C_n^{\ell_1}(G/A)$ by

$$\lambda_n([\underbrace{\{A\}|\cdots|\{A\}}_{n}]) = [\underbrace{A|\cdots|A}_{n}].$$

We omit the rest of the proof. $\square$

THEOREM 3.5. *Let A be an amenable subgroup of G, and let $\varphi\colon A \hookrightarrow G$ be an inclusion map. Then the seminorm $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ induced by the complex*

$$C_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A$$

coincides with the canonical seminorm for every $\omega \in [0, \infty]$.

Proof. Let $\|\cdot\|_1(\omega)$ denote the canonical seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ and let $\|\cdot\|_1^s(\omega)$ the seminorm on it induced by the complex $C_n(A \xrightarrow{\varphi} G)$. By definition of the canonical seminorm, we have $\|\cdot\|_1^s(\omega) \leq \|\cdot\|_1(\omega)$. By Theorem 3.3, the canonical seminorm on $H_*^{\ell_1}(A \xrightarrow{\varphi} G)$ coincides with the seminorm induced by the complex $C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$. We define

$$\beta_n \colon C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A \to C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$$

by the formula $\beta_n(x, a) = (q_n x,\, q'_{n-1} a)$, where q_* (similarly q'_*) is the map defined in Corollary 2.5. It is clear that β_n is a chain map. Also it is easy to check that $\|\beta_n\| \leq 1$ for the norm $\|\cdot\|_1(\omega)$. Thus $\|\cdot\|_1(\omega) \leq \|\cdot\|_1^s(\omega)$. $\square$

From now on, we always distinguish a homology class from a chain by using brackets: for example, $[x]$ stands for a homology class while x stands for a cycle.

THEOREM 3.6. *Let A be an amenable subgroup of G and let $\varphi\colon A \hookrightarrow G$ be an inclusion homomorphism. Then, for every $n \geq 2$, the groups $H_n^{\ell_1}(G)$ and $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ are isometrically isomorphic with respect to the norm $\|\cdot\|_1$.*

Proof. It is enough for us to consider the sequences $\{C_*^{\ell_1}(G/A), \partial_*\}$ and $\{C_*^{\ell_1}(\{A\}), \partial'_*\}$ in Lemma 3.4. So we have complexes

$$C_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A \quad \text{and}$$

$$EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(\{A\})_A \bigoplus C_n^{\ell_1}(G/A)_G \bigoplus C_{n-1}^{\ell_1}(\{A\})_A.$$

Then the exact sequence

$$0 \to C_n^{\ell_1}(\{A\})_A \xrightarrow{i_n} EC_n(A \xrightarrow{\varphi} G) \xrightarrow{p_n} C_n(A \xrightarrow{\varphi} G) \to 0$$

induces a long exact sequence

$$\cdots \to H_n^{\ell_1}(A) \to H_n^{\ell_1}(G) \xrightarrow{H_n(p)} H_n^{\ell_1}(A \xrightarrow{\varphi} G) \to H_{n-1}^{\ell_1}(A) \to \cdots .$$

Since $H_*^{\ell_1}(A) = 0$, the groups $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ and $H_n^{\ell_1}(G)$ are isomorphic. Remark that the induced map $\varphi_n\colon C_n^{\ell_1}(\{A\})_A \to C_n^{\ell_1}(G/A)_G$ is defined as an inclusion map. Since it is clear that $H_n(p)$ has the norm $\|H_n(p)\| \leq 1$, we show $H_n(p)^{-1}$ has the norm $\|H_n(p)^{-1}\| \leq 1$. Let $(b,\, x,\, a) \in EC_n(A \xrightarrow{\varphi} G)$ be a cycle. By definition of boundary operator, we have

$$\partial'_n b - a = 0, \quad \partial_n x + a = 0, \quad \text{and} \quad \partial'_{n-1} a = 0.$$

Recall that $\partial'_n = id$ if n is even and $\partial'_n = 0$ if n is odd (see the formula (2.6)).

Let n be even: $\partial'_n = id$ and $\partial'_{n-1} = 0$. Then $a \in \ker(\partial'_{n-1}) = \mathrm{Im}(\partial'_n)$. So there is an element $c \in C_n^{\ell_1}(\{A\})_A$ such that $\partial'_n c = a$ and $\|c\|_1 = \|a\|_1$. Notice that $d(0,\, x + c,\, 0) = (0,\, \partial_n x + \partial'_n c,\, 0) = (0,\, 0,\, 0)$, and also

$$(0,\, x + c,\, 0) + d(0,\, 0,\, -c) = (c,\, x,\, \partial'_n c) = (b,\, x,\, a).$$

Thus $(H_n(p))^{-1}([x,\, a])$ is represented by a cycle $(0,\, x{+}c,\, 0) \in EC_n(A \xrightarrow{\varphi} G)$. Also

$$\begin{aligned}\|(H_n(p))^{-1}([x,\, a])\|_1 &\leq \|(0,\, x + c,\, 0)\|_1 = \|x + c\|_1 \\ &\leq \|x\|_1 + \|c\|_1 = \|x\|_1 + \|a\|_1 = \|(x,\, a)\|_1.\end{aligned}$$

This shows that $\|(H_n(p))^{-1}\| \leq 1$ for every even n.

Similarly, we can prove that $\|(H_n(p))^{-1}\| \leq 1$ for every odd n. □

Now we prove Gromov's equivalence theorem for a case of groups with respect to the norms $\|\cdot\|_1(\omega)$.

THEOREM 3.7. *Let $\varphi\colon A \to G$ be a group homomorphism. If A is amenable, then for every $n \geq 2$ the norms $\|\cdot\|_1(\omega)$ on $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ are equal for all $\omega \in [0, \infty]$.*

Proof. Let $\omega \geq 0$. Recall that $\|\cdot\|_1 = \|\cdot\|_1(0) \leq \|\cdot\|_1(\omega)$. So we prove that $\|\cdot\|_1(\omega) \leq \|\cdot\|_1$. Let $(x, a) \in C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ be a cycle. Then $0 = d(x, a) = (\partial_n x + \varphi_{n-1} a,\, -\partial'_{n-1} a)$ and so $\partial'_{n-1} a = 0$, where $\varphi_n\colon C_n^{\ell_1}(A)_A \to C_n^{\ell_1}(G)_G$ is an induced map from φ, and ∂_n and ∂'_n are the boundary operators on $C_n^{\ell_1}(G)_G$ and $C_n^{\ell_1}(A)_A$ respectively. Since A

is amenable, we may define the boundary operator ∂'_* by the formula in (2.6).

Let $n-1$ be even: $\partial'_{n-1} = id$. Then $\partial'_{n-1}a = 0$ gives $a = 0$. Thus

$$\|(x,a)\|_1(\omega) = \|x\|_1 + (1+\omega)\|a\|_1 = \|x\|_1 = \|x\|_1 + \|a\|_1 = \|(x,a)\|_1.$$

If $n-1$ is odd: $\partial'_n = id$ and $\partial'_{n-1}a = 0$. Then $a \in \ker(\partial'_{n-1}) = \mathrm{Im}(\partial'_n)$. So there is an element $a_n \in C_n^{\ell_1}(A)_A$ such that $\partial'_n a_n = a$ and $\|a_n\|_1 = \|\partial'_n a_n\|_1 = \|a\|_1$. Then we have

$$(x,a) + d(0,a_n) = (x,a) + (\varphi_n a_n, -\partial'_n a_n) = (x + \varphi_n a_n,\ 0).$$

So we have

$$\begin{aligned}\|[x,a]\|_1(\omega) &\le \|(x+\varphi_n a_n,\ 0)\|_1(\omega) = \|x+\varphi_n a_n\|_1 \le \|x\|_1 + \|\varphi_n a_n\|_1 \\ &\le \|x\|_1 + \|a_n\|_1 = \|x\|_1 + \|a\|_1 = \|(x,\ a)\|_1.\end{aligned}$$

Thus we have $\|[x,\ a]\|_1(\omega) \le \|[x,a]\|_1$ for every $\omega \ge 0$. By passing to the limits, we have $\|[x,\ a]\|_1(\omega) \le \|[x,\ a]\|_1$ for all $\omega \in [0,\infty]$. □

In the rest of this section, A is a subgroup of G and $\varphi\colon A \hookrightarrow G$ is an inclusion homomorphism. We describe the relative ℓ_1 homology of G modulo A from the point of view of the ordinary relative case.

DEFINITION 3.4. Let

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be a co-allowable pair for $(G, A; \mathbf{R})$ with an A-morphism $\varphi_n\colon V_n \to U_n$ as in Definition 3.1. If φ_n induces an injective map $\varphi_n\colon (V_n)_A \to (U_n)_G$, then this pair of resolutions together with the A-morphisms φ_* is said to be co-proper.

PROPOSITION 3.8. *The bar resolutions of G and A are co-proper.*

Proof. It is clear that the inclusion homomorphism $A \hookrightarrow G$ induces an injective A-morphism $C_*^{\ell_1}(A) \to C_*^{\ell_1}(G)$ which is clearly injective. It is easy to check that the induced map $C_*^{\ell_1}(A)_A \to C_*^{\ell_1}(G)_G$ is injective. □

Let a pair of resolutions

$$\cdots \to U_1 \to U_0 \to \mathbf{R} \to 0 \quad \text{and} \quad \cdots \to V_1 \to V_0 \to \mathbf{R} \to 0$$

be co-proper as in Definition 3.4. Remark that there is an exact sequence

$$0 \to (V_n)_A \hookrightarrow (U_n)_G \to (U_n)_G/(V_n)_A \to 0. \tag{3.1}$$

It is easy to check that the induced sequence

$$\cdots \to (U_2)_G/(V_2)_A \to (U_1)_G/(V_1)_A \to (U_0)_G/(V_0)_A \to 0 \tag{3.2}$$

is a complex. The n-th homology of the complex (3.2) is denoted by $H_n^{\ell_1}(G,A)$. The sequence (3.1) induces an exact sequence
(3.3)
$$\to H_{n+1}^{\ell_1}(G,A) \to H_n^{\ell_1}(A) \to H_n^{\ell_1}(G) \to H_n^{\ell_1}(G,A) \to H_{n-1}^{\ell_1}(A) \to .$$

Notice that the bar resolutions of G and A induces an exact sequence

$$(3.4) \qquad 0 \to C_*^{\ell_1}(A)_A \xrightarrow{i_*} C_*^{\ell_1}(G)_G \xrightarrow{p_*} C_*^{\ell_1}(G)_G/C_*^{\ell_1}(A)_A \to 0.$$

We denote the quotient space $C_*^{\ell_1}(G)_G/C_*^{\ell_1}(A)_A$ by $C_*^{\ell_1}(G,A)$. Thus there is a complex

$$(3.5) \qquad \cdots \to C_2^{\ell_1}(G,A) \to C_1^{\ell_1}(G,A) \to C_0^{\ell_1}(G,A) \to 0.$$

By following Gromov's definition [4] of a norm $\|\cdot\|_1(\theta)$ on the relative ℓ_1 homology of a pair of spaces (X,Y) with $Y \subset X$ for $\theta \in [0,\infty]$, we define a norm $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(G,A)$: first we define a norm $\|\cdot\|_1(\theta)$ on $C_*^{\ell_1}(G)_G$ by putting

$$\|x\|_1(\theta) = \|x\|_1 + \theta\|\partial x\|_1.$$

Then, using the quotient homomorphism $p_*\colon C_*^{\ell_1}(G)_G \to C_*^{\ell_1}(G,A)$, we define the norm $\|\bar{c}\|_1(\theta)$ of $\bar{c} \in C_*^{\ell_1}(G,A)$ as the quotient norm, so that $\|\bar{c}\|_1(\theta) = \inf \|c\|_1(\theta)$, where the infimum is taken over $c \in p_*^{-1}(\bar{c}) \subset C_*^{\ell_1}(G)_G$. Then there is a corresponding seminorm $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(G,A)$. By passing θ to the limits, we define $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(G,A)$ for all $\theta \in [0,\infty]$.

THEOREM 3.9. *There is an isomorphism*

$$H_n(\beta)\colon H_n^{\ell_1}(A \xrightarrow{\varphi} G) \to H_n^{\ell_1}(G,A)$$

which carries the canonical seminorm $\|\cdot\|_1(\omega)$ *on* $H_n^{\ell_1}(A \xrightarrow{\varphi} G)$ *to the seminorm* $\|\cdot\|_1(\theta)$ *on* $H_n^{\ell_1}(G,A)$ *for* $\omega = \theta \in [0,\infty]$ *as follows: for* $[x,a] \in H_n^{\ell_1}(A \xrightarrow{\varphi} G)$,

$$\frac{1}{n+2}\|[x,a]\|_1(\omega) \le \|H_n(\beta)([x,a])\|_1(\theta) \le \|[x,a]\|_1(\omega).$$

Proof. We consider the following complexes

$$C_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$$
$$EC_n(A \xrightarrow{\varphi} G) = C_n^{\ell_1}(A)_A \bigoplus C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A.$$

Also we consider the following diagram

$$\begin{array}{ccccccc} 0 \to C_n^{\ell_1}(A)_A & \xrightarrow{i_n} & EC_n(A \xrightarrow{\varphi} G) & \xrightarrow{p_n} & C_n(A \xrightarrow{\varphi} G) \to 0 \\ \gamma_n \downarrow & & \alpha_n \downarrow & & \beta_n \downarrow \\ 0 \to C_n^{\ell_1}(A)_A & \xrightarrow{j_n} & C_n^{\ell_1}(G)_G & \xrightarrow{q_n} & C_n^{\ell_1}(G, A) \to 0, \end{array}$$

where $\gamma_n(a') = a'$, $\alpha_n(a', x, a) = x$, and $\beta_n(x, a) = x + C_n^{\ell_1}(A)_A$. It is clear that the diagram is commutative. It induces the commutative diagram

$$\begin{array}{ccccccc} \to H_n^{\ell_1}(A) & \longrightarrow & H_n^{\ell_1}(G) & \longrightarrow & H_n^{\ell_1}(A \xrightarrow{\varphi} G) & \longrightarrow & H_{n-1}^{\ell_1}(A) \to \\ H_n(\gamma) \downarrow & & H_n(\alpha) \downarrow & & H_n(\beta) \downarrow & & H_{n-1}(\gamma) \downarrow \\ \to H_n^{\ell_1}(A) & \longrightarrow & H_n^{\ell_1}(G) & \longrightarrow & H_n^{\ell_1}(G, A) & \longrightarrow & H_{n-1}^{\ell_1}(A) \to . \end{array}$$

Remark that $H_*(\gamma)$ and $H_*(\alpha)$ are isometric isomorphisms. So $H_n(\beta)$ is an isomorphism.

Let $\omega = \theta \geq 0$. Let $(x, a) \in C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ be a cycle. Then $d(x, a) = (\partial x + a, \, -\partial' a) = 0$ and $\partial x = -a$. Thus

$$\begin{aligned} \|\beta_n(x, a)\|_1(\theta) &= \|x + C_n^{\ell_1}(A)_A\|_1(\theta) \leq \|x\|_1(\theta) = \|x\|_1 + \theta\|\partial x\|_1 \\ &\leq \|x\|_1 + (1+\omega)\|a\|_1 = \|(x, a)\|_1(\omega). \end{aligned}$$

On the other hand, let $x \in C_n^{\ell_1}(G)_G$ be a relative cycle so that $\partial x \in C_{n-1}^{\ell_1}(A)_A$. Then $(x, -\partial x) \in C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$ and $d(x, -\partial x) = (\partial x - \partial x, \, \partial' \partial x) = (0, 0)$. It is easy to check that $(H_n^{\ell_1}(\beta))^{-1}[x]$ is represented by a cycle $(x, -\partial x)$. Also

$$\begin{aligned} \|(x, -\partial_n x)\|_1(\omega) &= \|x\|_1 + (1+\omega)\|\partial_n x\|_1 = \|x\|_1 + \|\partial_n x\|_1 + \omega\|\partial_n x\|_1 \\ &\leq (n+2)\|x\|_1 + \omega\|\partial_n x\|_1 \leq (n+2)(\|x\|_1 + \theta\|\partial_n x\|_1) \\ &= (n+2)\|x\|_1(\theta). \end{aligned}$$

Hence we have $\frac{1}{n+2}\|\cdot\|_1(\omega) \leq \|H_n(\beta)(\cdot)\|_1(\theta) \leq \|\cdot\|_1(\omega)$. □

4. Relative ℓ_1 homology of spaces

In this section every space is a connected countable cellular space.

Recall that ℓ_1 homology of a space X, denoted by $H_n^{\ell_1}(X)$, is the homology of the complex of Banach spaces

$$\cdots \to C_3^{\ell_1}(X) \xrightarrow{\partial_3} C_2^{\ell_1}(X) \xrightarrow{\partial_2} C_1^{\ell_1}(X) \xrightarrow{\partial_1} C_0^{\ell_1}(X) \to 0. \tag{4.1}$$

In [6] Matsumoto and Morita stated that *it is plausible that* $H_*^{\ell_1}(X)$ *depends only on its fundamental group* $\pi_1 X$. In the next theorem, we prove that the $H_*^{\ell_1}(X)$ does depend only on its fundamental group $\pi_1 X$.

THEOREM 4.1. *The group* $H_*^{\ell_1}(\pi_1 X)$ *is canonically isomorphic with* $H_*^{\ell_1}(X)$. *This isomorphism carries the canonical seminorm on* $H_*^{\ell_1}(\pi_1 X)$ *to the seminorm on* $H_*^{\ell_1}(X)$.

Proof. Let $\pi\colon \widetilde{X} \to X$ be the universal covering of X, so that $\pi_1 X$ acts freely on $\widetilde{X}$ and $\widetilde{X}/\pi_1 X = X$. The action of $\pi_1 X$ on $\widetilde{X}$ induces the action on $C_*^{\ell_1}(\widetilde{X})$ and thus turns them into bounded $\pi_1 X$-modules. We show that these $\pi_1 X$-modules are relatively projective. Let $\widetilde{X}_0 \subset \widetilde{X}$ consist one element from each $\pi_1 X$-orbit. As it is well known, the complex $C_*(\widetilde{X})$ is free on all simplexes the first vertices of which are in $\widetilde{X}_0$. Then, by Lemma 2.1, these $\pi_1 X$-modules $C_*^{\ell_1}(\widetilde{X})$ are relatively projective. We consider the sequence

$$\cdots \to C_2^{\ell_1}(\widetilde{X}) \to C_1^{\ell_1}(\widetilde{X}) \to C_0^{\ell_1}(\widetilde{X}) \to \mathbf{R} \to 0. \tag{4.1.1}$$

Since $\widetilde{X}$ is simply connected, $H_*^{\ell_1}(\widetilde{X}) = 0$ and so the sequence in (4.1.1) is exact. Thus the sequence (4.1.1) is a strong relatively projective $\pi_1 X$-resolution of the trivial $\pi_1 X$-module $\mathbf{R}$, where the fact that this resolution is strong is shown in the proof of Theorem 2.4 in [5]. Note that the map $\pi_*\colon C_*^{\ell_1}(\widetilde{X}) \to C_*^{\ell_1}(X)$ establishes an isometric isomorphism between $\left(C_*^{\ell_1}(\widetilde{X})\right)_{\pi_1 X}$ and $C_*^{\ell_1}(X)$ and it commutes with the boundary operators. Thus, as topological vector spaces, the ℓ_1 homology group of $\pi_1 X$ coincides with the homology of the complex

$$\cdots \to C_2^{\ell_1}(X) \to C_1^{\ell_1}(X) \to C_0^{\ell_1}(X) \to 0.$$

Now we prove that the isomorphism constructed between $H_*^{\ell_1}(\pi_1 X)$ and $H_*^{\ell_1}(X)$ is an isometry. Let $\|\cdot\|_1$ denote the canonical seminorm on $H_*^{\ell_1}(\pi_1 X)$ and $\|\cdot\|_1^s$ the seminorm on $H_*^{\ell_1}(X)$. By definition of the canonical seminorm, we have $\|\cdot\|_1 \geq \|\cdot\|_1^s$, so that it remains for us to prove that $\|\cdot\|_1 \leq \|\cdot\|_1^s$. Since the canonical seminorm is achieved by the bar resolution, it suffices to construct a $\pi_1 X$-morphism of the resolution (4.1.1) into the bar resolution of $\pi_1 X$ consisting of maps of norm ≤ 1.

Let $\sigma\colon \Delta_n \to \widetilde{X}$ be a singular n-simplex the first vertex of which is in $\widetilde{X}_0$, where $\Delta_n = [v_0, \dots, v_n]$. We define a map $f_n\colon C_n^{\ell_1}(\widetilde{X}) \to C_n^{\ell_1}(\pi_1 X)$ by $f_n(\sigma) = g_0[g_1|g_2|\cdots|g_n]$, where $g_i \in \pi_1 X$ such that $\sigma(v_i) = g_i \cdots g_0 \widetilde{X}_0$. It is easy to see that f_n commutes with the boundary operators and so it determines a $\pi_1 X$-morphism of the resolutions

$$\begin{array}{ccccccccc} \longrightarrow & C_1^{\ell_1}(\widetilde{X}) & \longrightarrow & C_0^{\ell_1}(\widetilde{X}) & \longrightarrow & \mathbf{R} & \longrightarrow & 0 \\ \downarrow & \downarrow f_1 & & \downarrow f_0 & & \downarrow id_{\mathbf{R}} & & \\ \longrightarrow & C_1^{\ell_1}(\pi_1 X) & \longrightarrow & C_0^{\ell_1}(\pi_1 X) & \longrightarrow & \mathbf{R} & \longrightarrow & 0 \end{array}$$

extending $id_{\mathbf{R}}$. By definition, it is clear that $\|f_*\| \leq 1$ and so $\|\cdot\|_1 \leq \|\cdot\|_1^s$. □

COROLLARY 4.2. *Let $\alpha\colon X_1 \to X_2$ be a continuous map such that the induced homomorphism $\alpha_*\colon \pi_1(X_1) \to \pi_1(X_2)$ is a surjection with an amenable kernel. Then the homomorphism $H_n(\alpha)\colon H_n^{\ell_1}(X_1) \to H_n^{\ell_1}(X_2)$ is an isometric isomorphism with respect to the norm $\|\cdot\|_1$ for every $n \geq 0$.*

Proof. This follows from Theorem 2.8 and Theorem 4.1. □

Now we define relative ℓ_1 homology of spaces.

DEFINITION 4.1. Let $\varphi\colon Y \to X$ be a continuous map of spaces. The mapping cone and the mapping cylinder of the chain complexes induced by φ are defined as follows:

$$C_n(Y \xrightarrow{\varphi} X) = C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y)$$
$$EC_n(Y \xrightarrow{\varphi} X) = C_n^{\ell_1}(Y) \bigoplus C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y),$$

where the boundary operators on $C_n(Y \xrightarrow{\varphi} X)$ and on $EC_n(Y \xrightarrow{\varphi} X)$ are defined by the same formulas in Definition 3.2.

Notice that there are complexes

$$(4.2)\quad \cdots \xrightarrow{d_3} C_2(Y \xrightarrow{\varphi} X) \xrightarrow{d_2} C_1(Y \xrightarrow{\varphi} X) \xrightarrow{d_1} C_0(Y \xrightarrow{\varphi} X) \to 0$$
$$(4.3)\quad \cdots \xrightarrow{d_3} EC_2(Y \xrightarrow{\varphi} X) \xrightarrow{d_2} EC_1(Y \xrightarrow{\varphi} X) \xrightarrow{d_1} EC_0(Y \xrightarrow{\varphi} X) \to 0.$$

DEFINITION 4.2. The n-th homology of the complex in (4.2) is called *the n-th relative ℓ_1 homology of X modulo Y* and is denoted by $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$. We denote the n-th homology of the complex in (4.3) by $H_n^{\ell_1}(EC(Y \xrightarrow{\varphi} X))$.

We define the norm $\|\cdot\|_1$ on $C_*(Y \xrightarrow{\varphi} X)$ (similarly on $EC_*(Y \xrightarrow{\varphi} X)$) by

$$\|(x, a)\|_1 = \|x\|_1 + \|a\|_1.$$

Also for every $\omega \geq 0$, we define a norm $\|\cdot\|_1(\omega)$ on $C_*(Y \xrightarrow{\varphi} X)$ by

$$\|(x, a)\|_1(\omega) = \|x\|_1 + (1+\omega)\|a\|_1.$$

There is the corresponding seminorm $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$. We define these norms $\|\cdot\|_1(\omega)$ on $H_*^{\ell_1}(Y \xrightarrow{\varphi} X)$ for all $\omega \in [0, \infty]$ by passing ω to the limits.

THEOREM 4.3. *Let $\varphi\colon Y \to X$ be a continuous map. Then the natural inclusion map $\rho_n\colon C_n^{\ell_1}(X) \to EC_n(Y \xrightarrow{\varphi} X)$ induces an isometric isomorphism $H_n(\rho)\colon H_n^{\ell_1}(X) \to H_n^{\ell_1}(EC(Y \xrightarrow{\varphi} X))$ with respect to the norm $\|\cdot\|_1$.*

We can prove Theorem 4.3 by the same method as the proof of Theorem 3.2.

Notice that there is an exact sequence

$$0 \to C_n^{\ell_1}(Y) \to EC_n(Y \xrightarrow{\varphi} X) \to C_n(Y \xrightarrow{\varphi} X) \to 0.$$

It induces a long exact sequence

$$\cdots \to H_n^{\ell_1}(Y) \to H_n^{\ell_1}(X) \to H_n^{\ell_1}(Y \xrightarrow{\varphi} X) \to H_{n-1}^{\ell_1}(Y) \to \cdots.$$

THEOREM 4.4. *Let $\varphi\colon Y \to X$ be a continuous map and $\varphi_*\colon \pi_1 Y \to \pi_1 X$ be an induced map. Then $H_n^{\ell_1}(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ and $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ are isometrically isomorphic. This isomorphism carries the canonical seminorm $\|\cdot\|_1(\omega)$ on $H_n^{\ell_1}(\pi_1 Y \xrightarrow{\varphi_*} \pi_1 X)$ to the seminorm on $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ for every $\omega \in [0, \infty]$.*

Proof. Let G and A denote $\pi_1 X$ and $\pi_1 Y$ respectively. By Theorem 3.3, the canonical seminorm on $H_n^{\ell_1}(A \xrightarrow{\varphi_*} G)$ is induced by the complex $C_n(A \xrightarrow{\varphi_*} G) = C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A$.

Let $\pi_1\colon \widetilde{X} \to X$ and $\pi_2\colon \widetilde{Y} \to Y$ be the universal coverings. As shown in Theorem 4.1, we can identify

$$C_n^{\ell_1}(X) \bigoplus C_{n-1}^{\ell_1}(Y) = \big(C_n^{\ell_1}(\widetilde{X})\big)_{\pi_1 X} \bigoplus \big(C_{n-1}^{\ell_1}(\widetilde{Y})\big)_{\pi_1 Y}.$$

By Theorem 2.2 and Theorem 4.1, there are chain maps

$$C_n^{\ell_1}(G)_G \underset{\zeta_n}{\overset{\alpha_n}{\rightleftarrows}} \big(C_n^{\ell_1}(\widetilde{X})\big)_{\pi_1 X} \quad \text{and} \quad C_n^{\ell_1}(A)_A \underset{\eta_n}{\overset{\gamma_n}{\rightleftarrows}} \big(C_n^{\ell_1}(\widetilde{Y})\big)_{\pi_1 Y}.$$

We define the maps

$$C_n^{\ell_1}(G)_G \bigoplus C_{n-1}^{\ell_1}(A)_A \underset{\Psi_n}{\overset{\Phi_n}{\rightleftarrows}} \big(C_n^{\ell_1}(\widetilde{X})\big)_{\pi_1 X} \bigoplus \big(C_{n-1}^{\ell_1}(\widetilde{Y})\big)_{\pi_1 Y}$$

by $\Phi_n(u,v) = (\alpha_n u, \gamma_{n-1} v)$ and $\Psi_n(u',v') = (\zeta_n u', \eta_{n-1} v')$. It is easy to check that Φ_n and Ψ_n are chain maps such that $\Psi_n \Phi_n$ is chain homotopic to *id*. Also it is easy to check that $\|\Phi_n\| \le 1$ and $\|\Psi_n\| \le 1$ with respect to the norm $\|\cdot\|_1(\omega)$ for every $\omega \in [0,\infty]$. □

COROLLARY 4.5. *Let $\varphi\colon Y \to X$ be a continuous map of spaces such that the fundamental group $\pi_1 Y$ is amenable. Then the norms $\|\cdot\|_1(\omega)$ on $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ are equal for every $\omega \in [0,\infty]$ and for every $n \ge 2$.*

Proof. It follows from Theorem 4.4 and Theorem 3.7. □

Let Y and $Y \subset X$ be a pair of spaces and let $\varphi\colon Y \to X$ be a natural inclusion map. As in the ordinary relative homology, the injective homomorphism $i_n\colon C_n^{\ell_1}(Y) \hookrightarrow C_n^{\ell_1}(X)$ induces an exact sequence

$$0 \to C_n^{\ell_1}(Y) \hookrightarrow C_n^{\ell_1}(X) \to C_n^{\ell_1}(X)/C_n^{\ell_1}(Y) \to 0. \tag{4.4}$$

We denote $C_n^{\ell_1}(X)/C_n^{\ell_1}(Y)$ by $C_n^{\ell_1}(X,Y)$. Notice that there is a complex

$$\cdots \to C_3^{\ell_1}(X,Y) \to C_2^{\ell_1}(X,Y) \to C_1^{\ell_1}(X,Y) \to 0. \tag{4.5}$$

The n-th homology of the complex in (4.5) is denoted by $H_n^{\ell_1}(X,Y)$.

The exact sequence (4.4) induces a long exact sequence

$$\to H_{n+1}^{\ell_1}(X,Y) \to H_n^{\ell_1}(Y) \to H_n^{\ell_1}(X) \to H_n^{\ell_1}(X,Y) \to H_{n-1}^{\ell_1}(Y) \to$$

For every $\theta \in [0,\infty]$, we define a seminorm $\|\cdot\|_1(\theta)$ on $H_*^{\ell_1}(X,Y)$ by the same formula as we defined on $H_*^{\ell_1}(G,A)$.

THEOREM 4.6. *There is an isomorphism*

$$H_n(\beta)\colon H_n^{\ell_1}(Y \xrightarrow{\varphi} X) \to H_n^{\ell_1}(X,Y)$$

which carries the canonical seminorm $\|\cdot\|_1(\omega)$ on $H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$ to the seminorm $\|\cdot\|_1(\theta)$ on $H_n^{\ell_1}(X,Y)$ for $\omega = \theta \in [0,\infty]$ as follows: for $[x,a] \in H_n^{\ell_1}(Y \xrightarrow{\varphi} X)$

$$\frac{1}{n+2}\|[x,a]\|_1(\omega) \le \|H_n(\beta)([x,a])\|_1(\theta) \le \|[x,a]\|_1(\omega).$$

The proof of Theorem 4.6 is referred to Theorem 3.9.

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