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GENERALIZATION OF WATSON'S THEOREM FOR DOUBLE SERIES

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Abstract. In 1965, Bhatt and Pandey obtained the Watson's theorem for double series by using Dixon's theorem on the sum of a ${}_3F_2$. The aim of this paper is to derive twenty three results for double series closely related to the Watson's theorem for double series obtained by Bhatt and Pandey. The results are derived with the help of twenty three summation formulas closely related to the Dixon's theorem on the sum of a ${}_3F_2$ obtained in earlier work by Lavoie, Grondin, Rathie and Arora.

1. Introduction

The generalized Kampé de Fériet function introduced by Srivastava and Panda [5] is defined and represented in the following manner:

$$
\begin{aligned}
&F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k)\\(\alpha_l):(\beta_m);(\gamma_n)\end{matrix}\middle| x,y\right] \\
&= \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}
\end{aligned}
\tag{1.1}
$$

where, for convergence,

(i) $p+q<l+m+1$, $p+k<l+n+1$, $|x|<\infty$, $|y|<\infty$, or

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(ii) $p+q=l+m+1$, $p+q=l+n+1$, and

$$\begin{cases} |x|^{\frac{1}{p-l}}+|y|^{\frac{1}{p-l}}<1, & \text{if } p>l \\ \max\{|x|,|y|\}<1, & \text{if } p\le l. \end{cases}$$

(1.1) reduces to the Kampé de Fériet function [3] in the special case $q=k$ and $m=n$.

In 1965, Bhatt and Pandey [2] obtained the Watson's theorem for double series by using Dixon's theorem on the sum of a ${}_3F_2$ with unit argument.

In 1994, Lavoie, Grondin, Rathie and Arora [4] generalized the Dixon's theorem which is given in the next section and they have obtained twenty three results closely related to the Dixon's theorem. In the same paper, they have obtained a large number of limiting cases of their results.

The aim of this paper is to derive twenty three results for the double series, in the form of a single result, closely related to the Watson's theorem for the following double series obtained earlier by Bhatt and Pandey [2]:

(1.2)

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1,a_2:b_1;b_1' \\ \frac{1}{2}(1+a_1+a_2),2b_1+2b_1':-;- \end{matrix}\middle|1,1\right]$$
$$=\frac{\Gamma(\frac{1}{2})\Gamma(b_1+b_1'+\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a_1+\frac{1}{2}a_2)\Gamma(\frac{1}{2}-\frac{1}{2}a_1-\frac{1}{2}a_2+b_1+b_1')}{\Gamma(\frac{1}{2}a_1+\frac{1}{2})\Gamma(\frac{1}{2}a_2+\frac{1}{2})\Gamma(\frac{1}{2}+b_1+b_1'-\frac{1}{2}a_1)\Gamma(\frac{1}{2}+b_1+b_1'-\frac{1}{2}a_2)}.$$

2. Results required

The following results will be required in our present investigation.

Transformation formula [1]:

(2.1)
$${}_3F_2\left[\begin{matrix} a,b,c \\ e,f \end{matrix}\middle|1\right]$$
$$=\frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}{}_3F_2\left[\begin{matrix} e-a,f-a,s \\ s+b,s+c \end{matrix}\middle|1\right],$$

where $s=e+f-a-b-c$ and $\Re e(s)>0$.

Generalized Dixon's theorem [4]:
(2.2)

$$
{}_3F_2\left[\begin{matrix} a, b, c \\ 1+a-b+i, 1+a-c+i+j \end{matrix}\middle| 1\right]
= \frac{2^{-2c+i+j}\Gamma(1+a-b+i)\Gamma(1+a-c+i+j)\Gamma(b-\frac{i}{2}-\frac{|i|}{2})}{\Gamma(a-2c+i+j+1)\Gamma(a-b-c+i+j+1)}
\times \frac{\Gamma\left(c-\frac{1}{2}(i+j+|i+j|)\right)}{\Gamma(b)\Gamma(c)}
\times \left\{A_{i,j}\frac{\Gamma\left(\frac{1}{2}a-c+\frac{1}{2}+\left[\frac{i+j+1}{2}\right]\right)\Gamma(\frac{1}{2}a-b-c+1+i+\left[\frac{j+1}{2}\right])}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}a-b+1+\left[\frac{i}{2}\right])}
+ B_{i,j}\frac{\Gamma(\frac{1}{2}a-c+1+\left[\frac{i+j}{2}\right])\Gamma(\frac{1}{2}a-b-c+\frac{3}{2}+i+\left[\frac{j}{2}\right])}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}a-b+\frac{1}{2}+\left[\frac{i+1}{2}\right])}\right\}
$$

for $i = 0, \pm1, \pm2, \pm3$ and $j = 0, 1, 2, 3$. Also $\Re e(a-2b-2c) > -2-2i-j$. $[x]$ is the greatest integer less than or equal to x and its absolute value is denoted by $|x|$. The coefficients $A_{i,j}$ and $B_{i,j}$ appear in the tables at the end of the paper.

3. Main result

The result to be proved is
(3.1)

$$
F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, a_2 : b_1; b_1' \\ 2b_1+2b_1'+j, \frac{1}{2}(1+a_1+a_2+i) : -; - \end{matrix}\middle| 1, 1\right]
= \frac{\Gamma(2b_1+2b_1'+j)\Gamma(\frac{a_1}{2}+\frac{a_2}{2}+\frac{i}{2}+\frac{1}{2})2^{-2b_1-2b_1'+a_1+a_2-j-1}}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1+b_1'+j)\Gamma(\frac{1}{2}-\frac{a_1}{2}+\frac{a_2}{2}+\frac{i}{2})}
\times \frac{\Gamma(2b_1+2b_1'-\frac{a_1}{2}-\frac{a_2}{2}+\frac{i}{2}+j+\frac{1}{2})\Gamma(b_1+b_1'+j+\frac{(-a_1+a_2+i+1)}{2})}{\Gamma(2b_1+2b_1'-\frac{a_1}{2}-\frac{a_2}{2}+\frac{i}{2}+j+\frac{1}{2})}
\times \frac{\Gamma(\frac{1}{2}-\frac{a_1}{2}+\frac{a_2}{2}-\frac{|i|}{2})\Gamma(b_1+b_1'-\frac{a_1}{2}-\frac{a_2}{2}+\frac{j}{2}+\frac{1}{2}-\frac{|i+j|}{2})}{\Gamma(b_1+b_1'-\frac{a_1}{2}+\frac{a_2}{2}+\frac{i}{2}+j+\frac{1}{2})}
\times \left\{A_{ij}\frac{\Gamma(\frac{a_1}{2}-\frac{j}{2}-1+\left[\frac{j+1}{2}\right])\Gamma(\frac{a_2}{2}-\frac{i}{2}-\frac{j}{2}+\left[\frac{i+j+1}{2}\right])}{\Gamma(b_1+b_1'+\frac{j}{2}-\frac{a_1}{2}+\frac{1}{2})\Gamma(b_1+b_1'-\frac{a_2}{2}-\frac{i}{2}+\frac{j}{2}+\frac{1}{2}+\left[\frac{i}{2}\right])}\right.
$$

$$+ B_{ij}\frac{\Gamma(\frac{a_1}{2}-\frac{j}{2}+\frac{1}{2}+[\frac{j}{2}])\Gamma(\frac{a_2}{2}-\frac{i}{2}-\frac{j}{2}+\frac{1}{2}+[\frac{i+j}{2}])}{\Gamma(b_1+b_1'+\frac{j}{2}-\frac{a_1}{2})\Gamma(b_1+b_1'-\frac{a_2}{2}-\frac{i}{2}+\frac{j}{2}+[\frac{i+1}{2}])}\Bigg\}$$

for $i=-3,-2,-1,0,1,2$ and $j=0,1,2,3$.

The coefficients $A_{i,j}$ and $B_{i,j}$ can be obtained from the tables of A_{ij} and B_{ij} given at the end of this paper by replacing a by $2b_1+2b_1'-a_1+j$, b by $\frac{1}{2}-\frac{1}{2}a_1+\frac{1}{2}a_2+\frac{1}{2}i$ and c by $b_1+b_1'-\frac{1}{2}a_1-\frac{1}{2}a_2+\frac{1}{2}i+j+\frac{1}{2}$, respectively.

4. Derivation

First we shall prove that

$$(4.1)\quad \begin{aligned} &F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1,a_2:b_1,b_1' \\ c_1,c_2:-,- \end{matrix}\Big| 1,1\right] \\ &=\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b_1-b_1')} \\ &\quad\times {}_3F_2\left[\begin{matrix} c_1-a_1,c_2-a_1,c_1+c_2-a_1-a_2-b_1-b_1'; \\ c_1+c_2-a_1-a_2,c_1+c_2-a_1-b_1-b_1'; \end{matrix}\Big| 1\right] \end{aligned}$$

PROOF. By noting $(\alpha)_{m+n}=(\alpha+m)_n(\alpha)_m$ and starting with (1.1), the left hand side of (4.1) is

$$\begin{aligned} &F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1,a_2;b_1,b_1' \\ c_1,c_2;-,- \end{matrix}\Big| 1,1\right] \\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}(b_1)_m(b_1')_n}{(c_1)_{m+n}(c_2)_{m+n}m!n!} \\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a_1+m)_n(a_1)_m(a_2+m)_n(a_2)_m(b_1)_m(b_1')_n}{(c_1+m)_n(c_1)_m(c_2+m)_n(c_2)_m m!n!} \\ &=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m(b_1)_m}{(c_1)_m(c_2)_m m!}\sum_{n=0}^{\infty}\frac{(a_1+m)_n(a_2+m)_n(b_1')_n}{(c_1+m)_n(c_2+m)_n n!} \\ &=\sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m(b_1)_m}{(c_1)_m(c_2)_m m!}\,{}_3F_2\left[\begin{matrix} a_1+m,a_2+m,b_1' \\ c_1+m,c_2+m \end{matrix}\Big| 1\right] \end{aligned}$$

Now using the second fundamental transformation formula (2.1), we get

$$= \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m (b_1)_m}{(c_1)_m (c_2)_m m!}$$
$$\times \frac{\Gamma(c_1+m)\Gamma(c_2+m)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1+m)\Gamma(c_1+c_2-a_1-b_1'+m)\Gamma(c_1+c_2-a_1-a_2)}$$
$$\times {}_3F_2\left[\begin{matrix} c_1-a_1, c_2-a_1, c_1+c_2-a_1-a_2-b_1' \\ c_1+c_2-a_1-b_1'+m, c_1+c_2-a_1-a_2 \end{matrix} \Big| 1\right]$$
$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)}$$
$$\times \sum_{m=0}^{\infty} \frac{(a_2)_m (b_1)_m}{m!(c_1+c_2-a_1-b_1')_m}$$
$$\times {}_3F_2\left[\begin{matrix} c_1-a_1, c_2-a_1, c_1+c_2-a_1-a_2-b_1' \\ c_1+c_2-a_1-b_1'+m, c_1+c_2-a_1-a_2 \end{matrix} \Big| 1\right]$$
$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_2)_m (b_1)_m}{m!(c_1+c_2-a_1-b_1')_m}$$
$$\times \frac{(c_1-a_1)_n (c_2-a_1)_n (c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-b_1'+m)_n (c_1+c_2-a_1-a_2)_n n!}.$$

But it is easy to see that

$$(c_1+c_2-a_1-b_1'+m)_n (c_1+c_2-a_1-b_1')_m$$
$$=(c_1+c_2-a_1-b_1'+n)_m (c_1+c_2-a_1-b_1')_n.$$

$$F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, a_2 : b_1, b_1' \\ c_1, c_2 : -, - \end{matrix} \Big| 1, 1\right]$$
$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)}$$
$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a_2)_m (b_1)_m}{m!(c_1+c_2-a_1-b_1'+n)_m}$$
$$\times \frac{(c_1-a_1)_n (c_2-a_1)_n (c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-b_1')_n (c_1+c_2-a_1-a_2)_n n!}$$

$$
\begin{aligned}
=&\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)}\\
&\times\sum_{n=0}^{\infty}\frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-b_1')_n(c_1+c_2-a_1-a_2)_n n!}\\
&\times\sum_{m=0}^{\infty}\frac{(a_2)_m(b_1)_m}{(c_1+c_2-a_1-b_1'+n)_m m!}\\
=&\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)}\\
&\times\sum_{n=0}^{\infty}\frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-b_1')_n(c_1+c_2-a_1-a_2)_n n!}\\
&\times{}_2F_1\left[\begin{matrix}a_2,b_1\\ c_1+c_2-a_1-b_1'+n\end{matrix}\Bigg|1\right].
\end{aligned}
$$

Using the well-known Gauss's theorem [1, p. 2], we have

$$
\begin{aligned}
=&\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b_1')\Gamma(c_1+c_2-a_1-a_2)}\\
&\times\sum_{n=0}^{\infty}\frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b_1')_n}{(c_1+c_2-a_1-b_1')_n(c_1+c_2-a_1-a_2)_n n!}\\
&\times\frac{\Gamma(c_1+c_2-a_1-b_1'+n)\Gamma(c_1+c_2-a_1-a_2-b_1-b_1'+n)}{\Gamma(c_1+c_2-a_1-b_1-b_1'+n)\Gamma(c_1+c_2-a_1-a_2-b_1'+n)}\\
=&\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b_1-b_1')}\\
&\times\sum_{n=0}^{\infty}\frac{(c_1-a_1)_n(c_2-a_1)_n(c_1+c_2-a_1-a_2-b_1-b_1')_n}{(c_1+c_2-a_1-a_2)_n(c_1+c_2-a_1-b_1-b_1')_n n!}\\
=&\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b_1-b_1')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b_1-b_1')}\\
&\times{}_3F_2\left[\begin{matrix}c_1-a_1,c_2-a_1,c_1+c_2-a_1-a_2-b_1-b_1'\\ c_1+c_2-a_1-a_2,c_1+c_2-a_1-b_1-b_1'\end{matrix}\Bigg|1\right].
\end{aligned}
$$

So we arrive at the right hand side of (4.1). This completes the proof of (4.1). □

Now, in (4.1), if we take $c_1=2b_1+2b_1'+j$ and $c_2=\frac{1}{2}(1+a_1+a_2+i)$, then for $i = 0, \pm 1, \pm 2, \pm 3$ and $j = 1, 2, 3$ we have

$$\begin{aligned}
&F^{2:1;1}_{2:0;0}\left[\begin{matrix} a_1, a_2 : b_1; b_1' \\ 2b_1 + 2b_1' + j, \frac{1}{2}(1 + a_1 + a_2 + i) : -; - \end{matrix} \middle| 1, 1\right] \\
=& \frac{\Gamma(2b_1 + 2b_1' + j)\Gamma(\frac{1}{2} + \frac{a_1}{2} + \frac{a_2}{2} + \frac{i}{2})}{\Gamma(a_1)\Gamma(2b_1 + 2b_1' - \frac{a_1}{2} - \frac{a_2}{2} + \frac{i}{2} + j + \frac{1}{2})} \\
&\times \frac{\Gamma(b_1 + b_1' - \frac{a_1}{2} - \frac{a_2}{2} + \frac{i}{2} + j + \frac{1}{2})}{\Gamma(b_1 + b_1' - \frac{a_1}{2} + \frac{a_2}{2} + \frac{i}{2} + j + \frac{1}{2})} \\
&\times {}_3F_2\left[\begin{matrix} 2b_1 + 2b_1' + j - a_1, A, b_1 + b_1' + j + B \\ 2b_1 + 2b_1' + j + B, b_1 + b_1' + A \end{matrix} \middle| 1\right],
\end{aligned}$$

where $A = \frac{(-a_1+a_2+i+1)}{2}$ and $B = \frac{(-a_1-a_2+i+1)}{2}$.

Now the ${}_3F_2$ on the right-hand side can be evaluated with the help of generalized Dixon's theorem (2.2) and after some simplification, we arrive at the desired result (3.1).

For $i = j = 0$, we get a known result due to Bhatt and Pandey's formula (1.2).

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