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SUFFICIENT CONDITIONS FOR OPTIMALITY IN DIFFERENTIAL INCLUSION UNDER STATE CONSTRAINTS

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ABSTRACT. We prove the sufficient conditions for optimality in differential inclusion problem by using the value function. For this purpose, we assume at first that the value function is locally Lipschitz. Secondly, without this assumption, we use the viability theory.

1. Introduction

In this article, we give the sufficient conditions for optimality of the following problem:

$$\min \left\{ \psi(x(T)) \;\middle|\; \begin{array}{l} x'(t) \in F(t, x(t)) \;\text{ a.e. in } [0,T], \\ x(0) = \xi_0, \\ g(t, x(t)) \leq 0 \;\text{ in } [0,T] \end{array} \right\}$$

For the necessary conditions, see [5] (Theorem 3.2.6, p.122). We define the feedback map by: (for the notations, see the next section)

$$G(t,x) := \{v \in F(t,x) \mid D_\uparrow V(t,x)(1,v) \leq 0\}.$$

By supposing that the value function is locally Lipschitz on the domain of definition, we prove that $x(\cdot)$ is optimal for the initial data (t_0, x_0) if and only if $x(\cdot)$ is the solution of the differential inclusion

$$\left\{ \begin{array}{l} x'(t) \in G(t, x(t)) \;\text{ a.e. in } [t_0, T] \\ x(t_0) = x_0. \end{array} \right.$$

The sufficient condition for optimality can also be formulated in the following way. Suppose once again that the value function is locally

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Lipschitz on the domain of definition. If for almost all t, there exists $q(t) \in \mathbf{R}^n$ such that

$$(1) \quad \begin{cases} \langle q(t), x'(t)\rangle = H(t, x(t), q(t)) \\ \Big\langle \Big(H(t, x(t), q(t)), -q(t)\Big), (1, x'(t))\Big\rangle \geq D_{\downarrow}V(t, x(t))(1, x'(t)), \end{cases}$$

then $x(\cdot)$ is optimal.

If the value function V is of class C^1, it verifies the Hamilton-Jacobi-Bellman equation:

$$H(t, x(t), -\frac{\partial V}{\partial x}(t, x(t))) = \frac{\partial V}{\partial t}(t, x(t)).$$

Furthermore, if

$$\langle x'(t), -\frac{\partial V}{\partial x}(t, x(t))\rangle = H(t, x(t), -\frac{\partial V}{\partial x}(t, x(t))),$$

then x is optimal because the above conditions (1) are satisfied. This is also the consequence of Verification Theorem ([6]). Therefore we can say that the above conditions (1) are more general than those of Verification Theorem.

In the above two cases, we assumed that the value function is locally Lipschitz. Without this assumption, by using the viability theory, we prove the sufficient conditions.

2. Optimal feedback and sufficient condition

2.1. Optimal feedback

Consider the following problem:

$$\min \left\{ \psi(x(T)) \;\middle|\; \begin{array}{l} x'(t) \in F(t, x(t)) \text{ a.e. in } [0, T], \\ x(0) = \xi_0, \\ g(t, x(t)) \leq 0 \text{ in } [0, T] \end{array} \right\}$$

where

$$\begin{array}{l} F : [0, T] \times \mathbf{R}^n \rightsquigarrow \mathbf{R}^n, \\ \xi_0 \in \mathbf{R}^n, \\ g : [0, T] \times \mathbf{R}^n \to \mathbf{R}, \\ \psi : \mathbf{R}^n \to \mathbf{R}. \end{array}$$

We assume that

(2)

$$\begin{cases} i)\ \forall (t,x) \in [0,T] \times \mathbf{R}^n,\ F(t,x) \text{ is nonempty, convex and compact} \\ ii)\ \forall x \in \mathbf{R}^n,\ F(\cdot,x) \text{ is measurable} \\ iii)\ \exists m \in L^1(0,T) \text{ such that for almost all } t \in [0,T],\ \forall x \in \mathbf{R}^n, \\ \qquad\qquad \sup_{v \in F(t,x)} \|v\| \le m(t)(1+\|x\|) \\ iv)\ \exists k \in L^1(0,T) \text{ such that } F(t,\cdot) \text{ is } k(t)-\text{Lipschitz a.e. in } [0,T] \\ v)\ g \text{ and } \psi \text{ are continuous} \end{cases}$$

We need the following notations:

$$B_R(x) = \{\, y \in \mathbf{R}^n \mid \|y-x\| \le R\},$$

$$\Omega = \{(t,x) \in [0,T] \times \mathbf{R}^n \mid g(t,x) \le 0\},$$

$$S^g_{[t_0,T]}(x_0) = \left\{ x(\cdot) \in AC(t_0,T) \;\middle|\; \begin{array}{l} x'(t) \in F(t,x(t)) \text{ a.e. in } [t_0,T], \\ g(t,x(t)) \le 0 \text{ in } [t_0,T], \\ x(t_0) = x_0 \end{array} \right\}$$

where $AC(t_0,T)$ is the set of absolutely continuous functions from $[t_0,T]$ to $\mathbf{R}^n$.

The value function associated to the above problem is defined by: for all $(t_0,x_0) \in [0,T] \times \mathbf{R}^n$ with $g(t_0,x_0) \le 0$,

$$V(t_0,x_0) = \inf\{\, \psi(x(T)) \mid x(\cdot) \in S^g_{[t_0,T]}(x_0)\}.$$

See [3] for the properties of the value function.

LEMMA 2.1. *Assume* (2). *Then for all* $R \ge 0$ *and for all* $(t_0,x_0) \in \Omega$ *with* $\|x_0\| \le R$, *there exists* $L_R \ge 0$ *such that*

$$\|x(t)\| \le L_R \quad \forall x \in S^g_{[t_0,T]}(x_0)\ \ \forall t \in [t_0,T].$$

PROOF. Let $x \in S^g_{[t_0,T]}(x_0)$. Then for almost all $t \in [t_0,T]$,

$$\begin{array}{rcl} x'(t) & \in & F(t,x(t)) \\ & \subset & F(t,0) + k(t)\|x(t)\|B \end{array}$$

where $B = \{x \in \mathbf{R}^n | \|x\| \le 1\}$. Hence for all $t \in [t_0,T]$,

$$\|x(t)\| \le \|x_0\| + \int_{t_0}^t m(s)ds + \int_{t_0}^t k(s)\|x(s)\|ds.$$

We can use the Gronwall's Lemma to end the proof. □

We can prove that the value function is lower semicontinuous and takes its values in $\mathbf{R} \cup \{\infty\}$. We define

$$Dom(V) := \{(t,x) \mid V(t,x) \in \mathbf{R}\}.$$

The proof of the next proposition is elementary, but we give it for the convenience of readers.

PROPOSITION 2.2. *Assume* (2). *Furthermore, suppose that for all* $t \in [0,T]$, $V(t,\cdot)$ *is* L_R*–Lipschitz on* $B_R(0) \cap Dom(V(t,\cdot))$. *Then for all* $(t_0,x_0) \in Dom(V)$, *for all* $x \in S^g_{[t_0,T]}(x_0)$, *the function*

$$[t_0,T] \ni t \to V(t,x(t))$$

is absolutely continuous.

PROOF. Let $x_1 \in S^g_{[t_0,T]}(x_0)$ and $t_0 \le t_1 \le t_2 \le T$. By dynamic programming principle, i.e., for all $t \in [t_0,T]$,

$$V(t_0,x_0) = \inf\{V(t,x(t)) \mid x \in S^g_{[t_0,T]}(x_0)\},$$

there exists $x_2 \in S^g_{[t_1,T]}(x_1(t_1))$ such that

$$V(t_2,x_2(t_2)) \le V(t_1,x_1(t_1)) + t_2 - t_1.$$

By Lemma 2.1, for all $i = 1,2$, we have

$$\begin{aligned} \|x_i(t_2) - x_i(t_1)\| &\le \int_{t_1}^{t_2} m(s)ds + \int_{t_1}^{t_2} k(s)\|x_i(s)\|ds \\ &\le \int_{t_1}^{t_2} m(s)ds + L_{\|x_0\|}\int_{t_1}^{t_2} k(s)ds. \end{aligned}$$

Hence,

$$\begin{aligned} 0 &\le V(t_2,x_1(t_2)) - V(t_1,x_1(t_1)) \\ &\le V(t_2,x_1(t_2)) - V(t_2,x_2(t_2)) + |t_2 - t_1| \\ &\le L_R\|x_1(t_2) - x_2(t_2)\| + |t_2 - t_1| \\ &\le L_R\Big(\|x_1(t_2) - x_1(t_1)\| + \|x_2(t_2) - x_1(t_1)\|\Big) + |t_2 - t_1| \\ &\le 2L_R\Big(\int_{t_1}^{t_2} m(s)ds + L_{\|x_0\|}\int_{t_1}^{t_2} k(s)ds\Big) + |t_2 - t_1|. \end{aligned}$$

This implies, by the definition of absolutely continuous functions, that

$$t \mapsto V(t,x(t))$$

is absolutely continuous. □

In general, the value function is not differentiable. Therefore we need to define the contingent derivative (see also [1]).

DEFINITION 2.3. Let X be a normed vector space, $\varphi : X \to R\cup\{\pm\infty\}$ be an extended function, $v \in X$ be a vector and $x_0 \in X$ be a vector such that $\varphi(x_0) \neq \pm\infty$.

The *contingent epiderivative* of φ at x_0 in the direction v is defined by:

$$D_\uparrow\varphi(x_0)(v) = \liminf_{h\to 0^+, v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}$$

and the *contingent hypoderivative* φ at x_0 in the direction v is defined by:

$$D_\downarrow\varphi(x_0)(v) = \limsup_{h\to 0^+, v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h}.$$

REMARK. When the function φ is locally Lipschitz, we have

$$\begin{aligned} D_\downarrow\varphi(x_0)(v) &= \limsup_{h\to 0^+, v'\to v} \frac{\varphi(x_0 + hv') - \varphi(x_0)}{h} \\ &= \limsup_{h\to 0^+} \frac{\varphi(x_0 + hv) - \varphi(x_0)}{h} \\ &\leq \limsup_{h\to 0^+, x\to x_0} \frac{\varphi(x + hv) - \varphi(x)}{h} \\ &= \varphi^0(x_0)(v). \end{aligned}$$

Therefore, in Theorem 2.6 and Theorem 3.3, the use of contingent hypoderivative is more general than that of Clarke's generalized directional derivative (see [5] for the definition).

We set

$$V(t,x) = \infty \;\; \forall (t,x) \notin \Omega.$$

To characterize the optimal solutions, we introduce the following feedback map G: for all $(t,x) \in \Omega$,

$$G(t,x) := \{v \in F(t,x) \mid D_\uparrow V(t,x)(1,v) \leq 0\}.$$

THEOREM 2.4. *Under the hypotheses of Proposition 2.2 if*

$$\text{(3)} \qquad \begin{cases} x'(t) \in G(t, x(t)) \;\; \text{a.e. in } [t_0, T] \\ x(t_0) = x_0, \end{cases}$$

then

$$V(t_0, x_0) = \psi(x(T)),$$

i.e., $x(\cdot)$ is optimal.

PROOF. By the definition of G, we have

$$x'(t) \in F(t, x(t)) \text{ a.e. in } [t_0, T]$$

and

$$g(t, x(t)) \leq 0 \text{ in } [t_0, T].$$

Set

$$\varphi(t) = V(t, x(t)).$$

Then Proposition 2.2 implies that φ is absolutely continuous. Hence $\varphi'(t)$ exists almost everywhere in $[t_0, T]$. On the other hand, φ is non-decreasing, i.e., $\varphi'(t) \geq 0$ almost everywhere. Hence to end the proof, it is sufficient to prove that

$$\varphi'(t) \leq 0 \text{ a.e. in } [t_0, T].$$

The condition (3) implies that there exists $h_i \to 0^+$ and $v_i \to x'(t)$ such that for almost all $t \in [t_0, T]$,

$$\begin{aligned} 0 &\geq D_\uparrow V(t, x(t))(1, x'(t)) \\ &= \liminf_{\substack{h \to 0^+ \\ v \to x'(t)}} \frac{V(t+h, x(t)+hv) - V(t, x(t))}{h} \\ &= \lim_{i \to \infty} \frac{V(t+h_i, x(t)+h_i v_i) - V(t, x(t))}{h_i}. \end{aligned} \tag{4}$$

Since for all $(t, x) \notin \text{Dom}(V)$, $V(t, x) = \infty$, we have for i sufficiently large,

$$(t + h_i, x(t) + h_i v_i) \in \text{Dom}(V).$$

Fix $t \in [t_0, T]$ such that $\varphi'(t)$ exists. Then

$$\varphi'(t) = \lim_{h \to 0^+} \frac{V(t+h, x(t+h)) - V(t, x(t))}{h} \leq 0$$

by (4) and the Lipschitz continuity of $V(t, \cdot)$. □

The converse of Theorem 2.4 is also true:

THEOREM 2.5. *If $x(\cdot)$ is optimal for $(t_0, x_0) \in$ Dom(V), then*

$$x'(t) \in G(t, x(t)) \text{ a.e. in } [t_0, T].$$

PROOF. Recall that $V(t,x) = \infty$ for all $(t,x) \notin \mathrm{Dom}(V)$. If $x(\cdot)$ is optimal, then $V(\cdot, x(\cdot))$ is constant and thereby for almost all t such that $x'(t)$ exists,

$$\begin{aligned} 0 &= \lim_{h\to 0^+} \frac{V(t+h, x(t+h)) - V(t,x(t))}{h} \\ &\geq \liminf_{h\to 0^+} \frac{V(t+h, x(t) + h\frac{x(t+h)-x(t)}{h}) - V(t,x(t))}{h} \\ &\geq \liminf_{\substack{h\to 0^+ \\ v \to x'(t)}} \frac{V(t+h, x(t)+hv) - V(t,x(t))}{h} \\ &= D_{\uparrow}V(t,x(t))(1,x'(t)). \end{aligned}$$

Therefore

$$x'(t) \in G(t,x(t)) \text{ a.e.}$$

□

2.2. Sufficient condition for optimality

In the above section, we obtained the sufficient and necessary conditions for optimality from the feedback map. In this section, we will give the sufficient condition by using the value function and the Hamiltonian. The Hamiltonian $H : [0,T] \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is defined by:

$$H(t,x,p) = \sup_{v\in F(t,x)} \langle p, v\rangle.$$

THEOREM 2.6. *Under the hypotheses of Proposition 2.2, if*

$$z \in S^g_{[t_0,T]}(x_0)$$

and for almost all $t \in [t_0,T]$, there exists $q(t) \in \mathbf{R}^n$ such that

$$(5) \quad \begin{cases} \langle q(t), z'(t)\rangle = H(t,z(t),q(t)) \\ \Big\langle \Big(H(t,z(t),q(t)), -q(t)\Big), (1,z'(t))\Big\rangle \geq D_{\downarrow}V(t,z(t))(1,z'(t)), \end{cases}$$

then

$$V(t_0,x_0) = \psi(z(T)),$$

i.e., $z(\cdot)$ is optimal.

PROOF. By Proposition 2.2, $V(\cdot, z(\cdot))$ is absolutely continuous. $z(\cdot)$ is also absolutely continuous. Hence $\frac{d}{dt}V(t,z(t))$ and $z'(t)$ exist almost everywhere. Fix a such $t \in [t_0, T]$. The Lipschitz continuity of the value function and (5), we have

$$\begin{aligned} 0 &= \left\langle \Big(\langle q(t), z'(t) \rangle, -q(t) \Big), (1, z'(t)) \right\rangle \\ &\geq D_{\downarrow} V(t, z(t))(1, z'(t)) \\ &= \limsup_{\substack{h \to 0^+ \\ v \to z'(t)}} \frac{V(t+h, z(t)+hv) - V(t, z(t))}{h} \\ &\geq \limsup_{h \to 0^+} \frac{V(t+h, z(t+h)) - V(t, z(t))}{h} \\ &= \frac{d}{dt} V(t, z(t)). \end{aligned}$$

We have proved that $V(\cdot, z(\cdot))$ is decreasing. But $V(\cdot, x(\cdot))$ is nondecreasing for all $x \in S^g_{[t_0,T]}(x_0)$. Therefore, $V(\cdot, z(\cdot))$ is constant and thereby

$$V(t_0, x_0) = \psi(z(T)).$$

□

3. Sufficient condition without Lipschitz continuity of value function

Let $z \in S^g_{[t_0,T]}(z(t_0)$ and set

$$\varphi(t) = V(t, z(t)), \quad t \in [t_0, T].$$

In this section, we suppose for almost all $t \in [t_0, T]$ and for all $x \in \mathbf{R}^n$

$$\sup_{v \in F(t,x)} \|v\| \leq M_1.$$

and there exists a constant $C > 0$ and $\rho(t) \in \mathbf{R}^{n+1}$ such that

$$\langle \rho(t), (1, \xi) \rangle \geq D_{\downarrow} V(t, z(t))(1, \xi) \ \ \forall \xi \in M_1 B$$

and

$$\|\rho(t)\| \leq C \ \ \forall t \in [t_0, T].$$

LEMMA 3.1. *There exists a constant M such that*

$$D_{\uparrow} \varphi(t)(1) \leq M \ \ \forall t \in [t_0, T].$$

PROOF. We have

$$
\begin{aligned}
D_{\uparrow}\varphi(t)(1) & \\
&= \liminf_{\substack{h\to 0^+\\ v\to 1}} \frac{V(t+h, z(t+hv)) - V(t,z(t))}{h} \\
&\le \liminf_{h\to 0^+} \frac{V(t+h, z(t)+h\frac{z(t+h)-z(t)}{h}) - V(t,z(t))}{h} \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{z(t+h)-z(t)}{h} &= \frac{1}{h}\int_t^{t+h} z'(s)ds \\
&\in \frac{1}{h}\int_t^{t+h} M_1 B ds \\
&= M_1 B.
\end{aligned}
$$

Hence there exists a sequence $h_n \to 0^+$ and $\xi \in M_1 B$ such that

$$
\frac{z(t+h_n)-z(t)}{h_n} \to \xi. \qquad (7)
$$

Hence (6) and (7) imply that

$$
\begin{aligned}
D_{\uparrow}\varphi(t)(1) &\le \limsup_{\substack{h\to 0^+\\ v\to \xi}} \frac{V(t+h, z(t)+hv) - V(t,z(t))}{h} \\
&= D_{\downarrow}V(t,z(t))(1,\xi) \\
&\le \langle \rho(t), (1,\xi)\rangle \\
&\le C(1+M_1) \\
&= M.
\end{aligned}
$$

□

Let E be a normed vector space and $K \subset E$. The contingent cone $T_K(x)$ of K at x is defined by:

$$
T_K(x) = \{v \in E \mid \liminf_{h\to 0^+} \frac{dist(x+hv, K)}{h} = 0\}.
$$

The epigraph of $f : K \to \mathbf{R} \cup \{\pm\infty\}$ is defined by

$$
Ep(f) := \{(x,\lambda) \in K \times \mathbf{R} \mid f(x) \le \lambda\}.
$$

We prove the next lemma by using the technique motivated by the proof of Proposition 2.8 of [4].

LEMMA 3.2. *$\varphi(\cdot)$ is Lipschitz in $[t_0, T]$.*

PROOF. Consider a set-valued map $F : \mathbf{R}^2 \rightsquigarrow \mathbf{R}^2$ such that $F(\tau, y) = \{(1, M)\}$ where M is the constant of Lemma 3.1. Set $K = Ep(\varphi)$. Note that K is closed. Fix $s \geq t_0$. Now consider the following differential inclusion:

$$(8) \qquad \begin{cases} (\tau, y)' \in F(\tau, y) \\ (\tau, y)(0) = (s, \varphi(s)) \in K. \end{cases}$$

By Lemma 3.1 and the fact that

$$T_K(\tau, y) \supset T_K(\tau, \varphi(\tau)) \quad \forall y \geq \varphi(\tau),$$

we have for all $(\tau, y) \in K$,

$$\begin{aligned} (1, M) &\in Ep(D_{\uparrow}\varphi(\tau)) \\ &= T_K(\tau, \varphi(\tau)) \\ &\subset T_K(\tau, y), \end{aligned}$$

i.e., for all $(\tau, y) \in K$,

$$F(\tau, y) \cap T_K(\tau, y) = \{(1, M)\} \neq \emptyset.$$

By Viability Theorem([1], [2], [7]), there exists a solution $(\tau, y)(\cdot)$ of (8) such that $(\tau, y)(r) \in K$ for all $0 \leq r \leq T - s$. On the other hand, (8) has only a solution. Hence

$$(\tau, y) = (s + r, \varphi(s) + Mr) \in K,$$

i.e.,

$$0 \leq \varphi(s + r) - \varphi(s) \leq Mr.$$

□

THEOREM 3.3. *If for almost all $t \in [t_0, T]$, there exist $q(t) \in \mathbf{R}^n$ and $\zeta(t) \in \mathbf{R}^n$ such that*

$$(9) \qquad \begin{cases} \langle q(t), z'(t) \rangle = H(t, z(t), q(t)), \\ \Big\langle \Big(H(t, z(t), q(t)) + \langle \zeta(t), z'(t) \rangle, -q(t) - \zeta(t)\Big), \Big(1, z'(t)\Big) \Big\rangle \\ \qquad \qquad \geq D_{\downarrow}V(t, z(t))(1, z'(t)), \end{cases}$$

then z is optimal.

PROOF. By Lemma 3.2, $\varphi(\cdot) = V(\cdot, z(\cdot))$ is Lipschitz. Let $t \in [t_0, T]$ be such that the derivatives $\varphi'(t)$ and $z'(t)$ exist and (9) is verified. Then

$$\begin{aligned} 0 &= \Big\langle \Big(\langle q(t) + \zeta(t), z'(t) \rangle, -q(t) - \zeta(t) \Big), \Big(1, z'(t)\Big) \Big\rangle \\ &= \Big\langle \Big(H(t, z(t), q(t)) + \langle \zeta(t), z'(t) \rangle, -q(t) - \zeta(t) \Big), \Big(1, z'(t)\Big) \Big\rangle \\ &\geq D_{\downarrow} V(t, z(t))(1, z'(t)) \\ &\geq \limsup_{h \to 0+} \frac{V(t+h, z(t+h)) - V(t, z(t))}{h} \\ &= \varphi'(t). \end{aligned}$$

This implies that $V(\cdot, z(\cdot))$ is decreasing. Because $V(\cdot, z(\cdot))$ is nondecreasing, $V(\cdot, z(\cdot))$ is constant, i.e., z is optimal. □

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