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REGULAR BRANCHED COVERING SPACES AND CHAOTIC MAPS ON THE RIEMANN SPHERE

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ABSTRACT. Let $(2,2,2,2)$ be ramification indices for the Riemann sphere. It is well known that the regular branched covering map corresponding to this, is the Weierstrass $\mathcal{P}$ function. Lattès [7] gives a rational function $R(z) = \frac{z^4+\frac{1}{2}g_2z^2+\frac{1}{16}g_2^2}{4z^3-g_2z}$ which is chaotic on $\overline{C}$ and is induced by the Weierstrass $\mathcal{P}$ function and the linear map $L(z) = 2z$ on complex plane C.

It is also known that there exist regular branched covering maps from T^2 onto $\bar{C}$ if and only if the ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ and $(3,3,3)$, by the Riemann-Hurwitz formula.

In this paper we will construct regular branched covering maps corresponding to the ramification indices $(2,4,4)$, $(2,3,6)$ and (3, 3, 3), as well as chaotic maps induced by these regular branched covering maps.

1. Introduction

Let $p : M \to \bar{C}$ be a regular branched covering map from a Riemann surface onto the Riemann sphere. Then we have the ramification indices corresponding to $p : M \to \bar{C}$. Conversely, the following shows when a regular branched covering exists for given ramification indices:

Let S be a compact Riemann surface and $(e_1, e_2, ..., e_s)$ ramification indices. Then, there is no complex manifold M and no regular branched covering map $p : M \to S$ with ramification indices $(e_1, e_2, ..., e_s)$ if and only if either (i) genus of S is 0 and $s = 1$ or (ii) genus of S is 0, $s = 2$ and $e_1 \neq e_2$. [1, 5].

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In particular, let M be simply connected. Then M is the unit disk D or the complex plane C or the Riemann sphere $\bar{C}$ according to $\rho = (2g(S)-2)+\sum_{j=1}^{s}(1-1/e_j) > 0$ or $\rho = 0$ or $\rho < 0$, respectively, where $g(S)$ is genus of S, (See [8](pg.231), [9] (pg.29)).

Let M and S be Riemann spheres. Then, we only have the following 5 orbifolds satisfying the Riemann-Hurwitz formula, (m,m), $(2,2,m)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$. It is also well known that the group of covering transformations is conjugate to the following groups, the cyclic, dihedral, tetrahedral, octahedral, and icosahedral group, respectively. And the branched covering maps corresponding to the above orbifolds can be given by rational functions. (See for details [9] (pg.32) and [6] (pg.124)).

Now let M be a torus T^2. Then, there exist regular branched covering maps from T^2 onto $\bar{C}$ if and only if the ramification indices are $(2,2,2,2)$, $(2,4,4)$, $(2,3,6)$ and $(3,3,3)$, by the Riemann-Hurwitz formula. Let $p : C \to T^2$ be the canonical covering map. Then the composition of $p : C \to T^2$ and the regular branched map from T^2 onto $\bar{C}$ is a universal branched regular covering map from C onto $\bar{C}$.

Let $(2,2,2,2)$ be ramification indices for the Riemann sphere. It is well known that the regular branched covering map corresponding to this, is the Weierstrass $\mathcal{P}$ function. Lattès [7] gives a rational function $R(z) = \frac{z^4+\frac{1}{2}g_2z^2+\frac{1}{16}g_2^2}{4z^3-g_2z}$ which is chaotic on $\overline{C}$ and is induced by the Weierstrass $\mathcal{P}$ function and the linear map $L(z) = 2z$ on the complex plane C.

The purpose of this paper is to construct regular branched covering maps corresponding to the ramification indices $(2,4,4)$, $(2,3,6)$, and $(3,3,3)$, as well as chaotic maps induced by these regular branched covering maps (Theorem 2.2,2.4 and 2.6).

2. Branched coverings and chaotic maps on the Riemann sphere

In this section we begin with brief review of Lattès' example which is one of the well known chaotic maps of $\bar{C}$ onto itself. The following are based on [2].

Let Λ be the lattice in C induced by 1 and the imaginary number i. We usually call this lattice a square lattice. Let $A(z) = 2z$ be a linear map of C onto itself. Since A preserves the lattice points, A induces a map of T^2 onto itself. We now project the dynamics of A on T^2 to a map on the Riemann sphere $\bar{C}$. Then the projection map $\pi : T^2 \to \bar{C}$ can be decided by an identification map with $\pi(z) = \pi(-z)$.

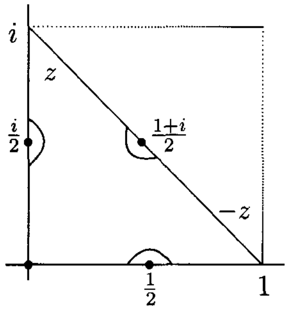

Figure 2.1

Then 0, $\frac{1}{2}$, $\frac{1}{2}i$, and $\frac{1+i}{2}$ have no partners in this identification. Hence the result of this identification is a map $\pi : T^2 \to \bar{C}$ which has 4 ramified values of index 2. And the map π can be given by the Weierstrass $\mathcal{P}$ function $\mathcal{P}(z) = \frac{1}{z^2} + \sum' \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$, where the sum is taken over the nonzero lattice points.

Then, by Addition Theorem of the Weierstrass $\mathcal{P}$ function, we have the map $R = \frac{z^4 + \frac{1}{2}g_2z^2 + 2g_3z + \frac{1}{16}g_2^2}{4z^3 - g_2z - g_3}$ of $\bar{C}$ onto itself so that the diagram, induced by A and the Weierstrass $\mathcal{P}$ function, commutes. Note that $g_3 = 0$ if the lattice is induced by 1 and the imaginary number i (See [4] pg.19). Consequently we have Lattès' chaotic map $R = \frac{z^4 + \frac{1}{2}g_2z^2 + \frac{1}{16}g_2^2}{4z^3 - g_2z}$ from $\bar{C}$ onto itself.

We remark that the Lattès' example also shows the following: Let $(2,2,2,2)$ be ramification indices on the Riemann sphere. Then the corresponding regular branched covering map with covering space T^2 is just the Weierstrass $\mathcal{P}$ function and it induces the chaotic map R as above with linear map $A(z) = 2z$ of C onto itself. Therefore we may say that the Weierstrass $\mathcal{P}$ function is a universal branched covering map corresponding to the ramification indices $(2,2,2,2)$. We refer to the reader [9] and ([8] pg.229-233) for detailed definitions and properties of regular branched coverings.

2.1. Ramification indices $(2,4,4)$

Let $(2,4,4)$ be ramification indices on the Riemann sphere $\bar{C}$. Then the discrete subgroup of covering transformations Γ of C corresponding to the ramification indices $(2,4,4)$ is generated by the maps $z \to z + a$ and $z \to iz$ for $a \in Z[i]$, where $Z[i]$ is a cyclic group generated by the imaginary number i [3]. Then C/Γ is biholomorphic to $\bar{C}$ and the quotient map $\mathcal{Q} : C \to \bar{C}$ is just the branched covering map corresponding to the ramified indices $(2,4,4)$. Note that the map $\mathcal{Q}$ is an even elliptic function with periods 1 and the imaginary number i, since $\mathcal{Q}(z) = \mathcal{Q}(iz) = \mathcal{Q}(i^2z) = \mathcal{Q}(-z)$.

Geometry of the branched covering space $(C/\Gamma, \mathcal{Q})$

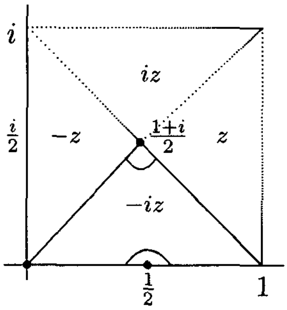

Figure 2.2

Now we may see that 0 is a pole of index 4 and is the only pole. Since $\mathcal{Q}(z)$ is an even function, $\mathcal{Q}(z) = \prod(\mathcal{P}(z) - \mathcal{P}(a_i))$ where $a_i's$ are the zeros of $\mathcal{Q}$. (See [10] pg.34, [4] pg.27). Notice that the sum of indices of zeros and the sum of indices of poles are equal. Therefore $\frac{1+i}{2}$ is the only zero of $\mathcal{Q}$ of index 4. Recall that $\mathcal{P}(\frac{1+i}{2}) = 0$. Consequently $\mathcal{Q}(z) = \mathcal{P}(z)^2$. Also recall that $\mathcal{P}(\frac{1}{2}) = e_1$ and $\mathcal{P}(\frac{i}{2}) = e_2$ are ramified points of index 2 for the Weierstrass $\mathcal{P}$ function and $e_1 = -e_2$ which is real number. So e_1^2 is the ramified point of index 2 for $\mathcal{Q}(z)$.

We now state the above argument as a theorem.

THEOREM 2.1. *Let $(2,4,4)$ be ramification indices on the Riemann sphere. Then the branched covering space corresponding to $(2,4,4)$ is the complex plane C with branched covering map $\mathcal{Q}(z) = \mathcal{P}(z)^2$. Moreover 0 and ∞ are the ramified points of index 4 and e_1^2 is the ramified point of index 2, where $e_1 = \mathcal{P}(\frac{1}{2})$.*

We now construct a chaotic map on $\bar{C}$ induced by the linear map $2z$ of C onto itself and the branched covering map $\mathcal{Q}(z)$ in Theorem 2.1. Recall that Lattès' example $R(z) = \frac{(z^2+\frac{g_2}{4})^2}{4z(z^2-g_2)}$, which is a chaotic map of $\bar{C}$ onto itself, is induced by the linear map $A(z) = 2z$ and the Weierstrass $\mathcal{P}$ function. Hence we have the following semi conjugate commutative diagram.

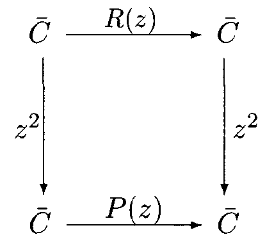

Figure 2.3

Consequently, we have the chaotic map $P(z) = \frac{(z+\frac{g_2}{4})^4}{16z(z-g_2)^2}$ of $\bar{C}$ onto itself corresponding to the ramified indices $(2,4,4)$. We now state the above as a theorem.

THEOREM 2.2. *Let $(2,4,4)$ be ramification indices on the Riemann sphere. And let $\mathcal{Q}(z) = \mathcal{P}(z)^2$ be the corresponding branched covering map. Then there exists a chaotic map $P(z) = \frac{(z+\frac{g_2}{4})^4}{16z(z-g_2)^2}$ of $\bar{C}$ onto itself corresponding to the ramified indices $(2,4,4)$, induced by the linear map $2z$ and the branched covering map $\mathcal{Q}(z) = \mathcal{P}(z)^2$.*

2.2. Ramification indices $(2,3,6)$

Let $(2,3,6)$ be ramification indices on $\bar{C}$. Then the transformation subgroup Γ corresponding to these ramification indices is generated by $z \to z+a$ and $z \to wz$ for $a \in Z[w]$ and $w = e^{\frac{\pi}{3}i}$ [3]. Then by the same argument as for ramification indices $(2,4,4)$, C/Γ is biholomorphic to $\bar{C}$ and the quotient map $\mathcal{Q}$ is just branched covering map.

Geometry of branched covering space $(C/\Gamma, \mathcal{Q})$

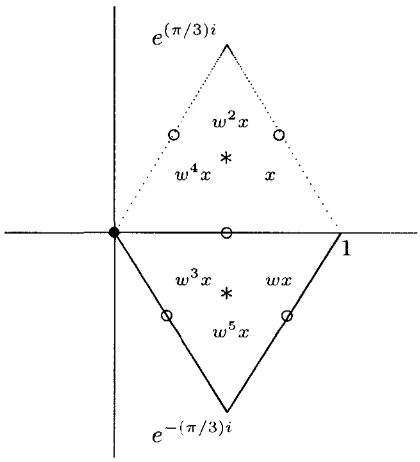

Figure 2.4

$\mathcal{Q}$ is an even elliptic function since $\mathcal{Q}(z) = \mathcal{Q}(w^3 z) = \mathcal{Q}(-z)$. Since we may take a pole at the point of lattice with index 6 and the pole is unique, the sum of indices of zeros is also 6. And therefore we may take zeros, $\frac{e^{\frac{\pi}{3}i}}{2}$, $\frac{e^{-\frac{\pi}{3}i}}{2}$ and $\frac{e^{\frac{\pi}{3}i}+e^{-\frac{\pi}{3}i}}{2} = \frac{1}{2}$ with index 2 and these are the all zeros of the elliptic function $\mathcal{Q}$ since the sum of indices of the poles and zeros are same.

Now, we have $\mathcal{Q}(z) = (\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3)$ where $e_1 = \mathcal{P}(\frac{e^{\frac{\pi}{3}i}}{2})$, $e_2 = \mathcal{P}(\frac{e^{-\frac{\pi}{3}i}}{2})$ and $e_3 = \mathcal{P}(\frac{e^{\frac{\pi}{3}i}+e^{-\frac{\pi}{3}i}}{2}) = \mathcal{P}(\frac{1}{2})$.

Therefore,

$$\mathcal{Q}(z) = (\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3) = \mathcal{P}(z)^3 - \frac{g_2}{4}\mathcal{P}(z) - \frac{g_3}{4}.$$

from relations between e_i's and g_i's. ([10] pg.36) But $g_2 = 0$ in the lattice Λ generated by $e^{\frac{\pi}{3}i}$ and $e^{-\frac{\pi}{3}i}$. Hence,

$$\mathcal{Q}(z) = \mathcal{P}(z)^3 - \frac{g_3}{4}.$$

Consequently the function $\mathcal{Q}(z)$ has ramified points at ∞ of index 6 whose branch point is 0, at 0 of index 2 whose branch points are $\frac{1}{2}e^{\frac{\pi}{3}i}$, $\frac{1}{2}e^{-\frac{\pi}{3}i}$ and $\frac{1}{2}(e^{\frac{\pi}{3}i} + e^{-\frac{\pi}{3}i}) = \frac{1}{2}$ and at $-\frac{g_3}{4}$ of index 3 whose branch points are $\frac{2}{3}e^{\frac{\pi}{6}i}$ and $\frac{2}{3}e^{-\frac{\pi}{6}i}$ which are zeros of the Weierstrass $\mathcal{P}$ function (Lemma 2.2).

We now state the above arguments as a theorem.

THEOREM 2.3. *Let $(2,3,6)$ be ramification indices on the Riemann sphere. Then the branched covering space corresponding to $(2,3,6)$ is the complex plane C with branched covering map $\mathcal{Q}(z) = \mathcal{P}(z)^3 - \frac{g_3}{4}$. Moreover 0, $-\frac{g_3}{4}$ and ∞ are the ramified points of index 2,3 and 6 respectively.*

Now to find the chaotic map induced by $\mathcal{Q}$ and the linear map $2z$ of C onto itself, we use Addition Theorem of the Weierstrass $\mathcal{P}$ function.

Recall that $g_2 = 0$ if the lattice Λ is induced by $e^{\frac{\pi}{3}i}$ and $e^{-\frac{\pi}{3}i}$ and therefore we have the following addition formula corresponding to this lattice Λ:

$$\mathcal{P}(2z) = \frac{\mathcal{P}(z)^4 + 2g_3\mathcal{P}(z)}{4\mathcal{P}(z)^3 - g_3}.$$

Now $\mathcal{Q}(z) = (\mathcal{P}(z))^3 - \frac{g_3}{4}$. Hence we have $\mathcal{Q}(2z) = (\mathcal{P}(2z))^3 - \frac{g_3}{4}$ and $\mathcal{P}(z) = (\mathcal{Q}(z) + \frac{g_3}{4})^{\frac{1}{3}}$.

So, $\mathcal{Q}(2z) = \left(\frac{(\mathcal{P}(z))^4+2g_3\mathcal{P}(z)}{4(\mathcal{P}(z))^3-g_3}\right)^3 - \frac{g_3}{4} = \left(\frac{\mathcal{P}(z)(\mathcal{P}^3(z)+2g_3)}{4\mathcal{Q}(z)}\right)^3 - \frac{g_3}{4} = \left(\frac{(\mathcal{Q}(z)+\frac{g_3}{4})^{\frac{1}{3}}(\mathcal{Q}(z)+\frac{9g_3}{4})}{4\mathcal{Q}(z)}\right)^3 - \frac{g_3}{4}$.

Consequently we have the map

$$R(z) = \left(\frac{(z+\frac{g_3}{4})^{\frac{1}{3}}(z+\frac{9g_3}{4})}{4z}\right)^3 - \frac{g_3}{4}$$

of $\bar{C}$ onto itself induced by the map $\mathcal{Q}$ and the linear map $2z$ of C onto itself such that the following diagram commutes.

$$\begin{array}{ccc} C & \xrightarrow{2z} & C \\ {\scriptstyle \mathcal{P}(z)^3-\frac{g_3}{4}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(z)^3-\frac{g_3}{4}} \\ \bar{C} & \xrightarrow{R(z)} & \bar{C} \end{array}$$

Figure 2.5

Now it is clear that the map R is chaotic from the semiconjugacy of the diagram. We now state the above argument as a theorem.

THEOREM 2.4. *Let $(2,3,6)$ be ramification indices on the Riemann sphere. Then there exists a chaotic map $R(z) = \frac{(z+\frac{g_3}{4})(z+\frac{9g_3}{4})^3}{4z} - \frac{g_3}{4}$ of $\bar{C}$ onto itself corresponding to the ramified indices $(2,4,4)$, induced by the linear map $2z$ and the branched covering map $\mathcal{Q}(z) = (\mathcal{P}(z))^3 - \frac{g_3}{4}$.*

2.3. The ramification indices $(3,3,3)$

Let $(3,3,3)$ be ramification indices on $\bar{C}$. Then also by [3] the transformation subgroup Γ corresponding to these indices is generated by $z \to z+a$ and $z \to w^2z$ for $a \in Z[w]$ and $w = e^{\frac{\pi}{3}i}$. Then by the same argument as the ramification indices $(2,4,4)$ or $(2,3,6)$ C/Γ is biholomorphic to $\bar{C}$ and the quotient map $\mathcal{Q}$ is just a branched covering map.

Geometry of Branched covering space $(C/\Gamma, \mathcal{Q})$

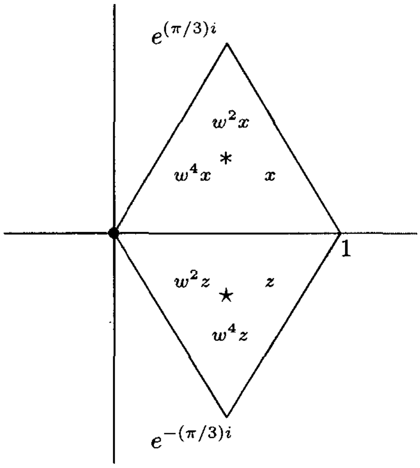

Figure 2.6

Now let Λ be the lattice induced by $e^{\frac{\pi}{3}i}$ and $e^{-\frac{\pi}{3}i}$, so called triangular lattice and let $1, \epsilon = e^{\frac{2}{3}\pi i} \epsilon^2 = e^{\frac{4}{3}\pi i}$ be the cube roots of unity. Then $\mathcal{P}(\epsilon z) = \epsilon \mathcal{P}(z)$ and $\mathcal{P}'(\epsilon z) = \mathcal{P}'(z)$ since $\epsilon\Lambda = \Lambda$.

This shows that z and ϵz have the same image under $\mathcal{P}'(z)$. Recall that the quotient map corresponding to the ramified indices $(3, 3, 3)$, $\mathcal{Q}$, is an elliptic function with this property. Therefore $\mathcal{Q}(z) = \mathcal{P}'(z)$. We can also show that $\mathcal{P}'(z)$ has exactly 3 ramified points with index 3, without using the above geometric argument, as the following lemma and proposition show. Hence we have a same result.

LEMMA 2.1. *let $\mathcal{P}(z)$ be the Weierstrass $\mathcal{P}$ function induced by the triangular lattice Λ. Then $w_4 = \frac{2}{3}e^{-\frac{\pi}{6}i}$ and $-w_4 = \frac{2}{3}e^{\frac{5\pi}{6}i} = \frac{2}{3}e^{\frac{\pi}{6}i} \bmod \Lambda$ are the only zeros of $\mathcal{P}(z)$.*

PROOF. Note that $w_4 = \epsilon w_4 = \epsilon^2 w_4 \bmod \Lambda$. Hence $\mathcal{P}(w_4) = \mathcal{P}(\epsilon w_4) = \epsilon \mathcal{P}(w_4)$ by the above. Since $\epsilon \neq 1$ we have $\mathcal{P}(w_4) = 0$. Recall that $\mathcal{P}(z)$ is an even function. And therefore $-w_4$ is also zero of $\mathcal{P}(z)$. And these are the only zeros of index 1 since $\mathcal{P}(z)$ has only one pole of index 2. □

PROPOSITION 2.1. *w_4 and $-w_4$ are branch points of $\mathcal{P}'(z)$ of order 3. Moreover w_4 and $-w_4$ are the only branch points of $\mathcal{P}'(z)$ except the pole.*

PROOF. Note that $\mathcal{P}''(z) = 6\mathcal{P}(z)^2$ in the triangular lattice Λ and w_4 and $-w_4$ are the only zeros of $\mathcal{P}(z)$ (Lemma 2.1). Therefore $\mathcal{P}''(w_4) = \mathcal{P}''(-w_4) = 0$. Moreover $\mathcal{P}'''(w_4) = \mathcal{P}'''(-w_4) = 0$ since $\mathcal{P}'''(z) = 12\mathcal{P}(z)\mathcal{P}'(z)$. Now $\mathcal{P}^{(4)}(z) = 12(\mathcal{P}''(z)\mathcal{P}(z) + (\mathcal{P}'(z))^2)$. So $\mathcal{P}^{(4)}(w_4) = (\mathcal{P}'(w_4))^2 \neq 0$ since $\mathcal{P}'(z_0) = 0$ if and only if $z_0 = \frac{1}{2}e^{\frac{\pi}{3}i}, \frac{1}{2}e^{-\frac{\pi}{3}i}$ or $\frac{1}{2}(e^{\frac{\pi}{3}i} + e^{-\frac{\pi}{3}i}) = \frac{1}{2}$. This is also true for $-w_4$. Consequently w_4 and $-w_4$ are the branch points of index 3. We now show that there is no branch point of index 2 for $\mathcal{P}'(z)$, i.e., there is no branch point with index 3 for $\mathcal{P}(z)$. In fact, if $\mathcal{P}''(z_0) = 0$ then $\mathcal{P}(z_0) = 0$. Therefore $\mathcal{P}'''(z_0) = 0$ by $\mathcal{P}'''(z) = 12\mathcal{P}(z)\mathcal{P}'(z)$. Consequently z_0 is a branch point whose index is more than 2. □

We now summarize the above arguments as a theorem.

THEOREM 2.5. *Let $(3,3,3)$ be the ramification indices on $\bar{C}$. Then the branched covering space corresponding to $(3,3,3)$ is the complex plane C with branched covering map $\mathcal{P}'(z)$. Moreover $\mathcal{P}'(z)$ has ramified points at ∞, $\mathcal{P}'(\frac{2}{3}e^{-\frac{\pi}{6}i})$, and $\mathcal{P}'(\frac{2}{3}e^{\frac{\pi}{6}i})$, each with index 3.*

We are now ready to construct a chaotic map corresponding to the ramified indices $(3,3,3)$.

Recall that $\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - g_3$ and $\mathcal{P}(2z) = \frac{\mathcal{P}(z)^4 + 2g_3\mathcal{P}(z)}{4\mathcal{P}(z)^3 - g_3}$. Hence,

$$\begin{aligned}
&\mathcal{P}'(2z)\\
&= (4\mathcal{P}(2z)^3 - g_3)^{\frac{1}{2}}\\
&= \left(4\left(\frac{\mathcal{P}(z)^4 + 2g_3\mathcal{P}(z)}{4\mathcal{P}(z)^3 - g_3}\right)^3 - g_3\right)^{\frac{1}{2}}\\
&= \left(4\left(\frac{\mathcal{P}(z)(\mathcal{P}(z)^3 + 2g_3)}{4\mathcal{P}(z)^3 - g_3}\right)^3 - g_3\right)^{\frac{1}{2}}\\
&= \left(4\left(\frac{\left(\frac{(\mathcal{P}'(z)^2 + g_3)}{4}\right)\left(\frac{(\mathcal{P}'(z)^2 + g_3)}{4} + 2g_3\right)^3}{\mathcal{P}'(z)^6}\right) - g_3\right)^{\frac{1}{2}}.
\end{aligned}$$

Consequently we have a map

$$R(z) = \left(4 \left(\frac{\frac{(z^2+g_3)}{4} \left(\frac{(z^2+g_3)}{4} + 2g_3 \right)^3}{z^6} \right) - g_3 \right)^{\frac{1}{2}}$$

such that the following diagram commutes.

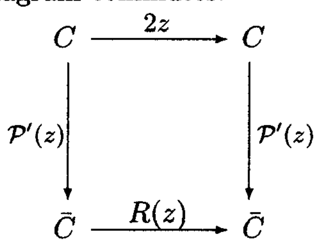

Figure 2.7

THEOREM 2.6. *Let $(3,3,3)$ be the ramification indices on $\bar{C}$. Then there exists a chaotic map $R(z)$ of $\bar{C}$ onto itself induced by the branched covering map $\mathcal{P}'(z)$ on the triangular lattice Λ and the linear map $2z$ of C onto itself such that*

$$R(z) = \left(4 \left(\frac{\frac{(z^2+g_3)}{4} \left(\frac{(z^2+g_3)}{4} + 2g_3 \right)^3}{z^6} \right) - g_3 \right)^{\frac{1}{2}}.$$

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