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OSCILLATION THEOREMS OF SOLUTIONS FOR SOME DIFFERENTIAL EQUATIONS

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Abstract. Some oscillation criteria are given for second order nonlinear differential equations by means of integral averaging technique.

§1. Introduction

The purpose of this paper is to study oscillatory properties of solutions with mixed argument

$$(1) \qquad \left[\frac{1}{p(t)}k(x'(t))\right]' + q(t)f(x(t), x(\phi(t)), x(\psi(t))) = 0,$$

$$(2) \qquad \left[\frac{1}{p(t)}k(x'(t))\right]' + q(t)f(x(t), x(\phi(t)), x(\psi(t)))g(x'(t)) = 0,$$

$$(3) \qquad \left[\frac{1}{p(t)}k(x'(t))\right]' + r(t)k(x'(t)) + q(t)f(x(t), x(\phi(t))) = 0,$$

where $t \geq t_0$ and $k(s) = |s|^\nu sgn\, s$ $(\nu \geq 1)$. Now f, g, p, q, ϕ, ψ are to be specified in the following text. In this paper we always define a function $P(t)$ as

$$(H) \qquad P(t) = \int_{t_0}^{t} p(s)^{1/\nu}\, ds, \qquad t_0 \leq t,$$

and assume that $P(t) \to \infty$ *as* $t \to \infty$.

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By a solution of (1), we mean a continuously differentiable function $x : [t_0, \infty) \to \mathbb{R}$ such that $x(t)$ satisfies (1) for all $t \geq t_0$. Let $\xi : [\phi(t_0), t_0] \to \mathbb{R}$ be a continuous function. By a solution of (2), we mean a continuously differentiable function $x : [\phi(t_0), \infty) \to \mathbb{R}$ such that $x(t) = \xi(t)$ for $\phi(t_0) \leq t_0$, and $x(t)$ satisfies (2) for all $t \geq t_0$. In the sequel it will be always assumed that nonconstant solutions of (1) exist on some ray $[T, \infty)$, $T \geq t_0$. A solution $x(t)$ is oscillatory if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of zeros of $x(t)$ such that $t_n \to \infty$ *as* $t \to \infty$. Otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all solutions are oscillatory.

Numerous oscillation criteria have been obtained ([1-13]). A half-linear differential equation

$$\left[\frac{1}{p(t)}k(x'(t))\right]' + q(t)k(x(t)) = 0, \tag{4}$$

a delay differential equation

$$\left[\frac{1}{p(t)}k(x'(t))\right]' + q(t)|x(t)|^{\alpha}|x(\phi(t))|^{\beta} sgn\ x(t) = 0$$

and an advanced differential equation

$$\left[\frac{1}{p(t)}k(x'(t))\right]' + q(t)|x(t)|^{\alpha}|x(\psi(t))|^{\beta} sgn\ x(t) = 0$$

are the particular cases of (1) where $\alpha + \beta = \nu$, $\alpha \geq 0$, $\beta \geq 0$.

In the study of oscillatory behavior of solutions for differential equations, the averaging technique (Winter [14]) is a very important tool. The Winter's results were improved by many authors including Philos [10].

Following Philos, we introduce a class of functions P. Let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. We say that a function $H \in C(D, (-\infty, \infty))$ is said to belong to a function class P if

(H_1) $H(t,t) = 0$ *for* $t \geq t_0$, $H(t,s) > 0$ on D_0

(H_2) $\dfrac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)}$

where h is a positive function defined on D. We note that $k^{-1}(t) = |t|^{1/\nu} sgn\, t$ is the inverse function of $k(s) = |s|^{\nu} sgn\, s = |s|^{\nu-1}s$.

§2. Main results

Hereinafter we assume that

(A_1) the differentiable function $p \in C[t_0, \infty)$ is positive and nonincreasing.

(A_2) the function $q \in C[t_0, \infty)$ is positive.

(A_3) $\phi(t)$ is nondecreasing and continuously differentiable, $\phi(t) \leq t$ and $\phi(t) \to \infty$ as $t \to \infty$.

(A_4) $\psi(t)$ is nondecreasing and continuously differentiable, $\psi(t) \geq t$.

(A_5) $a(t)$ is positive and continuously differentiable for all $t \in [t_0, \infty)$.

(A_6) $f(s,t,u) = |s|^{\alpha}|t|^{\beta}|u|^{\gamma}\, sgn\, s,\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,$ $\alpha + \beta + \gamma = \nu, \nu \geq 1$.

(A_7) $g(s) \geq M > 0$ for $s \neq 0$.

THEOREM 1. *Let the conditions* (A_1) – (A_6) *be satisfied. Assume that the following*

$$(5)\quad \limsup_{t \to \infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)a(s)q(s)\left[k\frac{\phi(s)}{s}\right]^{\beta} - V(t,s)^2 \right] ds = \infty$$

is valid, where

$$V(t,s) = \frac{a(t)^{1/(2\nu)}\left[h(t,s) - \dfrac{a'(t)}{a(t)}\sqrt{H(t,s)}\right]}{2\sqrt{\nu}p(t)^{1/(2\nu)}}.$$

Then the equation (1) *is oscillatory.*

PROOF. Assume that $x(t)$ is a nonoscillatory solution of equation (1) and that there exists $T_0 \geq t_0$ such that

$$(6)\qquad x(t) > 0 \quad \text{for all } t \geq T_0.$$

The similar argument holds also for the case when $x(t)$ is eventually negative. Then there exists a T_1 with $T_0 \geq T_0$ such that $x(\phi(t)) \geq 0$ for

$t \geq T_1 \geq T_0$. It follows from (6) that $\frac{1}{p(t)}|x'(t)|^{\nu}\, sgn\, x'(t)$ is decreasing for $t \geq T_1$. We may assume that there exists $T \geq T_1$ such that

$$x'(t) > 0 \text{ for all } t \geq T \geq T_1. \tag{7}$$

Otherwise, for every $T \geq T_1$ there exists $t_0 \geq T \geq T_1$ such that $x'(t_0) < 0$. Then for $t \geq t_0$ we have

$$\frac{1}{p(t)}|x'(t)|^{\nu}\, sgn\, x'(t) \leq C$$

where $C = \frac{1}{p(t_0)}|x'(t_0)|^{\nu}\, sgn\, x'(t_0) < 0$. Since g is increasing, it follows that

$$x'(t) \leq g^{-1}(Cp(t)) = -|Cp(t)|^{1/\nu} < 0.$$

Integrating from t_0 to t we obtain

$$x(t) \leq x(t_0) - \int_{t_0}^{t} |Cp(s)|^{1/\nu}\, ds,$$

which implies that $x(t)$ is eventually negative. Thus (7) follows. On the other hand, from (A_1), (A_2), (6), (7) and that

$$\frac{d}{dt}\left[\frac{1}{p(t)}x'(t)^{\nu}\right] = -\frac{p'(t)}{p(t)^2}x'(t)^{\nu} + \frac{1}{p(t)}\nu x'(t)^{\nu-1}x''(t) \leq 0$$

we obtain for $t \geq T_1$

$$x''(t) \leq 0. \tag{8}$$

Hence by [6, Lemma 2.1], for any $k \in (0,1)$ there exists a $T_2 \geq T_1$ such that for $t \geq T_2$

$$x(\phi(t)) \geq k\frac{\phi(t)}{t}\, x(t). \tag{9}$$

We note that for $t \geq T_2$

$$x(\phi(t)) \leq x(t) \leq x(\psi(t)) \tag{10}$$

because of (6). We consider a Riccati transform

$$W(t) = a(t)\frac{\frac{1}{p(t)}x'(t)^{\nu}}{x(t)^{\nu}}. \tag{11}$$

Since

$$\frac{d}{dt}\left[\frac{W(t)}{a(t)}\right] = -q(t)\left[\frac{x(\phi(t))}{x(t)}\right]^{\beta}\left[\frac{x(\psi(t))}{x(t)}\right]^{\gamma} - \nu p(t)^{1/\nu}|W(t)|^{1+1/\nu} \leq 0$$

we may assume that

$$0 < W(t) \leq 1. \tag{12}$$

By means of (8), (9) and (10) we have

$$\begin{aligned} W'(t) &= \frac{a'(t)}{a(t)}W(t) - a(t)q(t)\frac{f(x(t), x(\phi(t)), x(\psi(t)))}{x(t)^{\nu}} \\ &\quad - \nu a(t)^{-1/\nu}p(t)^{1/\nu}|W(t)|^{1+1/\nu} \\ &\leq \frac{a'(t)}{a(t)}W(t) - a(t)q(t)\left[k\frac{\phi(t)}{t}\right]^{\beta} - \nu a(t)^{-1/\nu}p(t)^{1/\nu}W^2(t). \end{aligned} \tag{13}$$

Integrating for $t \geq T \geq T_0$ after multiplying (11) by $H(t,s)$ we obtain, in view of (H_2),

$$\begin{aligned} &\int_T^t H(t,s)a(s)q(s)\left[k\frac{\phi(s)}{s}\right]^{\beta} ds \\ &\leq -\int_T^t H(t,s)W'(s)\,ds - \int_T^t \nu a(s)^{-1/\nu}p(s)^{1/\nu}H(t,s)W(s)^2\,ds \\ &\quad + \int_T^t \frac{a'(s)}{a(s)}H(t,s)W(s)\,ds \\ &= -H(t,s)W(s)\Big|_{s=T}^{s=t} + \int_T^t \frac{\partial H(t,s)}{\partial s}W(s)\,ds \\ &\quad - \int_T^t \left[\nu a(s)^{-1/\nu}p(s)^{1/\nu}H(t,s)W(s)^2 - \frac{a'(s)}{a(s)}H(t,s)W(s)\right] ds \end{aligned}$$

$$= H(t,T)W(T) - \int_T^t \Big[\nu a(s)^{-1/\nu} p(s)^{1/\nu} H(t,s) W(s)^2$$

$$+ \left\{ h(t,s) - \frac{a'(s)}{a(s)} \sqrt{H(t,s)} \right\} \sqrt{H(t,s)} W(s) \Big] \, ds$$

$$= H(t,T)W(T) - \int_T^t \left[\left\{ \nu a(s)^{-1/\nu} p(s)^{1/\nu} \right\}^{1/2} \sqrt{H(t,s)} W(s) \right.$$

$$\left. + V(t,s) \right]^2 ds + \int_T^t V(t,s)^2 \, ds$$

where

$$V(t,s) = \frac{a(t)^{1/(2\nu)} \left[h(t,s) - \frac{a'(t)}{a(t)} \sqrt{H(t,s)} \right]}{2\sqrt{\nu} p(t)^{1/(2\nu)}}.$$

From latter inequality and (H_2) it follows that

$$\int_T^t \left[H(t,s) a(s) q(s) \left[k \frac{\phi(s)}{s} \right]^\beta - V(t,s)^2 \right] ds \le H(t,T)W(T)$$

$$- \int_T^t \left[\left\{ \nu a(s)^{-1/\nu} p(s)^{1/\nu} \right\}^{1/2} \sqrt{H(t,s)} W(s) + V(t,s) \right]^2 ds.$$

Since this inequality is valid for all $t \ge T_0$, by (H_2) we have

$$\int_{T_0}^t \left[H(t,s) a(s) q(s) \left[k \frac{\phi(s)}{s} \right]^\beta - V(t,s)^2 \right] ds$$

$$\le \quad H(t,T_0)|W(T_0)| \le H(t,t_0)|W(T_0)|. \tag{14}$$

Consequently, by (14) and (H_2) we have

(15)

$$\int_{t_0}^t \left[H(t,s) a(s) q(s) \left[k \frac{\phi(s)}{s} \right]^\beta - V(t,s)^2 \right] ds$$

$$\le \int_{t_0}^{T_0} \left[H(t,s) a(s) q(s) \left[k \frac{\phi(s)}{s} \right]^\beta - V(t,s)^2 \right] ds + H(t,t_0)|W(T_0)|$$

$$\le H(t,t_0) \left\{ \int_{t_0}^{T_0} a(s) q(s) \left[k \frac{\phi(s)}{s} \right]^\beta ds + |W(T_0)| \right\}$$

which contradicts the assumption (5). Thus (1) is oscillatory. □

REMARK 1. In order for (1) to be oscillatory it is clear that (5) can be replaced by the conditions

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)a(s)q(s)\left[k\frac{\phi(s)}{s}\right]^{\beta} ds = \infty, \tag{16}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} V(t,s)^2\, ds < \infty. \tag{17}$$

COROLLARY 1. *If the equality*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)a(s)q(s) - V(t,s)^2\right] ds = \infty \tag{18}$$

is valid with $V(t,s)$ the same as in Theorem 1, then the differential equation (4) is oscillatory.

COROLLARY 2. *Let the assumptions $(A_1) - (A_6)$ be satisfied. For $n \geq 1$ if the inequality*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} \left[(kl)^{\beta}(t-s)^n a(s)q(s) - \frac{a(t)^{1/\nu}}{4\nu p(t)^{1/\nu}}(t-s)^{n-2}\left\{n - \frac{a'(t)}{a(t)}(t-s)\right\}^2\right] ds = \infty \tag{19}$$

is valid where a constant $k \in (0,1)$, then the equation (1) with $\phi(t) = lt$ $(0 < l \leq 1)$ is oscillatory.

PROOF. For $n \geq 1$ if we choose the functions $H(t,s)$ and $h(t,s)$ by

$$H(t,s) = (t-s)^n, \tag{20}$$

$$h(t,s) = n(t-s)^{(n-2)/2}, \tag{21}$$

the Corollary follows from Theorem 1. □

REMARK 2. We can make use of various form of $H(t,s)$. For $n \geq 1$ we may define the function $H(t,s)$ by

$$H(t,s) = \{P(t) - P(s)\}^n = \left\{\int_s^t p(\tau)^{1/\nu}\, d\tau\right\}^n,$$

$$h(t,s) = np(s)^{1/\nu}\{P(t) - P(s)\}^{(n-2)/2}.$$

REMARK 3. In the proof of Theorem 1 we assume that (12) is valid with $a(t) \equiv 1$. Then if we define the function $H(t,s)$ by (20), it follows that

$$V(t,s) = \frac{h(t,s)}{2\sqrt{\nu}p(t)^{1/(2\nu)}} = \frac{n(t-s)^{(n-2)/2}}{2\sqrt{\nu}p(t)^{1/(2\nu)}}.$$

Now it is obvious that

$$\lim_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^{n-2}\, ds = 0. \tag{22}$$

Thus if $p(t)$ is bounded below by a positive constant and if $\phi(t)/t \geq L > 0$ for $t \geq t_0$, the left side of (17) is equal to 0. On the other hand $H(t,s)$ satisfies the conditions $(K_1) - (K_3)$ in Wong [15]. Thus if the equality

$$\lim_{t\to\infty} \int_{t_0}^{t} q(s)\, ds = \infty \tag{23}$$

is valid, by Lemma [15] we obtain

$$\lim_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds = \infty. \tag{24}$$

Moreover, it is clear that

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)q(s) \left[k\frac{\phi(s)}{s}\right]^{\beta} ds \\ \geq\ &\lim_{t\to\infty} \frac{(kL)^{\beta}}{t^n} \int_{t_0}^{t} (t-s)^n q(s)\, ds. \end{aligned}$$

Therefore we conclude that both (1) and (2) are oscillatory if (23) is valid. We note that the left side of (24) is equal to 0 if $q(t) \in L^1[t_0,\infty)$ (see [15]).

REMARK 4. Let the function $H(t,s)$ be defined by (20) and put

$$U(t) \equiv \frac{a(t)^{1/\nu}}{4\nu p(t)^{1/\nu}}.$$

Then we obtain

$$\begin{aligned} V(t,s)^2 &= U(t)\left[h(t,s) - \frac{a'(t)}{a(t)}\sqrt{H(t,s)}\right]^2 \\ &\geq 2U(t)\left[h(t,s)^2 + \frac{a'(t)^2}{a(t)^2}H(t,s)\right]. \end{aligned}$$

We assume that $U(t,s)$ is bounded and that $\frac{a'(t)}{a(t)} \in L^2[t_0,\infty)$. If then the equality

$$\lim_{t\to\infty}\int_{t_0}^{t} a(s)q(s)\left[k\frac{\phi(s)}{s}\right]^{\beta} ds = \infty$$

is valid, by (21), (22) and Lemma [15] (1) is oscillatory.

THEOREM 2. *Under the conditions* $(A_1)-(A_7)$ *we assume that the following*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[MH(t,s)a(s)q(s)\left[k\frac{\phi(s)}{s}\right]^{\beta} - V(t,s)^2\right] ds = \infty \tag{25}$$

is valid where $V(t,s)$ *is the same as in Theorem 1. Then the equation* (2) *is oscillatory.*

PROOF. We define the function $W(t)$ by (11). Then it follows that

$$W'(t) \le \frac{a'(t)}{a(t)}W(t) - Ma(t)q(t)\left[k\frac{\phi(t)}{t}\right]^{\beta} - \nu a(t)^{-1/\nu}p(t)^{1/\nu}W^2(t). \tag{26}$$

The rest of proof is the same as in the proof of Theorem 1. □

THEOREM 3. *Under the conditions* $(A_1)-(A_5)$ *and* (A_6) *with* $\gamma = 0$ *we assume that the following*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)a(s)q(s)\left[k\frac{\phi(s)}{s}\right]^{\beta} - V_1(t,s)^2\right] ds = \infty \tag{27}$$

is valid where

$$V_1(t,s) = \frac{a(t)^{1/(2\nu)}\left[h(t,s) - \left\{\frac{a'(t)}{a(t)} - r(t)p(t)\right\}\sqrt{H(t,s)}\right]}{2\sqrt{\nu}p(t)^{1/(2\nu)}}.$$

Then the equation (3) *is oscillatory.*

PROOF. We define the function $W(t)$ by (11). Then it follows that

$$W'(t) = \frac{a'(t)}{a(t)}W(t) - a(t)\frac{r(t)x'(t)^\nu + q(t)f(x(t), x(\phi(t))}{x(t)^\nu}.$$

Thus we obtain

$$(28)\quad W'(t) \le \left[\frac{a'(t)}{a(t)} - r(t)p(t)\right] W(t) - a(t)q(t)\left[k\frac{\phi(t)}{t}\right]^\beta - \nu a(t)^{-1/\nu} p(t)^{1/\nu} W^2(t)$$

The rest of proof is the same as in the proof of Theorem 1. ☐

We consider a perturbed differential equation of the form

$$(29)\qquad \left[\frac{1}{p(t)}k(x'(t))\right]' + q(t)f_1(x(t)) = m(t)$$

with the condition

$$(A_8)\qquad \frac{f_1(s)}{s^\nu} \ge K \quad \text{for} \quad s \ne 0.$$

THEOREM 4. *Let the conditions* (A_1), (A_2), (A_4) *and* (A_8) *be satisfied. Assume that*

$$(30)\qquad \int^\infty a(s)m(s)\,ds < \infty,$$

and that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[KH(t,s)a(s)q(s) - V(t,s)^2\right] ds = \infty,$$

where $V(t,s)$ *is the same as in Theorem 1. Then the equation* (29) *is oscillatory.*

PROOF. Assume that $x(t)$ is a nonoscillatory solution. Then we may assume that there exist a positive constant C and $T_0 \ge t_0$ such that

$$x(t) > C \quad \text{for all } t \ge T_0.$$

We define the function $W(t)$ by (11). Then it follows that

$$W'(t) = \frac{a'(t)}{a(t)}W(t) + \frac{a(t)}{x(t)^\nu}\{-q(t)f_1(x(t)) + m(t)\}$$
$$- \nu a(t)^{-1/\nu}p(t)^{1/\nu}|W(t)|^{1+1/\nu}$$
$$\leq \frac{a'(t)}{a(t)}W(t) - Ka(t)q(t) + \frac{a(t)m(t)}{x(t)^\nu} - \nu a(t)^{-1/\nu}p(t)^{1/\nu}W(t)^2.$$

Thus for all $t \geq T \geq T_0$ we obtain

$$\int_T^t H(t,s)Ka(s)q(s)\,ds \leq -\int_T^t H(t,s)W'(s)\,ds$$
$$-\int_T^t \nu a(s)^{-1/\nu}p(s)^{1/\nu}H(t,s)W(s)^2\,ds$$
$$+\int_T^t H(t,s)\frac{a(s)m(s)}{x(s)^\nu}\,ds + \int_T^t \frac{a'(s)}{a(s)}H(t,s)W(s)\,ds$$
$$= H(t,T)W(T) - \int_T^t \Big[\nu a(s)^{-1/\nu}p(s)^{1/\nu}H(t,s)W(s)^2$$
$$+ \left\{h(t,s) - \frac{a'(s)}{a(s)}\sqrt{H(t,s)}\right\}\sqrt{H(t,s)}W(s)\Big]\,ds$$
$$+\frac{1}{C^\nu}\int_T^t H(t,s)a(s)m(s)\,ds$$
$$= H(t,T)|W(T)| - \int_T^t \Big[\left\{\nu a(s)^{-1/\nu}p(s)^{1/\nu}\right\}^{1/2}\sqrt{H(t,s)}W(s)$$
$$+ V(t,s)\Big]^2\,ds + \int_T^t V(t,s)^2\,ds + \frac{1}{C^\nu}\int_T^t H(t,s)a(s)m(s)\,ds$$

where $V(t,s)$ is the same as in Theorem 1. Consequently for each $t \geq T_0$ we get

$$\int_{T_0}^t \left[KH(t,s)a(s)q(s) - V(t,s)^2\right]\,ds \leq H(t,T_0)|W(T_0)|$$
$$+\frac{1}{C^\nu}H(t,T_0)\int_T^t a(s)m(s)\,ds.$$

The rest of proof is the same as in the proof of Theorem 1. □

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