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ON PRIME GAMMA-NEAR-RINGS WITH DERIVATIONS

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Dedicated to the memory of Mehmet Sapanci

Abstract. Conditions for a Γ-near-ring to be commutative are investigated.

1. Introduction

For preliminary definitions and results related to near-rings, we refer to Pilz [11]. The notion of a Γ-ring, a concept more general than a ring, was defined by Nobusawa [8]. Barnes [1] weakened the conditions slightly in the definition of Γ-ring in the sense of Nobusawa. Barnes [1], Kyuno [6], Luh [7] and Öztürk (together with Jun) [9] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. As a generalization of near-rings, Γ-near-rings were defined by Satyanarayana [13]. Recently, Booth, Groenewald and Satyanarayana studied several aspects in Γ-near-rings (see [2], [3], [12], [13], [14]). Also the third author (together with Cho and Kim) introduced the notion of Γ-derivations in Γ-near-rings and investigated basic properties (see [4], [5]). In this paper, we investigate some conditions for a Γ-near-ring to be commutative.

2. Preliminaries

All near-rings considered in this paper are left distributive. A Γ-*near-ring* is a triple $(M, +, \Gamma)$ where

(i) $(M, +)$ is a (not necessarily abelian) group,

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(ii) Γ is a non-empty set of binary operators on M such that $(M, +, \gamma)$ is a near-ring for each $\gamma \in \Gamma$,

(iii) $(x\beta y)\gamma z = x\beta(y\gamma z)$ for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$.

For a Γ-near-ring M, the set $M_0 := \{x \in M \mid 0\gamma x = 0 \text{ for all } \gamma \in \Gamma\}$ is called the *zero-symmetric part* of M. A Γ-near-ring M is said to be *zero-symmetric* if $M = M_0$. A subset U of a Γ-near-ring M is said to be *left* (resp. *right*) *invariant* if $x\gamma u \in U$ (resp. $u\gamma x \in U$) for all $x \in M, u \in U$ and $\gamma \in \Gamma$. If U is both left and right invariant, we say that U is *invariant.*

A Γ-near-ring M is said to be *prime* if $x\Gamma M\Gamma y = \{0\}$ implies $x = 0$ or $y = 0$ for all $x, y \in M$. If M and M' are Γ-near-rings, then a mapping $d : M \to M'$ such that $d(x + y) = d(x) + d(y)$ and $d(x\gamma y) = d(x)\gamma d(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$ is called a Γ-*near-ring homomorphism.* A Γ-*derivation* on a Γ-near-ring M is defined to be an additive endomorphism d of M satisfying the product rule $d(x\gamma y) = x\gamma d(y) + d(x)\gamma y$ for all $x, y \in M$ and $\gamma \in \Gamma$. As usual, an element $c \in M$ for which $d(c) = 0$ is called a *constant.* Let S be a non-empty subset of a Γ-near-ring M and d be a Γ-derivation on M. If $d(x\gamma y) = d(x)\gamma d(y)$ for all $x, y \in S$ and $\gamma \in \Gamma$, then d is said to *act as a* Γ-*homomorphism* on S.

Throughout this paper M will be a zero-symmetric Γ-near-ring. The symbol C will denote the multiplicative center of M, i.e.,

$$C := \{x \in M \mid x\gamma m = m\gamma x \text{ for all } m \in M \text{ and } \gamma \in \Gamma\}.$$

For $x, y \in M$ and $\gamma \in \Gamma$, the symbol $[x, y]_\gamma$ will denote the commutator $x\gamma y - y\gamma x$, while the symbol (x, y) will denote the additive-group commutator $x + y - x - y$.

LEMMA 2.1. [10, Lemma 3.1] *For all $x, y \in M$ and $\beta, \gamma \in \Gamma$, if $z \in C$ then $[z\beta x, y]_\gamma = z\beta[x, y]_\gamma = z\gamma[x, y]_\beta$.*

LEMMA 2.2. [10, Lemma 3.2] *Let M be a prime Γ-near-ring.*

(i) *If $z \in C \setminus \{0\}$, then z is not a zero divisor.*

(ii) *Let $z \in C \setminus \{0\}$ be an element such that $z + z \in C$. Then $(M, +)$ is abelian.*

(iii) *If $z \in C \setminus \{0\}$ and x is an element of M such that $x\gamma z \in C$ or $z\gamma x \in C$ where $\gamma \in \Gamma$, then $x \in C$.*

LEMMA 2.3. [5, Proposition 3.4] *Let M be a Γ-near-ring and d be a Γ-derivation on M. Then, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$*

(i) $(x\gamma d(y) + d(x)\gamma y)\beta z = x\gamma d(y)\beta z + d(x)\gamma y\beta z,$

(ii) $(d(x)\gamma y + x\gamma d(y))\beta z = d(x)\gamma y\beta z + x\gamma d(y)\beta z.$

3. Main results

LEMMA 3.1. *If M is prime and 2-torsion free and d is a Γ-derivation on M such that $d^2 = 0$, then $d = 0$.*

PROOF. Let $x, y \in M$ and $\gamma \in \Gamma$. From the hypothesis, we get $0 = d^2(x\gamma y) = 2d(x)\gamma d(y)$. Since M is 2-torsion free, we have,

$$d(x)\gamma d(y) = 0. \tag{1}$$

Taking $y\beta z$ instead of y in (1), we obtain

$$\begin{aligned} 0 &= d(x)\gamma d(y\beta z) = d(x)\gamma y\beta d(z) + d(x)\gamma d(y)\beta z \\ &= d(x)\gamma y\beta d(z) + 0\beta z = d(x)\gamma y\beta d(z) \end{aligned}$$

for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. Since M is prime, it follows that $d(x) = 0$ for all $x \in M$ or $d(z) = 0$ for all $z \in M$. Hence d is zero. □

LEMMA 3.2. *Let d be a Γ-derivation on M and suppose that $u \in M$ is not a left zero divisor. If $[u, d(u)]_\gamma = 0$ for all $\gamma \in \Gamma$, then (x, u) is a constant for all $x \in M$.*

PROOF. Let $x \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} d(u\gamma(u + x)) &= u\gamma d(u + x) + d(u)\gamma(u + x) \\ &= u\gamma(d(u) + d(x)) + d(u)\gamma u + d(u)\gamma x \\ &= u\gamma d(u) + u\gamma d(x) + d(u)\gamma u + d(u)\gamma x \end{aligned}$$

and

$$\begin{aligned} d(u\gamma u + u\gamma x) &= d(u\gamma u) + d(u\gamma x) \\ &= u\gamma d(u) + d(u)\gamma u + u\gamma d(x) + d(u)\gamma x. \end{aligned}$$

Since $d(u\gamma(u + x)) = d(u\gamma u + u\gamma x)$, it follows that

$$u\gamma d(x) + d(u)\gamma u = d(u)\gamma u + u\gamma d(x)$$

so from hypothesis that $u\gamma d((x, u)) = 0$. Since u is not a left zero divisor, we have that (x, u) is a constant. □

THEOREM 3.3. *Let d be a non-trivial Γ-derivation on M. Let M have no non-zero divisor of zero. If $[x, d(x)]_\gamma = 0$ for all $x \in M$ and $\gamma \in \Gamma$, then $(M, +)$ is abelian.*

PROOF. Let $c = (a, b)$ be any additive commutator where $a, b \in M$. Then c is a constant by Lemma 3.2, i.e. $d(c) = 0$. Moreover, since $w\gamma c = (w\gamma a, w\gamma b)$ where $w \in M$ and $\gamma \in \Gamma$, $w\gamma c$ is a constant by Lemma 3.2. In this case, we obtain $0 = d(w\gamma c) = w\gamma d(c) + d(w)\gamma c = d(w)\gamma c$ for all $w \in M$ and $\gamma \in \Gamma$. Since $d(w) \neq 0$ and M have no non-zero divisor of zero, we get that $c = 0$. This completes the proof. $\square$

THEOREM 3.4. *Let M be prime and let d be a non-trivial Γ-derivation on M. If $d(x) \in C$ for all $x \in M$, then $(M, +)$ is abelian. Furthermore, if M is 2-torsion free, then M is commutative.*

PROOF. Let c be an arbitrary constant and x be a non-constant. Then $d(x\gamma c) = x\gamma d(c) + d(x)\gamma c = d(x)\gamma c \in C$ where $\gamma \in \Gamma$ and so, since $d(x) \in C \setminus \{0\}$, we have that $c \in C \setminus \{0\}$ by Lemma 2.2(iii). Since $d(c + c) = 0$ for all constant c, we get that $(M, +)$ is abelian by Lemma 2.2(ii). Now, suppose that 0 is the only constant and u is not a zero divisor for $u \in M$. Let $x \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} d(u\gamma(u + x)) &= u\gamma d(u + x) + d(u)\gamma(u + x) \\ &= u\gamma d(u) + u\gamma d(x) + d(u)\gamma u + d(u)\gamma x \end{aligned}$$

and

$$\begin{aligned} d(u\gamma(u + x)) = d(u\gamma u + u\gamma x) &= d(u\gamma u) + d(u\gamma x) \\ &= u\gamma d(u) + d(u)\gamma u + u\gamma d(x) + d(u)\gamma x. \end{aligned}$$

Comparing above two expressions, we obtain, for all $x \in M$ and $\gamma \in \Gamma$

$$u\gamma d(x) + d(u)\gamma u = d(u)\gamma u + u\gamma d(x).$$

From the hypothesis and this equation, we have, for all $x \in M$ and $\gamma \in \Gamma$

$$\begin{aligned} 0 &= u\gamma(d(x) + d(u) - d(x) - d(u)) \\ &= u\gamma d(x + u - x - u) = u\gamma d((x, u)). \end{aligned}$$

Thus, since u is not a zero divisor, we get $d((x, u)) = 0$ for all $x \in M$ and so (x, u) is a constant, i.e., $(x, u) = 0$ for all $x \in M$. It follows that $u \in C(M)$ which is center of $(M, +)$.

Now, let x be a non-zero element of M. Since $d(M) \subseteq C$, it follows from Lemma 2.2(i) that $d(x)$ is not a zero divisor. This implies that $d(x) \in C(M)$ for all $0 \neq x \in M$. Let y be a non-zero element of M. Then $0 = d(x) + d(y) - d(x) - d(y) = d((x, y))$ and so $(x, y) = 0$ for all $x, y \in M$, that is, $(M, +)$ is abelian.

Taking $x\beta y$ instead of x in the hypothesis where $\beta \in \Gamma$, we obtain, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$

$$0 = [d(x\beta y), z]_\gamma = [x\beta d(y) + d(x)\beta y, z]_\gamma = [x\beta d(y), z]_\gamma + [d(x)\beta y, z]_\gamma.$$

Thus, from Lemma 2.1(i), we get, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$

$$d(x)\beta[y, z]_\gamma = d(y)\beta[z, x]_\gamma. \qquad (2)$$

Replacing x by $d(x)$ in (2) and using the hypothesis, we have, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$

$$d^2(x)\beta[y,z]_\gamma = 0.$$

Now, assume that M is not commutative. Choosing $y, z \in M$ with $[y,z]_\gamma \neq 0$ and since a central element $d^2(x)$ cannot be a non-zero divisor of zero, we obtain that $d^2(x) = 0$ for all $x \in M$. Then $d = 0$ by Lemma 3.1. But this is a contradiction. □

LEMMA 3.5. *Let M be prime and $x, y \in M$. If $x \in C$ and $x\Gamma y = \{0\}$, then $x = 0$ or $y = 0$.*

PROOF. Straightforward. □

THEOREM 3.6. *Let M be prime and let d be a non-trivial Γ-derivation on M. If $[d(x), d(y)]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, then $(M, +)$ is abelian. Moreover, if M is 2-torsion free, d acts as a Γ-homomorphism on M and $d^2(x) \in C$ for all $x \in M$, then M is commutative.*

PROOF. Since $[d(x), d(y)]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, if both w and $w + w$ commute elementwise with $d(x)$ for all $x \in M$, we get $0 = [w, d(x+y)]_\gamma = w\gamma d((x,y))$ for all $x, y \in M$ and $\gamma \in \Gamma$. Thus, taking $d(z)$ instead of w in the last equation, we have, for all $x, y, z \in M$ and $\gamma \in \Gamma$

$$d(z)\gamma d((x,y)) = 0. \tag{3}$$

Substituting $z\beta v$ for z in (3) and using Lemma 2.3(i), we obtain

$$d(z)\beta v\gamma d((x,y)) = 0$$

for all $x, y, z, v \in M$ and $\beta, \gamma \in \Gamma$. By primeness, we get that $d((x,y)) = 0$ for all $x, y \in M$. Since $z\gamma(x,y)$ is also an additive commutator for any $z \in M$ and $\gamma \in \Gamma$, we have $0 = d(z\gamma(x,y)) = d(z)\gamma(x,y)$ and by primeness $(x,y) = 0$ for all $x, y \in M$, i.e., $(M, +)$ is abelian.

Now, taking $x\beta z$ instead of x in the hypothesis that $[d(x), d(y)]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$, we obtain, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$

$$\begin{aligned} 0 &= d(x\beta z)\gamma d(y) - d(y)\gamma d(x\beta z) \\ &= (x\beta d(z) + d(x)\beta z)\gamma d(y) - d(y)\gamma(x\beta d(z) + d(x)\beta z) \end{aligned}$$

and so, from Lemma 2.3(i) and since $(M, +)$ is abelian, we get, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$

$$0 = x\beta d(z)\gamma d(y) - d(y)\gamma x\beta d(z) + d(x)\beta z\gamma d(y) - d(y)\gamma d(x)\beta z.$$

Replacing x by $d(x)$ in the previous equation and since d acts as a Γ-homomorphism on M, we have that

$$d^2(x)\beta z\gamma d(y) - d(y)\gamma d^2(x)\beta z = 0. \tag{4}$$

From (4) and since $d^2(x) \in C$ for all $x \in M$, we obtain, for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$

$$d^2(x)\gamma[z, d(y)]_\beta = 0.$$

Since $d^2(x) \in C$ for all $x \in M$ and from Lemma 3.5, we get that $d^2(x) = 0$ or $[z, d(y)]_\beta = 0$ for all $x, y, z \in M$ and $\beta, \gamma \in \Gamma$. If $d^2(x) = 0$ for all $x \in M$, then $d = 0$ by Lemma 3.1. But this is a contradiction. Thus we have $[z, d(y)]_\beta = 0$ for all $y, z \in M$ and $\beta \in \Gamma$, that is, $d(y) \in C$ for all $y \in M$. In this case, M is commutative by Theorem 3.4. □

LEMMA 3.7. *Let M be prime and let $U(\neq \{0\})$ be a left (resp. right) invariant subset of M. If $U \subseteq C$, then M is commutative.*

PROOF. Since a central left invariant subset of M is a right invariant subset of M, we may assume that $U \neq \{0\}$ is a right invariant subset of M. Let $x \in M$, $u \in U$ and $\gamma \in \Gamma$. From the hypothesis, we get that $[u, x]_\gamma = 0$. Replacing u by $u\beta y$ in the previous equation, we obtain $u\beta[y, x]_\gamma = 0$ for all $x, y \in M$, $u \in U$ and $\beta, \gamma \in \Gamma$ by Lemma 2.1(i). Thus, since $U \neq \{0\}$, we have $[y, x]_\gamma = 0$ for all $x, y \in M$ and $\gamma \in \Gamma$ by Lemma 3.5. This completes the proof. □

THEOREM 3.8. *Let M be prime and 2-torsion free and let $U(\neq \{0\})$ be a left (resp. right) invariant subset of M and d be a Γ-derivation on M. If $d(U) \subseteq C \setminus \{0\}$, then M is commutative.*

PROOF. Let $x \in M$, $u \in U$ and $\gamma \in \Gamma$. From the hypothesis, we get that $[d(u), x]_\gamma = 0$. Taking $u\beta u$ instead of u in the previous equation, we have $2[d(u)\beta u, x]_\gamma = 0$ for all $x \in M$, $u \in U$ and $\gamma \in \Gamma$. Since M is 2-torsion free, we obtain, for all $x \in M$, $u \in U$ and $\gamma \in \Gamma$

$$[d(u)\beta u, x]_\gamma = 0. \tag{5}$$

From (5) and Lemma 2.1(i), we get, for all $x \in M$, $u \in U$ and $\beta, \gamma \in \Gamma$

$$d(u)\beta[u, x]_\gamma = 0. \tag{6}$$

From (6) and the hypothesis, we have that $[u, x]_\gamma = 0$ for all $x \in M$, $u \in U$ and $\gamma \in \Gamma$ by Lemma 3.5. Thus M is commutative by Lemma 3.7. □

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