Further Improvements on Bose's 2D Stability Test

Li Xu, Minoru Yamada, Zhiping Lin, Osami Saito, and Yoshihisa Anazawa

Abstract: This paper proposes some further improvements on N.K. Bose's 2D stability test for polynomials with real coefficients by revealing symmetric properties of the polynomials, resultants occurring in the test and by generalizing Sturm's method. The improved test can be fulfilled by a totally algebraic algorithm with a finite number of steps and the computational complexity is largely reduced as it involves only certain real variable polynomials with degrees not exceeding half of their previous complex variable counterparts. Nontrivial examples for 2D polynomials having both numerical and literal coefficients are also shown to illustrate the computational advantage of the proposed method.

Keywords: Discrete systems, 2D systems, stability test, Sturm's method.

1. INTRODUCTION

Two-dimensional (2D) systems have wide applications in areas such as signal and image processing, multipass processes and iterative algorithm designs. As stability is the first necessary requirement for a 2D system to work properly, the problem of algebraic tests for 2D system stability has been continuously attracting considerable attention (see, e.g., [1-8] and the references therein). A 2D discrete system given by the transfer function

$$G(z_1,z_2)=\frac{N(z_1,z_2)}{D(z_1,z_2)} \tag{1}$$

is (structurally) stable by definition if and only if

$$D(z_1,z_2)\neq 0,\quad |z_1|\leq 1, |z_2|\leq 1, \tag{2}$$

where N and D are assumed to be 2D factor coprime polynomials. It is well known that the condition of (2) is equivalent to the conditions [9]

$$D(z_1,0)\neq 0,\quad |z_1|\leq 1, \tag{3}$$

$$D(z_1,z_2)\neq 0,\quad |z_1|=1, |z_2|\leq 1. \tag{4}$$

The condition of (3) can be checked by well-known 1D methods, and the main difficulty is how to test the condition (4). Based on Schüssler's 1D stability criterion, N. K. Bose has developed a resultant-based algebraic test for (4) and also a simplified version of it as follows [1,2].

Let $D(z_1,z_2)$ be a 2D polynomial in two complex variables z_1, z_2 with real or complex coefficients, n_2 the degree of $D(z_1,z_2)$ in z_2, and write $D_1(z_1,z_2)=z_2^{n_2}D(z_1,z_2^{-1})$ in the form

$$D_1(z_1,z_2)=\sum_{k=0}^{n_2} d_k(z_1)z_2^k, \tag{5}$$

where $d_k(z_1)$ are 1D polynomials in z_1 with real or complex coefficients. Furthermore, define

$$D_{1s}(z_1,z_2)=\frac{1}{2}\left[\sum_{k=0}^{n_2} d_k(z_1)z_2^k + z_2^{n_2}\sum_{k=0}^{n_2}\overline{d_k(z_1)}z_2^{-k}\right], \tag{6}$$

$$D_{1a}(z_1,z_2)=\frac{1}{2}\left[\sum_{k=0}^{n_2} d_k(z_1)z_2^k - z_2^{n_2}\sum_{k=0}^{n_2}\overline{d_k(z_1)}z_2^{-k}\right], \tag{7}$$

where $\overline{d_k(z_1)}$ denotes the complex conjugates of both the coefficients and the complex variable z_1. Then, the simplified Bose's test for (4) is given by the following theorem.

Theorem 1 [2]: The polynomial $D_1(z_1,z_2)\neq 0$ for $|z_1|=1$, $|z_2|\geq 1$, or equivalently, $D(z_1,z_2)\neq 0$ for

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$|z_1|=1$, $|z_2|\le 1$ if and only if

(a) $\left|d_0(1)/d_{n_2}(1)\right|<1$;

(b) all the zeros of $D_{1s}(1,z_2)$ and $D_{1a}(1,z_2)$ are located on the unit circle $|z_2|=1$;

(c) the zeros of $D_{1s}(1,z_2)$ and $D_{1a}(1,z_2)$ are simple and alternate on $|z_2|=1$; and

(d) the resultant $R(z_1)$ of $D_{1s}(z_1,z_2)$ and $D_{1a}(z_1,z_2)$ satisfies that $R(z_1)\ne 0$ for $z_1=e^{j\theta_1}$, $0\le\theta_1<2\pi$, i.e., $|z_1|=1$.

Since $D_{1s}(1,z_2)$ and $D_{1a}(1,z_2)$ are 1D polynomials, it is possible to test the conditions of (b) and (c) by using, e.g., the 1D continued fraction expansion (CFE) method [10]. On the other hand, the resultant $R(e^{j\theta})$ is a trigonometric function and it may not be as straightforward to simplify this trigonometric function and carry out the test of condition (d) [2].

As the use of resultants is an effective method for stability tests, the purpose of this paper is to propose some further improvements on Bose's stability test for the case where $D(z_1,z_2)$ has real coefficients, such that the test can be fulfilled more efficiently by a totally algebraic algorithm and can be implemented more easily and systematically using a 2D CAD system [12]. In the paper, it is first shown that $D_{1s}(1,z_2)$, $D_{1a}(1,z_2)$ can be expressed in the forms of $D_{1s}(1,z_2)=L_{1s}(z_2)\tilde{D}_{1s}(z_2)$ and $D_{1a}(1,z_2)=L_{1a}(z_2)\tilde{D}_{1a}(z_2)$ where $\tilde{D}_{1s}(z_2)$, $\tilde{D}_{1a}(z_2)$ are certain self-inversive polynomials. Thus they can be easily converted into polynomials $\hat{D}_{1s}(x_2)$, $\hat{D}_{1a}(x_2)$ with degrees in the real variable $x_2=(z_2+z_2^{-1})/2$ not exceeding $n_2/2$. This fact then leads to a result that the conditions of (b) and (c) hold true if and only if the zeros of $\hat{D}_{1s}(x_2)$ and $\hat{D}_{1a}(x_2)$ are located within $|x_2|<1$ and are simple and alternate in a certain order, which can be tested by generalizing Sturm's method. In fact, this is a significant new result even for the 1D stability test as it clarifies that by exploiting the symmetric properties of the associated polynomials we only need to carry out the test for certain real variable polynomials with degrees not exceeding half of the degrees of the original polynomials. Further, it is shown that the resultant $R(z_1)$ is in fact a self-inversive polynomial itself and thus can be converted into a polynomial $\hat{R}(x)$ with the degree in $x_1=(z_1+z_1^{-1})/2$ being only half of the degree of $R(z_1)$ in z_1. Therefore, the test of (d) can be readily accomplished by directly applying Sturm's method to $\hat{R}(x_1)$ on $|x_1|\le 1$. We note that some related results on using Chebyshev polynomials in 1D stability tests have recently been given in [11].

The paper is organized as follows. In Section 2, improvements on the 1D stability test portion will be given, while in Section 3 the self-inversive property of the resultant $R(z_1)$ will be shown and possible improvements for Bose's 2D stability test will be presented. Nontrivial examples for polynomials having both numerical and literal coefficients will also be shown in Section 4 to illustrate the computational advantage of the improved test. Finally, some concluding remarks are given in Section 5.

2. IMPROVEMENTS ON THE 1D STABILITY TEST PORTION

Let $F(z)$ be a 1D polynomial of degree n described by

$$F(z)=\sum_{k=0}^{n} f_k z^k \tag{8}$$

where f_k, $k=1,\dots,n$ are real coefficients and define $F_1(z)$, $F_2(z)$ as

$$F_1(z)=\frac{1}{2}(F(z)+z^nF(z^{-1}))=\frac{1}{2}\sum_{k=0}^{n} f_{1,k}z^k, \tag{9}$$

$$F_2(z)=\frac{1}{2}(F(z)-z^nF(z^{-1}))=\frac{1}{2}\sum_{k=0}^{n} f_{2,k}z^k. \tag{10}$$

For the stability of $F(z)$, the following result stated in Lemma 1 is widely known [13,14]. It should be noted here, however, that the stability conditions for 1D polynomials are usually given with respect to a forward operator, while in the 2D context delay operators are usually used. Though this difference in stability conditions may cause confusion, to maintain consistency with the results in the literature, we use z and z_1 to denote the forward and delay operators respectively throughout this paper. Therefore, whenever we mention 1D stability, we imply the corresponding definition with respect to z or z_1, i.e., $F(z)\ne 0$, $\forall |z|\ge 1$ or $F(z_1)\ne 0$, $\forall |z_1|\le 1$.

Lemma 1: The polynomial $F(z)$ is stable, i.e., $F(z)\ne 0$, $\forall |z|\ge 1$ if and only if

1) $|f_0/f_n|<1$;

2) all the zeros of $F_1(z)$ and $F_2(z)$ are located on $|z|=1$; and

3) the zeros of $F_1(z)$ and $F_2(z)$ are simple and alternate on $|z|=1$.

This stability criterion can be tested by the CFE approach [10]. In this section, however, we want to show the following results: by revealing and utilizing the symmetrical properties of $F_1(z), F_2(z)$, two real variable polynomials $\hat{F}_1(x), \hat{F}_2(x)$ can be constructed that have degrees not exceeding half of the ones for $F_1(z), F_2(z)$, and the stability of $F(z)$ can now be tested by simply checking if $\hat{F}_1(x), \hat{F}_2(x)$ satisfy the conditions of Lemma 1 within the real interval $|x|<1$, which can be easily fulfilled by using some extended results of Sturm's method.

Lemma 2: $F_1(z)$, $F_2(z)$ can be expressed as follows.

$$F_1(z)=\begin{cases} z^{n/2}\tilde{F}_1(z) & n:\text{even} \\ z^{(n-1)/2}(z+1)\tilde{F}_1(z) & n:\text{odd}\end{cases} \tag{11}$$

$$F_2(z)=\begin{cases} z^{n/2}(z-z^{-1})\tilde{F}_2(z) & n:\text{even} \\ z^{(n-1)/2}(z-1)\tilde{F}_2(z) & n:\text{odd}\end{cases} \tag{12}$$

where $\tilde{F}_i(z),\ i=1,2$ are self-inversive polynomials given by

$$\tilde{F}_i(z)=\frac{1}{2}\left[\tilde{f}_{i,0}+\sum_{k=1}^{m_i}\tilde{f}_{i,k}(z^k+z^{-k})\right] \tag{13}$$

and when n is even,

$$\tilde{f}_{1,k}=f_{1,n/2+k},\quad k=0,1,\ldots,n/2,\ m_1=n/2, \tag{14}$$

$$\tilde{f}_{2,k}=f_{2,n/2+k+1}+f_{2,n/2+k+3}+\ldots+f_{2,n/2+l}, \tag{15}$$
$$k=0,1,\ldots,n/2-1,\quad m_2=n/2-1,$$
$$l=\begin{cases} n/2-1 & n/2+k:\mathit{even}, \\ n/2 & n/2+k:\mathit{odd};\end{cases}$$

when n is odd,

$$\begin{aligned}\tilde{f}_{1,k}&=f_{1,(n+1)/2+k}-f_{1,(n+1)/2+k+1} \\ &\quad+f_{1,(n+1)/2+k+2}-\cdots+(-1)^{(n-1)/2+k}f_{1,n},\end{aligned} \tag{16}$$
$$k=0,1,\ldots,(n-1)/2,\quad m_1=(n-1)/2,$$

$$\tilde{f}_{2,k}=f_{2,(n+1)/2+k}+f_{2,(n+1)/2+k+1}+\cdots+f_{2,n} \tag{17}$$
$$k=0,1,\ldots,(n-1)/2,\quad m_2=(n-1)/2.$$

Proof: First, note that $F_1(z), F_2(z)$ can always be written as

$$\begin{aligned}F_1(z)&=\frac{1}{2}\left[\sum_{k=0}^{n}f_kz^k+z^n\sum_{k=0}^{n}f_kz^{-k}\right] \\ &=\frac{1}{2}\left[\sum_{k=0}^{n}f_kz^k+\sum_{k=0}^{n}f_kz^{n-k}\right] \\ &=\frac{1}{2}\left[\sum_{k=0}^{n}f_kz^k+\sum_{k=0}^{n}f_{n-k}z^k\right] \\ &=\frac{1}{2}\sum_{k=0}^{n}(f_k+f_{n-k})z^k,\end{aligned} \tag{18}$$

$$\begin{aligned}F_2(z)&=\frac{1}{2}\left[\sum_{k=0}^{n}f_kz^k-z^n\sum_{k=0}^{n}f_kz^{-k}\right] \\ &=\frac{1}{2}\sum_{k=0}^{n}(f_k-f_{n-k})z^k.\end{aligned} \tag{19}$$

Comparing (18), (19) with (9), (10), we see

$$f_{1,k}=f_{1,n-k}=f_k+f_{n-k}, \tag{20}$$

$$f_{2,k}=-f_{2,n-k}=f_k-f_{n-k}. \tag{21}$$

If n is even, multiplying (9) by $z^{-n/2}$ yields

$$\begin{aligned}z^{-n/2}F_1(z)&=\frac{1}{2}\sum_{k=0}^{n}f_{1,k}z^{k-n/2} \\ &=\frac{1}{2}\left[\sum_{k=0}^{n/2-1}f_{1,k}z^{k-n/2}+f_{1,n/2}+\sum_{k=n/2+1}^{n}f_{1,k}z^{k-n/2}\right] \\ &=\frac{1}{2}\left[\sum_{k=1}^{n/2}f_{1,n/2-k}z^{-k}+f_{1,n/2}+\sum_{k=1}^{n/2}f_{1,n/2+k}z^k\right] \\ &=\frac{1}{2}\left[\sum_{k=1}^{n/2}f_{1,n/2+k}z^{-k}+f_{1,n/2}+\sum_{k=1}^{n/2}f_{1,n/2+k}z^k\right] \\ &=\frac{1}{2}\left[f_{1,n/2}+\sum_{k=1}^{n/2}f_{1,n/2+k}(z^k+z^{-k})\right],\end{aligned} \tag{22}$$

where $f_{1,k}=f_{1,n-k}$ is used. Setting the right-hand side of (22) as $\tilde{F}_1(z)$, we get the results of (11), (13) and (14).

Further multiplying (10) by $z^{-n/2}$ gives

$$\begin{aligned}z^{-n/2}F_2(z)&=\frac{1}{2}\sum_{k=0}^{n}f_{2,k}z^{k-n/2} \\ &=\frac{1}{2}\left[\sum_{k=0}^{n/2-1}f_{2,k}z^{k-n/2}+f_{2,n/2}+\sum_{k=n/2+1}^{n}f_{2,k}z^{k-n/2}\right]\end{aligned}$$

$$=\frac{1}{2}\left[\sum_{k=1}^{n/2} f_{2,n/2-k}z^{-k}+f_{2,n/2}+\sum_{k=1}^{n/2} f_{2,n/2+k}z^{k}\right]$$
$$=\frac{1}{2}\left[-\sum_{k=1}^{n/2} f_{2,n/2+k}z^{-k}+\sum_{k=1}^{n/2} f_{2,n/2+k}z^{k}\right]$$
$$=\frac{1}{2}\sum_{k=1}^{n/2} f_{2,n/2+k}(z^{k}-z^{-k}), \tag{23}$$

where the relations $f_{2,k}=-f_{2,n-k}$, $f_{2,n/2}=f_{n/2}-f_{n/2}=0$ are applied. Since $z^k-z^{-k}=(z-z^{-1})(z^{k-1}+z^{k-3}+\cdots+z^{-k+3}+z^{-k+1})$ for $k\geq 2$, (23) can be expanded as

$$z^{-n/2}F_2(z)=\frac{1}{2}(z-z^{-1})\Big[f_{2,n/2+1}(\quad 1 \quad) + f_{2,n/2+2}(\ z+z^{-1}\ ) + f_{2,n/2+3}(z^2+1+z^{-2}) \ \vdots \ +\begin{cases} f_{2,n}(z^{n/2-1}+\cdots+\quad 1 \quad+\cdots+z^{n/2-1}), & \frac{n}{2}: odd \\ f_{2,n}(z^{n/2-1}+\cdots+z+z^{-1}+\cdots+z^{n/2-1}), & \frac{n}{2}: even\end{cases}\Big]. \tag{24}$$

By collecting all the coefficients of the terms of 1 and (z^k+z^{-k}), $k=1,\ldots,n/2-1$ in the parentheses and denoting the calculated coefficients as $\tilde{f}_{2,k}$, respectively, the relation

$$z^{-n/2}F_2(z)=\frac{1}{2}(z-z^{-1})\left[\tilde{f}_{2,0}+\sum_{k=1}^{n/2-1}\tilde{f}_{2,k}(z^k+z^{-k})\right] \tag{25}$$

can be obtained that implies (12), (13). It is ready to verify that $\tilde{f}_{2,k}$ satisfies (15).

When n is odd, the following result can be obtained by multiplication of $z^{-(n-1)/2}$ to $F_1(z)$.

$$z^{-(n-1)/2}F_1(z)=\frac{1}{2}\sum_{k=0}^{n} f_{1,k}z^{k-(n-1)/2} = \frac{1}{2}\sum_{k=0}^{(n-1)/2} f_{1,(n+1)/2+k}(z^{k+1}+z^{-k}), \tag{26}$$

where the fact $f_{1,k}=f_{1,n-k}$ is used again. Noting that $z^{k+1}+z^{-k}=z^{-k}(z^{2k+1}+1)$ and $2k+1$ is always an odd number, we can show $z^{k+1}+z^{-k}=(z+1)(z^k-z^{k-1}+z^{k-2}-\cdots-z^{-k+1}+z^{-k})$. Substituting this result into (26) yields

$$z^{-(n-1)/2}F_1(z)=\frac{1}{2}(z+1)\big\{f_{1,(n+1)/2}(\quad 1 \quad) + f_{1,(n+1)/2+1}(\quad z-1+z^{-1}\quad) + f_{1,(n+1)/2+2}(z^2-z+1-z^{-1}+z^{-2}) \ \vdots \ + f_{1,n}(z^{(n-1)/2}-z^{(n-1)/2-1}+\cdots+(-1)^{(n-1)/2} +\cdots-z^{-(n-1)/2+1}+z^{-(n-1)/2})\big\}. \tag{28}$$

Again, collecting the coefficients corresponding to the terms of 1 and (z^k+z^{-k}), $k=1,\ldots,(n-1)/2$ and denoting the obtained results by $\tilde{f}_{1,k}$, the results of (11), (13) and (16) are obtained.

Similarly, noting that $f_{2,k}=-f_{2,n-k}$, we can show

$$z^{-(n-1)/2}F_2(z)=\frac{1}{2}\sum_{k=0}^{n} f_{2,k}z^{k-(n-1)/2} =\frac{1}{2}\sum_{k=0}^{(n-1)/2} f_{2,(n+1)/2+k}(z^{k+1}-z^{-k}). \tag{29}$$

Since $z^{k+1}-z^{-k}=(z-1)(z^k+z^{k-1}+\cdots+z^{-k})$, the following equation holds.

$$z^{-(n-1)/2}F_2(z)=\frac{1}{2}(z-1)\big\{f_{2,(n+1)/2}(\quad 1 \quad) + f_{2,(n+1)/2+1}(\quad z+1+z^{-1}\quad) + f_{2,(n+1)/2+2}(z^2+z+1+z^{-1}+z^{-2}) \ \vdots \ + f_{2,n}(z^{(n-1)/2}+z^{(n-1)/2-1}\cdots+1+\cdots +z^{-(n-1)/2+1}+z^{-(n-1)/2})\big\}. \tag{30}$$

In the same way as above, we see that the coefficients $\tilde{f}_{2,k}$ for (12), (13) summarized from (30) are the ones given in (17).

Corollary 1: The coefficients $\tilde{f}_{i,k}$, $i=1,2$ for $\tilde{F}_i(z)$ defined by (13) can be calculated recursively as follows.

$$\tilde{f}_{1,k}=\begin{cases} f_{1,n/2+k}, \quad k=0,1,\ldots,n/2 & n: even \\ f_{1,(n+1)/2+k}-\tilde{f}_{1,k+1}, & n: odd \\ \quad k=0,1,\ldots,(n-1)/2-1 & \\ \quad (\text{with } \tilde{f}_{1,(n-1)/2}=f_{1,n}) & \end{cases} \tag{31}$$

$$\tilde{f}_{2,k}=\begin{cases} f_{2,n/2+k+1}+\tilde{f}_{2,k+2}, \\ \quad k=0,1,\ldots,\frac{n}{2}-3, \qquad n:\ even \\ \quad (\text{with } \tilde{f}_{2,n/2-1}=f_{2,n},\ \tilde{f}_{2,n/2-2}=f_{2,n-1}) \\ f_{2,(n+1)/2+k}-\tilde{f}_{2,k+1}, \\ \quad k=0,1,\ldots,(n-1)/2-1,\ n:\ odd \\ \quad (\text{with } \tilde{f}_{2,(n-1)/2}=f_{2,n}). \end{cases} \tag{32}$$

Proof: It follows directly from (14)–(17).

As $\tilde{F}_i(z),\ i=1,2$, are self-inversive polynomials and $\bar{z}=z^{-1}$ on the unit circle $|z|=1$, $\tilde{F}_i(z),\ i=1,2,$ can be converted into polynomials in a real variable $x=(z+z^{-1})/2$ in the following way [15].

$$\tilde{F}_i(z)=\boldsymbol{f}_i\boldsymbol{Z}^{(m_i)}=\boldsymbol{f}_i\boldsymbol{D}_{m_i}\boldsymbol{x}_{m_i}\triangleq\hat{F}_i(x),\quad i=1,2 \tag{33}$$

where $\boldsymbol{f}_i=[\tilde{f}_{i,0}\ \tilde{f}_{i,1}\ \cdots\ \tilde{f}_{i,m_i}]$,

$$\boldsymbol{Z}^{(m_i)}=\begin{bmatrix}1\\ z^{-1}+z^{1}\\ z^{-2}+z^{2}\\ \vdots\\ z^{-m_i}+z^{m_i}\end{bmatrix}=\begin{bmatrix}1\\ \boldsymbol{d}_1\boldsymbol{x}_1\\ \boldsymbol{d}_2\boldsymbol{x}_2\\ \vdots\\ \boldsymbol{d}_{m_i}\boldsymbol{x}_{m_i}\end{bmatrix}=\boldsymbol{D}_{m_i}\boldsymbol{x}_{m_i},$$

$$\boldsymbol{x}_k=[1\ x\ x^2\cdots x^k]^T,$$
$$\boldsymbol{d}_k=[d_{k,0}\ d_{k,1}\cdots d_{k,k}]$$

which is defined by

$$z^k+z^{-k}=\sum_{j=0}^{k}d_{k,j}x^j=\boldsymbol{d}_k\boldsymbol{x}_k \tag{34}$$

and calculated by the recursion

$$d_{k,0}=-d_{k-2,0},\ d_{k,j}=2d_{k-1,j-1}-d_{k-2,j},\quad k\geq 3,\quad j=1,2,\ldots,k \tag{35}$$

with $[d_{1,0}\ d_{1,1}]=[0\ 2]$, $[d_{2,0}\ d_{2,1}\ d_{2,2}]=[-2\ 0\ 4]$ and $d_{i,j}=0, \forall i<j$. Further, it is easy to see that

$$\boldsymbol{D}_{m_i}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ d_{1,0} & d_{1,1} & 0 & \cdots & 0\\ d_{2,0} & d_{2,1} & d_{2,2} & & \vdots\\ \vdots & \vdots & \vdots & \ddots & 0\\ d_{m_i,0} & d_{m_i,1} & d_{m_i,2} & \cdots & d_{m_i,m_i}\end{bmatrix}.$$

Now, for $\hat{F}_i(x),\ i=1,2$ obtained above, the following result can be given.

Theorem 2: The polynomial $F(z)$ is stable, i.e., $F(z)\neq 0,\ \forall\,|z|\geq 1$, if and only if

(a) $|f_0/f_n|<0$;

(b) all the real zeros of $\hat{F}_1(x)$ and $\hat{F}_2(x)$ are located within the interval $|x|<1$; and

(c) the real zeros of $\hat{F}_1(x)$ and $\hat{F}_2(x)$ are simple and alternate within $|x|<1$. Further, when n is odd and if we order all these zeros within $|x|<1$ according to their values starting from the smallest, then the last (first) zero belongs to $\hat{F}_1(x)$ ($\hat{F}_2(x)$).

Proof: Denote by $\Gamma(F(z))$ the set of zeros of $F(z)$. Then, due to (11) and (12), we see that, for $z\neq 0$,

$$\Gamma(F_1(z))=\Gamma(\tilde{F}_1(z)), \tag{36}$$
$$\Gamma(F_2(z))=\Gamma(\tilde{F}_2(z))\cup\{\pm 1\}, \tag{37}$$

when n is even, while

$$\Gamma(F_1(z))=\Gamma(\tilde{F}_1(z))\cup\{-1\}, \tag{38}$$
$$\Gamma(F_2(z))=\Gamma(\tilde{F}_2(z))\cup\{1\}, \tag{39}$$

when n is odd.

Based on these relations, we first show the necessity. In view of the fact that $F_i(z)\neq 0$ for $|z|\neq 1$ implies that $\tilde{F}_i(z)\neq 0$ for $|z|\neq 1$, if $F_i(z),\ i=1,2,$ satisfy the conditions 2), 3) of Lemma 1, then the zeros of $\tilde{F}_i(z),\ i=1,2$, must be on $|z|=1$ except $z=\pm 1$, and are simple and alternate. Noting that $x=(z+z^{-1})/2$ maps the unit circle $|z|=1$ of the complex plane to the real interval $-1\leq x\leq 1$, and that the complex zeros of a polynomial with real coefficients appear only in pairs of conjugates, we conclude that $\hat{F}_i(x),\ i=1,2$, have zeros between $-1<x<1$ and the zeros are alternate. Further, when n is even, as $F_2(z)$ has certainly the zeros of $z=\pm 1$ as shown in (11) or (37), $\hat{F}_1(x)$ must have zeros which locate, due to the condition 3) of Lemma 1, at the farthest two sides in $|x|<1$. As the degrees of $\hat{F}_1(x)$ and $\hat{F}_2(x)$ in x are respectively $n/2$ and $n/2-1$, the condition that the zeros of $\hat{F}_1(x)$ and $\hat{F}_2(x)$ are alternate implies that $\hat{F}_1(x)$ has zeros which locate at the farthest two sides in $|x|<1$. On the other hand, when n is odd, in view of that $F_1(z)$ and $F_2(z)$ have surely zeros at $z=-1$ and

$z=1$, respectively, and both of $\hat{F}_1(x)$ and $\hat{F}_2(x)$ have the degree of $(n-1)/2$, we conclude that $\hat{F}_1(x)$ has a zero on the farthest right side, or equivalently, $\hat{F}_2(x)$ has a zero on the farthest left side in $|x|<1$. In consequence, if $F_i(z),\ i=1,2$ satisfy the conditions of Lemma 1, then $\hat{F}_i(x),\ i=1,2$ must satisfy the conditions of this theorem.

The sufficiency can be shown similarly by simply reversing the process for the necessity proof except for the case of $z=0$, which should receive special attention as $\tilde{F}_i(z),\ i=1,2$ can never be zero at the point $z=0$ and $\tilde{F}_i(z)\neq 0$ does not necessarily imply that $F_i(z)\neq 0$ at $z=0$. Therefore, what we have to do now is to show that if $\hat{F}_i(x),\ i=1,2$ satisfy the conditions of this theorem, then $F_i(z)\neq 0$ at $z=0$. When n is even, the degrees of $\hat{F}_i(x)$, $i=1,2$ are respectively $n/2$ and $n/2-1$, so if $\hat{F}_i(x),\ i=1,2$ satisfy the conditions of the theorem, it is meant that $\tilde{F}_i(z),\ i=1,2$ have respectively n and $n-2$ zeros on $|z|=1$. This, along with the other additional two zeros of $F_2(z)$ at $z=\pm 1$, signifies that both $F_i(z),\ i=1,2$ have n zeros on $|z|=1$, therefore $F_i(z)\neq 0,\ i=1,2$ at $z=0$. The case when n is odd can be shown similarly, thus the proof is completed.

In what follows we show that the conditions of Theorem 2 can be easily tested by using the extended Sturm's theorem given by Lemma 3. Let $g(x),\ g_1(x)$ be two given polynomials with real coefficients, and without loss of generality let $g(x)$ be the one whose degree is not less than the other one, i.e., $\deg g(x)\geq \deg g_1(x)$. Further, for simplicity, suppose that $g(x)$ and $g_1(x)$ possess no common zeros. Define the following polynomial sequence starting from $g(x),\ g_1(x)$.

$$g(x),\ g_1(x),\ g_2(x),\cdots,\ g_l(x) \tag{40}$$

where

$$\begin{aligned} g_{i-1}(x) &= h_i(x)g_i(x)-g_{i+1}(x), \\ &i=1,2,\ldots,l-1,\ g_0(x)=g(x) \end{aligned} \tag{41}$$

with $\deg g_i(x)\geq \deg g_{i+1}(x)$ ($i=2,\ldots,l$). Note that the above polynomial sequence can be calculated by using the Euclidean Division Algorithm.

Denote by $V(x)$ the number of changes of signs in the sequence of values of polynomials from (40) at the point x. For the point $x=x_0$ such that $g(x_0)/g_1(x_0)=0$, define $\chi(x_0)$ as follows. When x changes from $x<x_0$ to $x>x_0$ near x_0,

- if the sign of $g(x)/g_1(x)$ does not change, $\chi(x_0)=0$;
- if the sign of $g(x)/g_1(x)$ changes from $-$ to $+$, $\chi(x_0)=1$;
- if the sign of $g(x)/g_1(x)$ changes from $+$ to $-$, $\chi(x_0)=-1$.

Lemma 3: Suppose that $g(x)/g_1(x)\neq 0$ at the points $x=a$, $x=b$. Then,

$$V(a)-V(b)=\sum_{x_0\in\{x|g(x)/g_1(x)=0;\,a<x<b\}}\chi(x_0). \tag{42}$$

Proof: It can be shown in the same way as in the proof of Sturm's Theorem [16,17].

In fact, the only difference of this lemma to Sturm's Theorem is that, for the case of Sturm's Theorem $g_1(x)=dg(x)/dx$ and therefore we always have that $\chi(x_0)=1$ for every x_0 such that $g(x_0)=0$. $\chi(x_0)=1$ means by definition that $V(x)$ surely decreases by 1 at every zero of $g(x)$ so that the number of zeros of $g(x)$ is equal to $V(a)-V(b)$. However, for the case considered in this lemma, $\chi(x_0)$ is not necessarily always equal to 1, so $V(a)-V(b)$ is just the sum total of the actual values of $\chi(x_0)$ at every zero of $g(x)/g_1(x)$.

Based on the above lemma, the following results can be given.

Lemma 4: Let $g(x)=\hat{F}_1(x),\ g_1(x)=\hat{F}_2(x)$, and denote by $V(x)$ the number of variation of the signs in the sequence of values of polynomials from (40) at the point x. Then, polynomial $F(z)$ satisfies the conditions of Theorem 2, i.e., $F(z)$ is stable, if and only if

(a) $|f_0/f_n|<1$,

(b)

$$V(-1)-V(1)=\text{sign}(f_{1,n}\cdot f_{2,n})\cdot\frac{n}{2},\quad n:\text{ even} \tag{43}$$

and

$$V(-1)-V(1)=-\text{sign}(f_{1,n}\cdot f_{2,n})\cdot\frac{n-1}{2},\ n:\text{odd}. \tag{44}$$

Proof: Consider first the sufficiency under the assumption that $\hat{F}_1(\pm 1)\neq 0,\ \hat{F}_2(\pm 1)\neq 0$, i.e., $\hat{F}_1(\pm 1)/\hat{F}_2(\pm 1)\neq 0$. It will be seen later in the proof of necessity that $\hat{F}_1(\pm 1)/\hat{F}_2(\pm 1)\neq 0$ is necessary for the conditions of (43) and (44) to hold, thus it is implicitly required by these conditions.

Suppose that (43), (44) are satisfied. When n is even, the degrees of $\hat{F}_i(x),\ i=1,2$ are $n/2,\ n/2-1$, respectively. Neglecting the sign of the term on the right-hand side of (43) and just supposing that it is $\pm n/2$, we see that, due to the results of Lemma 3, $\hat{F}_1(x)/\hat{F}_2(x)$, equivalently $\hat{F}_1(x)$, has $n/2$ zeros in the interval $|x|<1$ and $\hat{F}_1(\pm 1)/\hat{F}_2(\pm 1)$ always changes sign either from $-$ to $+$ or from $+$ to $-$ at each of its zeros, namely, $\chi(x_0)$ always takes the same value of either 1 or -1 at each of the zeros. This means that the $n/2$ zeros of $\hat{F}_1(x)$ are simple and are separated by the $n/2-1$ simple zeros of $\hat{F}_2(x)$. It should also be clear that the two zeros of $\hat{F}_1(x)$ are located on the side furthest left and the side furthest right within the interval of $|x|<1$, respectively.

Next we determine the sign of $\chi(x_0)$ for all $x_0 \in \{x \mid \hat{F}_1(x)\ /\ \hat{F}_2(x)=0; -1<x<1\}$. Since $\hat{F}_i(x)$, $i=1,2$ possess no zeros for $x>1$ as shown above, for sufficiently large x, the signs of $\hat{F}_i(x),\ i=1,2$ must be the same as their leading coefficients $\tilde{f}_{1,n/2}$, $\tilde{f}_{2,n/2-1}$, respectively, thus $\mathrm{sign}(\hat{F}_1(x)\ /\ \hat{F}_2(x)) = \mathrm{sign}(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1})$. For the case when $\mathrm{sign}(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1})=1$, one can conclude that $\hat{F}_1(x)\ /\hat{F}_2(x)$ changes sign from $-$ to $+$, i.e., $\chi(x_0)=1(=\mathrm{sign}(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1}))$ at the largest zero of $\hat{F}_1(x)\ /\ \hat{F}_2(x)$. Similarly, it can be seen that $\mathrm{sign}(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1})=-1$ implies that $\chi(x_0)=-1$ $(=\mathrm{sign}(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1}))$ at the largest zero of $\hat{F}_1(x)/\hat{F}_2(x)$. As $\chi(x_0)$ has the same value for all zeros of $\hat{F}_1(x)\ /\ \hat{F}_2(x)$, we have that $\chi(x_0)=\mathrm{sign}$ $(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1})$ at each of such zeros. Further, in view of Corollary 1, we see that $\tilde{f}_{1,n/2}=f_{1,n}$ and $\tilde{f}_{2,n/2-1}=f_{2,n}$ and the sign of the term on the right-hand side of (43) is equal to $\mathrm{sign}(\tilde{f}_{1,n/2}\cdot \tilde{f}_{2,n/2-1})$ $=\mathrm{sign}(f_{1,n}\cdot f_{2,n})$.

Conversely, when n is odd, both $\hat{F}_1(x)$ and $\hat{F}_2(x)$ are of degree $(n-1)/2$. Neglecting the sign of the term on the right-hand side of (44) and supposing that it is $\pm\ (n-1)/2$, we can show similarly as above that both $\hat{F}_1(x)$ and $\hat{F}_2(x)$ have $(n-1)/2$ simple zeros in $|x|<1$, and these zeros are alternate. Now, consider the condition for the requirement that a zero of $\hat{F}_1(x)$ is located on the side furthest right in $|x|<1$. Let x_0 be the largest zero of $\hat{F}_2(x)$ and denote by $\mathrm{sign}(\hat{F}_1(x_0))$ the sign of $\hat{F}_1(x)$ at points near x_0 with $x>x_0$. Then, if $\chi(x_0)=1$ which means that $\hat{F}_1(x)$ has the opposite sign with $\hat{F}_2(x)$, i.e., $\tilde{f}_{2,(n-1)/2}$, for a point x immediately after x_0, then $\mathrm{sign}(\hat{F}_1(x_0)) = -\mathrm{sign}(\tilde{f}_{2,(n-1)/2})$. In the same way, we see that if $\chi(x_0)=-1$, then $\mathrm{sign}(\hat{F}_1(x_0))=\mathrm{sign}(\tilde{f}_{2,(n-1)/2})$. Suppose that

$$\chi(x_0)=-\mathrm{sign}(\tilde{f}_{1,(n-1)/2}\cdot \tilde{f}_{2,(n-1)/2}). \qquad (45)$$

Then, if $\tilde{f}_{1,(n-1)/2}$ and $\tilde{f}_{2,(n-1)/2}$ have different signs, it follows from (45) that $\chi(x_0)=1$, and this in turn implies that $\mathrm{sign}(\hat{F}_1(x_0))=\mathrm{sign}(\tilde{f}_{2,(n-1)/2})$ $=-\mathrm{sign}(\tilde{f}_{1,(n-1)/2})$. Similarly, we can see that, if $\tilde{f}_{1,(n-1)/2}$ and $\tilde{f}_{2,(n-1)/2}$ have the same sign, then $\chi(x_0)=-1$, and $\mathrm{sign}(\hat{F}_1(x_0))=-\mathrm{sign}(\tilde{f}_{2,(n-1)/2})=-$ $\mathrm{sign}(\tilde{f}_{1,(n-1)/2})$. In both cases, we have the result that $\mathrm{sign}(\hat{F}_1(x_0))=-\mathrm{sign}(\tilde{f}_{1,(n-1)/2})$, i.e., $\hat{F}_1(x)$ has different signs for a point x immediately after x_0 (<1) and for a sufficiently large x $(>>1)$. This fact signifies that $\hat{F}_1(x)$ must have another zero in $x_0<x<1$. Now substituting $\tilde{f}_{1,(n-1)/2}=f_{1,n}$, $\tilde{f}_{2,(n-1)/2}=f_{2,n}$ into (45) yields that $\chi(x_0)=-\mathrm{sign}$ $(f_{1,n}\cdot f_{2,n})$. As $\chi(\hat{x}_0)$ has the same value for all zeros of $\hat{F}_1(x)/\hat{F}_2(x)$, we see that this relation holds true for all the zeros of $\hat{F}_1(x)/\hat{F}_2(x)$, which is just the sign given in the right-hand side of (44).

Therefore, if the condition (b) of Lemma 4 holds true, then the conditions (b) and (c) of Theorem 2 are satisfied. Thus the proof of the sufficiency is completed.

The necessity is clear from the process of the proof of sufficiency, since, if $\hat{F}_i(x),\ i=1,2$ have any zero(s) outside the interval $|x|<1$, including the zeros of $x=\pm 1$, the conditions of (43) and (44) can never be satisfied.

Theorem 3: Let $g(x)$, $g_1(x)$ and $V(x)$ be the same as those in Lemma 4. Then, $F(x)$ is stable if and only if,

$$V(-1)-V(1)=\frac{n}{2},\quad n:\ \text{even} \tag{46}$$

and

$$V(-1)-V(1)=-\frac{n-1}{2},\quad n:\ \text{odd}. \tag{47}$$

Proof: It follows from (20), (21) that $f_{1,n} = f_n + f_0$ and $f_{2,n} = f_n - f_0$. So, we have $f_{1,n} \cdot f_{2,n} = f_n^2 - f_0^2$. If $|f_0/f_n| < 1$, then $f_n^2 - f_0^2 > 0$ and

$$\text{sign}(f_{1,n} \cdot f_{2,n}) = \text{sign}(f_n^2 - f_0^2) = 1. \tag{48}$$

Conversely, if (48) is true, $|f_0/f_n| < 1$ must also hold true. Therefore, it is clear that the conditions of this theorem are equivalent to those of Lemma 4.

3. IMPROVEMENTS ON THE 2D STABILITY TEST PORTION

Another important step in Bose's 2D stability test is investigating whether the resultant $R(z_1)$ of $D_{1s}(z_1,z_2)$ and $D_{1a}(z_1,z_2)$ is devoid of zeros on $|z_1|=1$. In this section, we first show that $R(z_1)$ is in fact a self-inversive polynomial itself on $|z_1|=1$. Therefore, it can also be converted into a real variable polynomial with a reduction in the degree to half of the degree for $R(z_1)$. Then, improved versions of Bose's 2D stability test will be provided based on the results obtained in this and the previous sections.

Lemma 5: The resultant $R(z_1)$ is self-inversive on $|z_1|=1$.

Proof: Since $\bar{z}_1 = z_1^{-1}$ on $|z_1|=1$, $D_{1s}(z_1,z_2)$, $D_{1a}(z_1,z_2)$ can be respectively written as

$$D_{1s}(z_1,z_2)=\frac{1}{2}\sum_{k=0}^{n_2}\left\{d_k(z_1)+d_{n_2-k}(z_1^{-1})\right\}z_2^k, \tag{49}$$

$$D_{1a}(z_1,z_2)=\frac{1}{2}\sum_{k=0}^{n_2}\left\{d_k(z_1)-d_{n_2-k}(z_1^{-1})\right\}z_2^k. \tag{50}$$

Therefore, defining $h_{s,k}(z_1)=\frac{1}{2}(d_k(z_1)+d_{n_2-k}(z_1^{-1}))$, $h_{a,k}(z_1)=\frac{1}{2}(d_k(z_1)-d_{n_2-k}(z_1^{-1}))$, the resultant $R(z_1)$ of $D_{1s}(z_1,z_2)$ and $D_{1a}(z_1,z_2)$ with respect to z_2 is given by

$$R(z_1)=\begin{vmatrix} h_{s,n_2}(z_1) & h_{s,n_2-1}(z_1) & \cdots & h_{s,1}(z_1) & h_{s,0}(z_1) & & & 0 \\ & h_{s,n_2}(z_1) & \cdots & h_{s,2}(z_1) & h_{s,1}(z_1) & h_{s,0}(z_1) & & \\ & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & & & h_{s,n_2}(z_1) & h_{s,n_2-1}(z_1) & h_{s,n_2-2}(z_1) & \cdots & h_{s,0}(z_1) \\ h_{a,n_2}(z_1) & h_{a,n_2-1}(z_1) & \cdots & h_{a,1}(z_1) & h_{a,0}(z_1) & & & 0 \\ & h_{a,n_2}(z_1) & \cdots & h_{a,2}(z_1) & h_{a,1}(z_1) & h_{a,0}(z_1) & & \\ & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & & & h_{a,n_2}(z_1) & h_{a,n_2-1}(z_1) & h_{a,n_2-2}(z_1) & \cdots & h_{a,0}(z_1) \end{vmatrix} \tag{51}$$

Then, reversing the order of all the $2n_2$ columns, and reversing the order of the upper n_2 rows and the lower n_2 rows respectively, we can obtain the following result for all n_2.

$$R(z_1)=(-1)^{n_2}\begin{vmatrix} h_{s,0}(z_1) & h_{s,1}(z_1) & \cdots & h_{s,n_2-1}(z_1) & h_{s,n_2}(z_1) & & & 0 \\ & h_{s,0}(z_1) & \cdots & h_{s,n_2-2}(z_1) & h_{s,n_2-1}(z_1) & h_{s,n_2}(z_1) & & \\ & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & & & h_{s,0}(z_1) & h_{s,1}(z_1) & h_{s,2}(z_1) & \cdots & h_{s,n_2}(z_1) \\ h_{a,0}(z_1) & h_{a,1}(z_1) & \cdots & h_{a,n_2-1}(z_1) & h_{a,n_2}(z_1) & & & 0 \\ & h_{a,0}(z_1) & \cdots & h_{a,n_2-2}(z_1) & h_{a,n_2-1}(z_1) & h_{a,n_2}(z_1) & & \\ & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ 0 & & & h_{a,0}(z_1) & h_{a,1}(z_1) & h_{a,2}(z_1) & \cdots & h_{a,n_2} \end{vmatrix} \tag{52}$$

It follows from $h_{s,k}(z_1)=h_{s,n_2-k}(z_1^{-1})$, $h_{a,k}(z_1)=-h_{a,n_2-k}(z_1^{-1})$ that

$$R(z_1^{-1})=(-1)^{n_2}\begin{vmatrix} h_{s,n_2}(z_1) & h_{s,n_2-1}(z_1) & \cdots & h_{s,0}(z_1) & & 0 \\ & h_{s,n_2}(z_1) & \cdots & h_{s,1}(z_1) & h_{s,0}(z_1) & \\ & & \ddots & \ddots & \ddots & \ddots \\ 0 & & & h_{s,n_2}(z_1) & h_{s,n_2-1}(z_1) & \cdots & h_{s,0}(z_1) \\ -h_{a,n_2}(z_1) & -h_{a,n_2-1}(z_1) & \cdots & -h_{a,0}(z_1) & & 0 \\ & -h_{a,n_2}(z_1) & \cdots & -h_{a,1}(z_1) & -h_{a,0}(z_1) & \\ & & \ddots & \ddots & \ddots & \ddots \\ 0 & & & -h_{a,n_2}(z_1) & -h_{a,n_2-1}(z_1) & \cdots & -h_{a,0}(z_1) \end{vmatrix} \tag{53}$$

Multiplying the lower n_2 rows by -1 respectively, we see that $R(z_1)=R(z_1^{-1})$ signifying that $R(z_1)$ is self-inversive.

Now, using the relation of (33), we can transform $R(z_1)$ into a polynomial $\hat{R}(x_1)$ in the real variable $x_1=(z_1+z_1^{-1})/2$. It should be clear that to test the condition (d) of Theorem 1 we only need to see whether or not $\hat{R}(x_1)\neq 0$ on $|x_1|\le 1$.

Define $\tilde{D}_{1s}(z_2)$, $\tilde{D}_{1a}(z_2)$ by

$$D_{1s}(1,z_2)=\begin{cases} z_2^{n_2/2}\tilde{D}_{1s}(z_2) & n:\text{even} \\ z_2^{(n_2-1)/2}(z_2+1)\tilde{D}_{1s}(z_2) & n:\text{odd} \end{cases} \tag{54}$$

$$D_{1a}(1,z_2)=\begin{cases} z_2^{n_2/2}(z_2-z_2^{-1})\tilde{D}_{1a}(z_2) & n:\text{even} \\ z_2^{(n_2-1)/2}(z_2-1)\tilde{D}_{1a}(z_2) & n:\text{odd} \end{cases} \quad (55)$$

If we put $F_1(z_2)=D_{1s}(1,z_2)$ and $F_2(z_2)=D_{1a}(1,z_2)$ then Lemma 2 gives us that $\tilde{D}_{1s}(z_2)$, $\tilde{D}_{1a}(z_2)$ are self-inversive polynomials. Further, transform by (33) $\tilde{D}_{1s}(z_2)$ and $\tilde{D}_{1a}(z_2)$ into real variable polynomials $\hat{D}_{1s}(x_2)$ and $\hat{D}_{1a}(x_2)$, respectively, where $x_2=(z_2+z_2^{-1})/2$.

Based on the above results and those given in the previous section, we can now give the following improved versions of Bose's 2D stability test.

Theorem 4: The polynomial $D_1(z_1,z_2)\neq 0$ for $|z_1|=1, |z_2|\geq 1$, or equivalently, $D(z_1,z_2)\neq 0$ for $|z_1|=1$, $|z_2|\leq 1$ if and only if

(a) $|d_0(1)/d_{n_2}(1)|<1$;

(b) all the zeros of $\hat{D}_{1s}(x_2)$ and $\hat{D}_{1a}(x_2)$ are located within $|x_2|<1$; and

(c) the zeros of $\hat{D}_{1s}(x_2)$ and $\hat{D}_{1a}(x_2)$ are simple and alternate within $|x_2|<1$. Further, when n_2 is odd and if we order all these zeros within $|x|<1$ according to their values starting from the smallest, then the last (first) zero belongs to $D_{1s}(x_2)$ ($D_{1a}(x_2)$).

(d) $\hat{R}(x_1)\neq 0$ for $x_1\leq 1$.

Proof: It follows immediately from the results of Theorems 1, 2 and Lemma 5.

Let $g(x_2)=\hat{D}_{1s}(x_2)$, $g_1(x_2)=\hat{D}_{1a}(x_2)$ and define the polynomial sequence (40) and $V(x_2)$ for $g(x_2)$, $g_1(x_2)$ in the same way as stated in Section 2. Then, the following theorem can be given.

Theorem 5: The polynomial $D_1(z_1,z_2)\neq 0$ for $|z_1|=1, |z_2|\geq 1$, or equivalently, $D(z_1,z_2)\neq 0$ for $|z_1|=1$, $|z_2|\leq 1$ if and only if

(a)

$$V(-1)-V(1)=\frac{n}{2},\quad n:\text{ even} \quad (56)$$

and

$$V(-1)-V(1)=-\frac{n-1}{2},\quad n:\text{ odd.} \quad (57)$$

(b) $\hat{R}(x_1)\neq 0$ for $x_1\leq 1$.

Proof: It is obvious from Lemma 4 and Theorems 3 and 4.

It is noted that the condition (b) of Theorem 5 can be checked directly by Sturm's method [16,17]. A totally algebraic algorithm for the stability test of a 2D polynomial $D(z_1,z_2)$ with real coefficients can now be summarized as follows.

Algorithm:

Step 1. Verify the condition $D(z_1,0)\neq 0$, $|z_1|\leq 1$ by using the procedure proposed in Section 2 or any other 1D stability test. If this condition is not valid then exit with $D(z_1,z_2)$ unstable.

Step 2. Calculate $d_k(z_1)$ defined in (5) and verify the necessary condition $|d_0(1)/d_{n_2}(1)|<1$. If this condition is not valid then exit with $D(z_1,z_2)$ unstable.

Step 3. Calculate $D_{1s}(z_1,z_2)$, $D_{1a}(z_1,z_2)$ by (49) and (50), and calculate $\tilde{D}_{1s}(z_2)$, $\tilde{D}_{1a}(z_2)$ defined in (54), (55) by applying the results of Corollary 1 for $D_{1s}(1,z_2)$, $D_{1a}(1,z_2)$. Further, construct $\hat{D}_{1s}(x_2)$, $\hat{D}_{1a}(x_2)$ from $\tilde{D}_{1s}(z_2)$, $\tilde{D}_{1a}(z_2)$ by (33) and verify if $\hat{D}_{1s}(x_2)$, $\hat{D}_{1a}(x_2)$ satisfy the condition (a) of Theorem 5 by the extended Sturm's method shown in Section 2. If this condition is not valid then exit with $D(z_1,z_2)$ unstable.

Step 4. Calculate the resultant $R(z_1)$ by (51) and transform it to $\hat{R}(x_1)$ by (33). Then test the condition (b) of Theorem 5 by using Sturm's method [16,17]. If $\hat{R}(x_1)\neq 0$ on $|x_1|\leq 1$, then the 2D polynomial $D(z_1,z_2)$ is stable.

4. ILLUSTRATIVE EXAMPLES

Several nontrivial examples are given to show the effectiveness and computational advantage of the proposed method.

Example 1: Consider the stability of the following 2D polynomial used in [1,2].

$$D(z_1,z_2)=(12+10z_1+2z_1^2)+(6+5z_1+z_1^2)z_2 \quad (58)$$

Step 1. Obviously, $D(z_1,0)=12+10z_1+2z_1^2=2(z_1+2)(z_1+3)$ has zeros of -2, -3, thus is 1D stable.

Step 2. Since

$$D_1(z_1,z_2)=z_2D(z_1,z_2^{-1})$$
$$=(6+5z_1+z_1^2)+(12+10z_1+2z_1^2)z_2$$

and

$$\left|\frac{d_0(1)}{d_1(1)}\right|=\left|\frac{12}{24}\right|=\frac{1}{2}<1,$$

the condition (a) of Theorems 1 and 4 is satisfied.

Step 3. It is easy to see that

$$D_{1s}(z_1,z_2)=\frac{1}{2}[(18+5z_1+z_1^2+10z_1^{-1}+2z_1^{-2}) +(18+5z_1^{-1}+z_1^{-2}+10z_1+2z_1^2)z_2] \tag{59}$$

$$D_{1a}(z_1,z_2)=\frac{1}{2}[(-6+5z_1+z_1^2-10z_1^{-1}-2z_1^{-2}) -(-6+5z_1^{-1}+z_1^{-2}-10z_1-2z_1^2)z_2] \tag{60}$$

and $D_{1s}(1,z_2)=18(1+z_2)$, $D_{1a}(1,z_2)=6(z_2-1)$. Thus, we have that $\tilde{D}_{1s}(z_2)=18$, $\tilde{D}_{1a}(z_2)=6$. Clearly, the condition (a) of Theorem 5 is satisfied as $V(-1)=V(1)=0$, $n_2=1$ and $V(-1)-V(1)=-(n_2-1)/2=0$.

Step 4. The resultant $R(z_1)$ is calculated as

$$R(z_1)=-186-105(z_1+z_1^{-1})-18(z_1^2+z_1^{-2}) =\begin{bmatrix}-186 & -105 & -18\end{bmatrix}\begin{bmatrix}1\\ z_1+z_1^{-1}\\ z_1^2+z_1^{-2}\end{bmatrix}.$$

Let $x_1=(z_1+z_1^{-1})/2$, then it follows from (33) that

$$\begin{bmatrix}1\\ z_1+z_1^{-1}\\ z_1^2+z_1^{-2}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ -2 & 0 & 4\end{bmatrix}\begin{bmatrix}1\\ x_1\\ x_1^2\end{bmatrix}$$

and

$$\hat{R}(x_1)=-150-210x_1-72x_1^2. \tag{61}$$

Then, the Sturm's polynomial sequence can be constructed as

$$g_0(x_1)=\frac{1}{2}\hat{R}(x_1)=-36x_1^2-105x_1-75, \tag{62}$$

$$g_1(x_1)=dg_0(x_1)/dx_1=-72x_1-105, \tag{63}$$

$$g_2(x_1)=-\frac{25}{16}, \tag{64}$$

and it is ready to verify that $g_0(\pm 1)/g_1(\pm 1)\neq 0$ and $V(-1)=V(1)=0$. Then, due to Sturm's Theorem we conclude that $\hat{R}(x_1)\neq 0$, for $|x_1|\leq 1$, and therefore the 2D polynomial given in (58) is stable.

Noting that the above $\hat{R}(x_1)$ is only a second degree polynomial, while the resultant $R(x)$ obtained in [1] is an 8th degree polynomial and the one in [2] is a trigonometric function. It should be noted that the correct results for Example 1 of [2] are as follows:

$$B_{1s}(e^{j\theta_1},z_2)=\frac{1}{2}[(18+15\cos\theta_1+3\cos 2\theta_1)(1+z_2) - j(5\sin\theta_1+\sin 2\theta_1)(1-z_2)]$$

$$B_{1a}(e^{j\theta_1},z_2)=\frac{1}{2}[(-6-5\cos\theta_1-\cos 2\theta_1) + j(15\sin\theta_1+3\sin 2\theta_1) + ((6+5\cos\theta_1+\cos 2\theta_1) + j(15\sin\theta_1+3\sin 2\theta_1))z_2]$$

and

$$\begin{aligned}R(e^{j\theta_1}) &=[(18+15\cos\theta_1+3\cos 2\theta_1)+j(5\sin\theta_1+\sin 2\theta_1)]\\ &\quad\cdot[-(6+5\cos\theta_1+\cos 2\theta_1)+j(15\sin\theta_1+3\sin 2\theta_1)]\\ &\quad-[(18+15\cos\theta_1+3\cos 2\theta_1)-j(5\sin\theta_1+\sin 2\theta_1)]\\ &\quad\cdot[(6+5\cos\theta_1+\cos 2\theta_1)+j(15\sin\theta_1+3\sin 2\theta_1)]\\ &=-[(18+15\cos\theta_1+3\cos 2\theta_1)(6+5\cos\theta_1+2\cos 2\theta_1)\\ &\quad+(5\sin\theta_1+2\sin 2\theta_1)(15\sin\theta_1+3\sin 2\theta_1)].\end{aligned} \tag{65}$$

By making use of various formulas of trigonometric functions, it is possible to reduce (65) to the form of

$$\begin{aligned}R(e^{j\theta_1}) &=-(186+210\cos\theta_1+36\cos 2\theta_1)\\ &=-(150+210\cos\theta_1+72\cos^2\theta_1),\end{aligned} \tag{66}$$

which corresponds to the result of (61) with $x_1=(e^{j\theta_1}+e^{-j\theta_1})/2=\cos\theta_1$. However, in general it would be rather complicated to reduce a polynomial of trigonometric functions ($\sin k\theta$ and $\cos k\theta$) to a polynomial of a single real variable x. Thus, it is difficult to implement such operations by a computer program, while the procedure proposed in this paper can be implemented quite easily.

Example 2: Test the stability of the following 2D polynomial used in [7,15]

$$D(z_1,z_2)=\left(\frac{1}{4}\right)z_2^3+\left(\frac{1}{4}z_1+\frac{1}{2}\right)z_2^2 +\left(\frac{1}{4}z_1^2+\frac{1}{2}z_1+1\right)z_2+\left(\frac{1}{4}z_1^3+\frac{1}{2}z_1^2+z_1+2\right). \tag{67}$$

It should be noted, however, that the polynomial $F(z_1,z_2)$ given in [7] is in fact the reciprocal polynomial of $D(z_1,z_2)$ with respect to z_1, i.e., $F(z_1,z_2)=z_1^3D(z_1^{-1},z_2)$ and it corresponds to the stability criterion $F(z_1,z_2)\neq 0$ for $|z_1|\geq 1$, $|z_2|\leq 1$.

Step 1. It is easy to see that $D(z_1,0)=(1/4)z_1^3$

$+(1/2)z_1^2 + z_1 + 2$ is 1D stable. In fact, it has the zeros of -2, $\pm 2i$.

Step 2. It is also ready to verify that

$$\left|\frac{d_0(1)}{d_3(1)}\right| = \left|\frac{1/4}{15/4}\right| = \frac{1}{15} < 1.$$

Step 3. It can be shown that

$$\begin{aligned} D_{1s}(z_1,z_2) &= \left(\frac{1}{8}+\frac{1}{4}z_1+\frac{1}{2}z_1^2+z_1^3+\frac{1}{8}z_1^{-3}\right)z_2^3 \\ &+\left(\frac{1}{8}z_1+\frac{1}{4}z_1^2+\frac{1}{2}z_1^3+\frac{1}{8}z_1^{-2}+\frac{1}{4}z_1^{-3}\right)z_2^2 \\ &+\left(\frac{1}{8}z_1^{-1}+\frac{1}{4}z_1^{-2}+\frac{1}{2}z_1^{-3}+\frac{1}{8}z_1^2+\frac{1}{4}z_1^3\right)z_2 \\ &+\frac{1}{8}+\frac{1}{4}z_1^{-1}+\frac{1}{2}z_1^{-2}+z_1^{-3}+\frac{1}{8}z_1^3, \\ D_{1a}(z_1,z_2) &= \left(\frac{1}{8}+\frac{1}{4}z_1+\frac{1}{2}z_1^2+z_1^3-\frac{1}{8}z_1^{-3}\right)z_2^3 \\ &+\left(\frac{1}{8}z_1+\frac{1}{4}z_1^2+\frac{1}{2}z_1^3-\frac{1}{8}z_1^{-2}-\frac{1}{4}z_1^{-3}\right)z_2^2 \\ &+\left(-\frac{1}{8}z_1^{-1}-\frac{1}{4}z_1^{-2}-\frac{1}{2}z_1^{-3}+\frac{1}{8}z_1^2+\frac{1}{4}z_1^3\right)z_2 \\ &-\frac{1}{8}-\frac{1}{4}z_1^{-1}-\frac{1}{2}z_1^{-2}-z_1^{-3}+\frac{1}{8}z_1^3, \end{aligned}$$

and

$$\begin{aligned} D_{1s}(1,z_2) &= 2z_2^3+\frac{5}{4}z_2^2+\frac{5}{4}z_2+2, \\ D_{1a}(1,z_2) &= \frac{7}{4}z_2^3+\frac{1}{2}z_2^2-\frac{1}{2}z_2-\frac{7}{4}. \end{aligned}$$

The self-inversive polynomials $\tilde{D}_{1s}(z_2)$, $\tilde{D}_{1a}(z_2)$ are then obtained as

$$\begin{aligned} \tilde{D}_{1s}(z_2) &= 2(z_2+z_2^{-1})+\frac{1}{4}, \\ \tilde{D}_{1a}(z_2) &= \frac{7}{4}(z_2+z_2^{-1})+\frac{9}{4}. \end{aligned}$$

which can be converted by setting $x_2 = (z_2 + z_2^{-1})/2$ into

$$\begin{aligned} \hat{D}_{1s}(x_2) &= 4x_2+\frac{1}{4}, \\ \hat{D}_{1a}(x_2) &= \frac{7}{2}x_2+\frac{9}{4}. \end{aligned}$$

For this simple case it is trivial to see that the zeros of $\hat{D}_{1s}(x_2)$ and $\hat{D}_{1a}(x_2)$ are respectively $-1/16$ and $-9/14$, both located within $|x_2|<1$, and $-9/14 < -1/16$, i.e., the zero of $\hat{D}_{1s}(x_2)$ is located on the right side.

Step 4. The resultant $R(z_1)$ can be shown to be as

$$\begin{aligned} R(z_1) = -\frac{1}{4096}\{&64(z_1^9+z_1^{-9})+448(z_1^8+z_1^{-8}) \\ &+1968(z_1^7+z_1^{-7})+6976(z_1^6+z_1^{-6}) \\ &+19100(z_1^5+z_1^{-5})+43228(z_1^4+z_1^{-4}) \\ &+83265(z_1^3+z_1^{-3})+131818(z_1^2+z_1^{-2}) \\ &+176393(z_1+z_1^{-1})+197850\} \end{aligned} \quad (68)$$

and it can be transformed into $\hat{R}(x_1)$ with $x_1 = (z_1 + z_1^{-1})/2$ as follows.

$$\begin{aligned} \hat{R}(x_1) = &-8x_1^9-28x_1^8-\frac{87}{2}x_1^7-53x_1^6 \\ &-\frac{1763}{32}x_1^5-\frac{2583}{64}x_1^4-\frac{13397}{512}x_1^3 \\ &-\frac{7261}{512}x_1^2-\frac{4449}{1024}x_1-\frac{3807}{2048}. \end{aligned}$$

By applying Sturm's method it can be seen that $\hat{R}(x_1) \neq 0$, $|x_1| \leq 1$. Therefore, we have the conclusion that the 2D polynomial of (67) is stable.

Example 3: Show the conditions for the following 2D polynomial to be stable [2], where a, b, c are real numbers.

$$D(z_1,z_2) = 1 + az_1 + (b + cz_1)z_2. \quad (69)$$

Step 1. To ensure that $D(z_1,0) = 1 + az_1 \neq 0$ for $|z_1| \leq 1$, it is necessary to have that $|a| > 1$.

Step 2. Since $d_0(z_1) = b + cz_1$ and $d_1(z_1) = 1 + az_1$, the following relation must hold.

$$\left|\frac{d_0(1)}{d_1(1)}\right| = \left|\frac{b+c}{1+a}\right| < 1. \quad (70)$$

Step 3. Since

$$D_1(z_1,z_2) = (b+cz_1)+(1+az_1)z_2, \quad (71)$$

$$\begin{aligned} D_{1s}(z_1,z_2) = \frac{1}{2}\{&(cz_1+b+1+az_1^{-1}) \\ &+(az_1+1+b+cz_1^{-1})z_2\}, \end{aligned} \quad (72)$$

$$D_{1a}(z_1,z_2)=\frac{1}{2}\{(cz_1+b-1-az_1^{-1})+(az_1+1-b-cz_1^{-1})z_2\}, \tag{73}$$

it follows that

$$D_{1s}(1,z_2)=\frac{1}{2}(z_2+1)(1+a+b+c), \tag{74}$$

$$D_{1a}(1,z_2)=\frac{1}{2}(z_2-1)(1+a+b+c), \tag{75}$$

which have already satisfied the condition (a) of Theorem 5.

Step 4. The resultant $R(z_1)$ is calculated by

$$R(z_1)=\begin{vmatrix} h_{s1}(z_1) & h_{s0}(z_1) \\ h_{a1}(z_1) & h_{a0}(z_1) \end{vmatrix} = -\frac{1}{4}\left\{(1+a^2-b^2-c^2)+(a-bc)(z_1+z_1^{-1})\right\},$$

where

$$h_{s1}(z_1)=\frac{1}{2}(az_1+1+b+cz_1^{-1}),$$
$$h_{s0}(z_1)=\frac{1}{2}(cz_1+b+1+az_1^{-1}),$$
$$h_{a1}(z_1)=\frac{1}{2}(az_1+1-b-cz_1^{-1}),$$
$$h_{a0}(z_1)=\frac{1}{2}(cz_1+b-1-az_1^{-1}).$$

Letting $x_1=(z_1+z_1^{-1})/2$, we have the result

$$\hat{R}(x_1)=-\frac{1}{4}\left\{(1+a^2-b^2-c^2)+2(a-bc)x_1\right\}. \tag{76}$$

Following Sturm's method, set $g_0(x_1)=\hat{R}(x_1)$, $g_1(x_1)=dg_0(x_1)/dx_1=-(a-bc)/2$. Since $g_1(x_1)$ is already a constant, to see if $\hat{R}(x_1)\neq 0$ for $|x_1|\leq 1$ we only need to see if $V(-1)-V(1)=0$ where $V(x_1)$ denotes the number of the sign variation in the polynomial sequence $g_0(x_1)$, $g_1(x_1)$. It is easy to see that

$$g_0(-1)=-\frac{1}{4}\left\{(1+a^2-b^2-c^2)-a(a-bc)\right\},$$
$$g_1(-1)=2(a-bc),$$
$$g_0(1)=-\frac{1}{4}\left\{(1+a^2-b^2-c^2)+a(a-bc)\right\},$$
$$g_1(1)=2(a-bc).$$

As $g_1(-1)=g_1(1)$, to satisfy that $V(-1)-V(1)=0$, $g_0(-1)$ and $g_0(1)$ must have the same sign which is equivalent to requiring that

$$\frac{g_0(1)}{g_0(-1)}=\frac{1+a^2-b^2-c^2-2(a-bc)}{1+a^2-b^2-c^2+2(a-bc)}=\frac{(1-a)^2-(b-c)^2}{(1+a)^2-(b+c)^2}>0. \tag{77}$$

Condition (70) implies that $(1+a)^2-(b+c)^2>0$. Hence, for (77) to be true, we must have

$$(1-a)^2-(b-c)^2>0 \tag{78}$$

which is equivalent to

$$\left|\frac{b-c}{1-a}\right|<1. \tag{79}$$

Therefore, the necessary and sufficient condition for the polynomial of (69) to be stable is

$$|a|>1,\ \left|\frac{b+c}{1+a}\right|<1,\ \left|\frac{b-c}{1-a}\right|<1. \tag{80}$$

This result is the same as the one given in [2,9]. Note that the result of Example 3 is a special case of Bose's result [2] by restricting the coefficients to be real.

5. CONCLUDING REMARKS

Some improvements have been proposed for N.K. Bose's 2D stability test, for polynomials with real coefficients, by revealing symmetric properties of the polynomials and resultants occurring in the test and by generalizing Sturm's method. Consequently, the improved test can be fulfilled by a totally algebraic algorithm and the computational complexity is significantly reduced as it involves only certain real variable polynomials with degrees not exceeding half of their previous complex variable counterparts. Nontrivial examples have also been illustrated.

Kurosawa *et al.* [18] have proposed an efficient algorithm for calculating the determinant of a matrix with 1D polynomial entries by making use of DFT (or FFT). It should be clear that this algorithm can be directly applied to compute the resultant $R(z_1)$ so that the complexity of Bose's 2D stability test can be improved even further.

An interesting question is whether the proposed improvements for Bose's 2D stability test could be extended to the general nD $(n>2)$ case. As it has recently been pointed out by the authors [19], it is a nontrivial task to generalize 2D stability testing methods to the general nD $(n > 2)$ case. We will look into this problem in the future. Another possible future research topic is to apply the proposed improvements in the investigation of the stability of interval 2D systems, which is important because practical systems are usually subject to parameter uncertainty.

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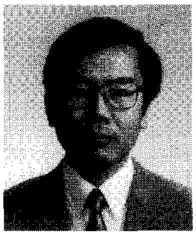

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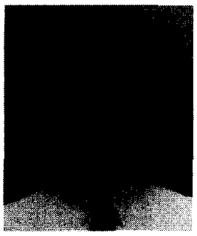

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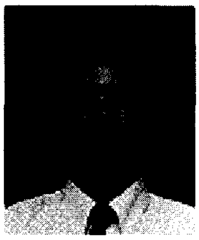

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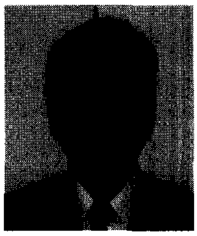

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