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Ljung-Box Test in Unit Root AR-ARCH Model[1]

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Abstract

In this paper, we investigate the limiting distribution of the Ljung-Box test statistic in the unit root AR models with ARCH errors. We show that the limiting distribution is approximately chi-square distribution with the degrees of freedom only depending on the number of autocorrelation lags appearing in the test. Some simulation results are provided for illustration.

Keywords : Unit root model, AR-ARCH model, Ljung-Box test statistic

1. Introduction

In time series analysis, selecting a suitable model for given time series is a primary task since further inferences are based on it. In model selection procedure, one chooses a tentative model through the model selection criteria such as AIC (Akaike's information criteria) and BIC (Bayesian information criterion), and then conducts a diagnostic check for the tentative model. For a diagnostic test in stationary ARMA models, Box and Pierce (1970) proposed a test statistic based on the sum of autocorrelations of residuals, and showed that it is approximately distributed as a chi-square distribution. Later, Davis, Triggs and Newbold (1977) and Ljung and Box (1978) demonstrated that a modified statistic gives a closer small sample approximation to the chi-square distribution. More recently, Monti (1994) suggested a diagnostic test based on the residual partial autocorrelation, and Kwan and Wu (1997) investigated the finite sample performance of Monti's test.

The Ljung-Box (LB) test has long been popular among practitioners deeming its convenience. Basically, it tests for the uncorrelatedness of the errors in stationary ARMA models, and has become a very useful diagnostic tool. However, one should be careful in utilizing the chi-square approximation result since it can be affected by models and parameter estimations. For instance, one can easily see this phenomenon when a trimmed mean estimator of the autoregressive parameter in stationary AR(1) models is utilized. Therefore, one must check whether the

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assumptions for the Ljung-Box conditions are satisfied or not.

As with the estimators, the number of lags of autocorrelations in the LB test plays an important role as to its performance. For instance, if the number is too large, the test suffers from severe size distortions and the power of the test decreases. To overcome such a problem, Hong (1996) proposed a test statistic using all available autocorrelations, where he gave lower weights to large lags for alleviating the influence from the large lags. Hong's method also merits to remove the influence of the parameter estimators.

Meanwhile, Mcleod and Li (1983) proposed to use the test based on squared residuals in ARMA models motivated by an intent to emphasize its usefulness in detecting nonlinear type statistical dependency (cf. Granger and Anderson, 1978). According to their analysis, the limiting distribution does not depend on the orders of ARMA models, but on the number of autcorrelations appearing in the LB test. See also the discussion in Tong (1990), page 225.

The goal of this paper is not to generalize the tests mentioned above but to investigate the limiting distribution of the LB test statistic when the given model is the unit root AR(1) model with ARCH (autoregressive conditional heteroscedastic) errors. Both unit root and ARCH models are very famous in econometrics, but to our knowledge there are no results on LB test for unit root AR(1)-ARCH models. In Section 2, we show that the LB test statistic is approximately chi-square distributed. Some simulations are provided for illustration in Section 3.

2. LB Test in Unit Root AR-ARCH Model

We consider the unit root model with the ARCH error,

$$y_t - \phi y_{t-1} = \epsilon_t,$$

where $\phi = 1$ and $\{\epsilon_t\}$ follows the first-order ARCH model, denoted as

$$\epsilon_t = \xi_t \sigma_t, \quad \sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2,$$

and $\omega > 0$, $\alpha \geqq 0$ and ξ_t's are iid random variables with zero mean and unit variance.

Let $\hat{\phi}_n$ be the least squares estimator of ϕ. Based on the least squares residual

$$\hat{\epsilon}_t = y_t - \hat{\phi}_n y_{t-1},$$

we calculate the residual correlation at the lag h,

$$\hat{r}(h) = \frac{\sum_{t=1}^{n-h} (\hat{\epsilon}_{t+h} - \bar{\hat{\epsilon}})(\hat{\epsilon}_t - \bar{\hat{\epsilon}})}{\sum_{t=1}^{n} (\hat{\epsilon}_t - \bar{\hat{\epsilon}})^2}, \quad h = 1, \cdots, m,$$

where $\bar{\hat{\epsilon}} = \sum_{t=1}^{n} \hat{\epsilon}_t / n$ and m is a positive integer. We define the LB test statistic

$$Q_n(m) = n(n+2) \sum_{h=1}^{m} (n-h)^{-1} \hat{r}(h).$$

Then it turns out that

$$Q_n(m) \rightarrow \chi^2_m. \tag{2.1}$$

The above means that the limiting distribution of the LB test statistic in unit root AR(1) model with ARCH errors does not depend on the order of autoregressive, whereas the LB test statistic in the stationary ARMA(p,q) model follows approximately a chi-square distribution with degrees of freedom $m-p-q$ when both m and n are large enough (cf. Ljung and Box, 1978). The following theorem deals with the residuals of AR(1) model and directly ensures the result as in (2.1).

Theorem. Suppose that $\{y_t\}$ satisfies that

$$y_t - \phi y_{t-1} = \epsilon_t \ , \quad y_0 \equiv 0$$

where $\{\epsilon_t\}$ forms a stationary martingale difference sequence with $\sigma^2 = E(\epsilon_1^2) < \infty$. If $-1 < \phi \leq 1$, and $\hat{\phi}_n$ is an estimator of ϕ such that

$$\sqrt{n}(\hat{\phi}_n - \phi) = O_P(1) \quad \text{for} \ |\phi| < 1$$

and

$$n(\hat{\phi}_n - \phi) = O_P(1) \quad \text{for} \ \phi = 1$$

then

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} \hat{\epsilon}_{t+h}\hat{\epsilon}_t = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \epsilon_{t+h}\epsilon_t + o_P(1),$$

where $\hat{\epsilon}_t = y_t - \hat{\phi}_n y_{t-1}$.

Proof. We only prove the case that $\phi = 1$ since the other case is simple. Write that for the fixed h,

$$\begin{aligned} &\frac{1}{\sqrt{n}}\sum_{t=1}^{n-h} \hat{\epsilon}_{t+h}\hat{\epsilon}_t \\ &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n-h} \epsilon_{t+h}\epsilon_t - \frac{1}{\sqrt{n}}(\hat{\phi}_n - \phi)\sum_{t=1}^{n-h} y_{t+h-1}\epsilon_t \\ &\quad + \frac{1}{\sqrt{n}}(\hat{\phi}_n - \phi)\sum_{t=1}^{n-h} y_{t-1}\epsilon_{t+h} + \frac{1}{\sqrt{n}}(\hat{\phi}_n - \phi)^2\sum_{t=1}^{n-h} y_{t+h-1}y_{t-1}. \end{aligned} \tag{2.2}$$

Note that

$$E(\sum_{t=1}^{n-h} y_{t+h-1}\epsilon_t)^2 = E[\sum_{t=1}^{n-h} y_{t-1}\epsilon_t + \sum_{t=1}^{n-h}(\epsilon_t + \dots + \epsilon_{t+h-1})\epsilon_t]^2 \leq O(n^2).$$

Hence,

$$\frac{1}{\sqrt{n}}(\hat{\phi}_n-\phi)\sum_{t=1}^{n-h} y_{t+h-1}\epsilon_t = \frac{1}{\sqrt{n}}O_P(1) = o_P(1).$$

Similarly,

$$\frac{1}{\sqrt{n}}(\hat{\phi}_n-\phi)\sum_{t=1}^{n-h} y_{t-1}\epsilon_{t+h} = o_P(1),$$

and the last term in (2.2) is $o_P(1)$, too. Therefore, the theorem is proved.

Remark. It is well-known that the least squares estimator for the autoregressive coefficient satisfies the stochastic boundedness condition (cf. Hamilton (1994)). A typical and important example for the stationary martingale difference error process is the ARCH process. The above result is somewhat interesting since the nonstationary feature does not affect the limiting distribution.

3. Simulation Study

We consider the model

$$y_t = y_{t-1} + \epsilon_t,$$

where $\{\epsilon_t\}$ is an ARCH(1) process. For calculating the empirical sizes and powers at the significance level 0.05, sets of 200, 500 and 1000 observations are generated from the above model. In each simulation, 200 observations are discarded to remove initialization effects. For the simulation, we use independent 1000 replications. The figures of Table 1 indicate the ratios of the number of rejections of the null hypothesis,

$$H_0:\ \epsilon_t\text{'s are martingale difference sequences.}$$

In order to investigate the empirical sizes, we consider

$$\epsilon_t = \xi_t\sigma_t,\ \sigma_t^2 = 0.3 + 0.5\epsilon_{t-1}^2,$$

where ξ_t are iid random variables and follow a standard normal distribution. Meanwhile, to see the power, we focus on the alternative

$$\epsilon_t = 0.2\epsilon_{t-1} + \xi_t,$$

where ξ_t are iid random variables and follow a standard normal distribution.

As mentioned above, under H_0, the LB test statistic, $Q_n(m)$ follows approximately a chi-square distribution with m degrees of freedom, for which we choose 2, 5, 10 and 15. From Table 1, we can see that the LB test statistic has some size distortions for larger lags, but these size distortions tend to decrease as n increases. We observe that the test statistic yields good empirical powers as n increases. However, when the sample size is relatively small, the empirical powers seriously decrease as m increases. Accordingly, for utilizing the LB test, we need to employ the large sample. Especially, larger sample should be adopted in the case of large lags.

From our results, we conclude that the LB test statistic can be employed as a suitable diagnostic tool for model checking in unit root AR models with ARCH errors. We leave the case of the unstable AR(p) model with GARCH(p, q) errors as a task of future study.

<Table 1> Empirical sizes and powers in unit root model with ARCH(1) errors

	Size				Power			
m	2	10	15	20	2	10	15	20
n=200	.060	.074	.069	.075	.760	.523	.459	.423
n=500	.058	.069	.072	.078	.989	.928	.846	.819
n=1000	.052	.051	.060	.065	.999	.997	.996	.991

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