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Fractional Integration in the Context of Deterministic Trends[1]

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Abstract

In this article we show that the tests of Robinson (1994) may have serious problems in distinguishing between fractionally integrated processes in the context of deterministic trends. The results are obtained via Monte Carlo experiments. A simple procedure, based on the t-values of the coefficients from the differenced regression, is presented to correctly specify the time series of interest and, an empirical application, using data of the US GNP is also carried out at the end of the article.

Keywords : Misspecification; Fractional integration; Deterministic trends

1. Introduction

The presence of deterministic regressors in stochastic models is not new in econometrics. Thus, for example, in the context of unit root tests, Dickey and Fuller (1979) had to tabulate different critical values depending on whether the raw time series contains no regressors, an intercept, or an intercept and a linear time trend. The same happens with other unit root tests (e.g., Phillips, 1987; Phillips and Perron, 1988; Sargan and Bhargava, 1983; etc.), the null limit distribution varying with features of the regressors (see, also Schmidt and Phillips, 1992). Robinson (1994) proposed tests for unit roots and other hypotheses, which are embedded in fractional alternatives and, unlike these other previous procedures, they have standard null and local limit distributions independently of the regressors used in the model. In this article, we show however that the tests of Robinson (1994) may have serious problems if we misspecify these deterministic components.

The structure of the paper is as follows: Section 2 starts by describing the tests of Robinson (1994), and several Monte Carlo experiments are conducted to show how spurious regressions can arise under model misspecification of the deterministic regressors. Section 3 presents a simple procedure to determine the correct model specification. This method is based on the t-values of the coefficients from the differenced regression. In Section 4, the procedure

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is applied to the US GNP data. Section 5 concludes.

2. The Tests of Robinson(1994) and Spurious Regressions

Following discussions of Bhargava (1986), Schmidt and Phillips (1992) and others on parameterization of unit root models, Robinson (1994) considers the regression model,

$$y_t = \beta' z_t + x_t, \quad t = 1, 2, \ldots \tag{1}$$

where y_t is the time series we observe; $\beta = (\beta_1, \cdots, \beta_k)'$ is a $k\times1$ vector of unknown parameters; z_t is a $k\times1$ vector of deterministic regressors that may include, for example, an intercept ($z_t \equiv 1$) or a linear time trend ($z_t = (1, t)'$), and the regression errors x_t are such that:

$$(1 - L)^{d+\theta} x_t = u_t, \quad t = 1, 2, \ldots \tag{2}$$

where d is a given real value, L is the lag operator ($Lx_t = x_{t-1}$) and u_t is an I(0) process, defined for the purpose of the present paper, as a covariance stationary process with spectral density function that is positive and finite at the zero frequency. Thus, under the null hypothesis, defined by:

$$H_o : \theta = 0, \tag{3}$$

we test for a unit root if d = 1, though other types of long memory processes can also be tested with d > 0. Specifically, the score test statistic proposed by Robinson (1994) is given by:

$$\hat{R} = \frac{T}{\hat{\sigma}^4} \hat{a}' \hat{A}^{-1} \hat{a} = \hat{r}^2; \quad \hat{r} = \left(\frac{T}{\hat{A}}\right)^{1/2} \frac{\hat{a}}{\hat{\sigma}^2}, \tag{4}$$

where T is the sample size and

$$\hat{a} = \frac{-2\pi}{T} \sum_j^{T-1} \psi(\lambda_j) g(\lambda_j; \hat{\tau})^{-1} I_{\hat{u}}(\lambda_j); \quad \psi(\lambda_j) = \log\left|2 \sin \frac{\lambda_j}{2}\right|;$$

$$\hat{A} = \frac{2}{T}\left(\sum_j^{T-1} \psi(\lambda_j) \psi(\lambda_j)' - B\right);$$

$$B = \sum_j^{T-1} \psi(\lambda_j)\psi(\lambda_j)' - \sum_j^{T-1} \psi(\lambda_j)\hat{\varepsilon}(\lambda_j)' x \left(\sum_j^{T-1} \hat{\varepsilon}(\lambda_j)\hat{\varepsilon}(\lambda_j)'\right)^{-1} x \sum_j^{T-1} \hat{\varepsilon}(\lambda_j)\psi(\lambda_j)';$$

$$\hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial \tau} \log g(\lambda_j; \hat{\tau}); \quad \hat{\sigma}^2 = \sigma^2(\hat{\tau}) = \frac{2\pi}{T} \sum_j^{T-1} g(\lambda_j; \hat{\tau})^{-1} I_{\hat{u}}(\lambda_j); \quad \lambda_j = \frac{2\pi j}{T},$$

$g(\lambda; \tau)$ is the function appearing in the spectral density of u_t: $f(\lambda; \tau) = (\sigma^2/2\pi)\, g(\lambda; \tau)$, evaluated at $\hat{\tau}$ = arg min $\sigma^2(\tau)$, and $I(\lambda)$ is the periodogram of $\hat{u}_t$ defined as:

$$I(\lambda) = \frac{1}{2\pi T}\left|\sum_{j=1}^{T} \hat{u}_t e^{i\lambda t}\right|^2;$$

$$\hat{u}_t = (1-L)^d y_t - \hat{\beta}' w_t,$$

with

$$w_t = (1-L)^d z_t;$$

$$\hat{\beta} = \left(\sum_{t=1}^{T} w_t w_t'\right)^{-1} \sum_{t=1}^{T} w_t (1-L)^d y_t. \quad (5)$$

Based on H_o (3), Robinson (1994) established that under certain regularity conditions:

$$\hat{R} \to \chi_1^2, \qquad as\ T \to \infty, \quad (6)$$

These conditions are very mild regarding technical assumptions which are satisfied by (1) and (2). Thus, we are in a classical large sample testing situation by reasons described in Robinson (1994). Because $\hat{R}$ involves a ratio of quadratic forms, its exact null distribution can be obtained via Imhof's algorithm. However, a simple asymptotically valid test can be described under much wider distributional assumptions. A test of H_o (3) against H_a: $\theta > 0$ ($\theta < 0$) will be given by the rule: "Reject H_o (3) if $\hat{r} > z_\alpha$ ($\hat{r} < z_\alpha$), where the probability that a standard normal variate exceeds z_α is α. He also showed that the test is efficient in the Pitman sense, that is, that against local alternatives, $\hat{R}$ has a limit distribution given by a $\chi_1^2(\nu)$, with a non centrality parameter ν, that is optimal under Gaussianity on u_t. Tests based on (1) and (2) with z_t in (1) equal to 0 (i.e., with no regressors in the undifferenced regression); $z_t \equiv 1$ (with an intercept), and $z_t = (1, t)'$ (an intercept and a linear time trend) were applied to US historical annual macroeconomic data in Gil Alana and Robinson (1997). They test the same null hypothesis as in (3), and it is shown in that paper that the non rejection values of d substantially vary depending on the deterministic regressors included in (1). The same happens with other versions of Robinson's (1994) tests based on seasonal (quarterly and monthly) and cyclical data (see, eg, Gil Alana and Robinson, 2001, and Gil Alana, 1999, 2001), with the non rejection values of d being affected by the deterministic trends.

In this section we investigate how misspecification in (1) may affect the statistical properties of Robinson's (1994) tests and, for this purpose, we conduct several experiments via Monte Carlo simulations. First, we assume that the true model is given by:

$$(1-L)^d x_t = u_t, \quad t = 1, 2, ... \quad (7)$$

for a given value d and white noise u_t. We perform Robinson's (1994) tests against alternatives of form as in (1) and (7) with z_t = (1,t)′. Then, the opposite experiment will be conducted, i. e., assuming that the true model is:

$$y_t = \beta_0 + \beta_1 t + x_t, \quad t = 1, 2, \ldots \tag{8}$$

and (7), for given values β_0, β_1 and d, while the alternatives will be given exclusively by (7).

In both cases we look at the rejection frequencies, generating Gaussian series, obtained by the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Vetterling (1986), with 50,000 replications of each case. We should expect these rejection probabilities to be, if not 1, at least a value relatively close to it, since they are subject to model misspecification. The sample sizes are 50, 100, 200 and 300 and the nominal size is 5% in all cases.

Table 1

Rejection frequencies of $\hat{R}$ in (4)

True model: $(1-L)^d y_t = u_t$; and white noise u_t.
Alternative: $y_t = \beta_0 + \beta_1 t + x_t$; $(1-L)^d x_t = u_t$; and white noise u_t.

T / d	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
50	0.175	0.167	0.161	0.160	0.156	0.154	0.153	0.152	0.152	0.153	0.153
100	0.137	0.135	0.134	0.131	0.131	0.129	0.127	0.126	0.126	0.126	0.128
200	0.124	0.121	0.118	0.117	0.115	0.116	0.116	0.115	0.117	0.117	0.116
300	0.116	0.115	0.112	0.111	0.112	0.111	0.112	0.112	0.112	0.112	0.112

The rejection frequencies are two-sided at the 5% level. 50,000 replications were used in each case.

Table 1 reports the rejection frequencies of the two sided statistic $\hat{R}$ in (4) in a model given by (7) with d = 0.50, (0.10), 1.50. The alternatives are given by (7) and (8), with the same d values under both the null and the alternative hypotheses. We see that the rejection frequencies are very low in all cases, never exceeding 0.300. These probabilities are even smaller when d and T are relatively high, being, for example, never superior to 0.120 if T = 300.

Table 2

]Rejection frequencies of $\hat{R}$ in (4)

True model: $y_t = 1 + t + x_t$; $(1-L)^d x_t = u_t$; and white noise u_t.
Alternative: $(1-L)^d y_t = u_t$; and white noise u_t.

T / d	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
50	1.000	0.996	0.642	0.177	0.149	0.155	0.152	0.149	0.150	0.153	0.157
100	1.000	1.000	0.998	0.482	0.130	0.129	0.126	0.129	0.128	0.128	0.127
200	1.000	1.000	1.000	0.956	0.157	0.116	0.116	0.118	0.117	0.116	0.116
300	1.000	1.000	1.000	0.999	0.214	0.112	0.113	0.114	0.113	0.113	0.114

The rejection frequencies are two-sided at the 5% level. 50,000 replications were used in each case.

Table 2 resumes the rejection probabilities of the opposite experiment, i.e., calculating $\hat{R}$ in a model given by (7) and (8) with the same values of d as in Table 1 and β_0 = β_1 = 1. The alternatives are now exclusively given by (7) with the same values of d as in the true model. We see that the rejection frequencies are now very high if d = 0.50, 0.60, 0.70 and 0.80, however, for values of d constrained between 0.90 and 1.50, these probabilities considerably reduce, ranging between 0.10 and 0.15 in practically all cases. This apparent different behaviour between the cases when d is higher than and smaller than 1 may be related to the fact that whereas $(1 - L)^d t$ tends to zero for d > 1 as t increases, it continues to trend with t for d < 1 (whereas $(1 - L)^d t^d$ tends to a non zero constant for all d). Though not reported in the paper, performing the tests under an adequate specification of the deterministic trends, the results were completely satisfactory in terms of both the size and the power properties. An extensive Monte Carlo work is conducted in this context in Robinson (1994).

We can summarize the results in Tables 1 and 2 by saying that the tests of Robinson (1994) have serious difficulties in distinguishing between different I(d) processes in the context of deterministic trends. This is particularly worrisome in those cases where we include an unneccessary trend in the model, for all values of d, or if we do not include it when it is required for values of d ≥ 1.

3. A Simple Procedure of Statistical Modelling

We present in this section a very simple procedure for choosing an adequate model specification with the tests of Robinson (1994) in the presence of deterministic trends. This procedure is based on the t-values of the coefficients of the null differenced regression in (1) and (2). Note that under the null hypothesis H_o (3), (1) and (2) can be rewritten as:

$$(1 - L)^d y_t \;=\; \beta' w_t \;+\; u_t, \qquad t \;=\; 1, 2, \ldots, \tag{9}$$

with w_t as defined in (5) and $\hat{\beta}$ obtained via least squared estimation. Of course, in the context of autocorrelated disturbances, the estimate of β can be improved by means of GLS.

We could start from a general model, including an intercept and a linear time trend, (i.e. (8)) and (2), and test H_o (3) for different values of d. Then, from the non rejection values of d, we only choose as potential model specifications, those models with significant coefficients in terms of the t-values of β_0 and β_1. If H_o (3) is rejected for all d, or if some of the d's cannot be rejected but both β_0 and β_1 (or only β_1) are insignificantly different from 0, we perform again Robinson's (1994) tests but this time in a model given by (2) and

$$y_t \;=\; \beta_0 \;+ x_t\,, \qquad t \;=\; 1, 2, \ldots, \tag{10}$$

and follow the same procedure based on $\hat{R}$ (or $\hat{r}$) in (4) and the t value of β_0. In a final step, if once more H_o (3) is rejected for all d or if it cannot be rejected for a given d but β_0 is a non significant coefficient, Robinson's (1994) tests is then performed exclusively based on (2), choosing as a model specification the one with a non rejected value of the test statistic.

We should also mention that Robinson's (1994) tests proposed in this article have nothing to do with the estimation of the fractional differencing parameter d and thus, it is not surprising that, in a given empirical application, many non-rejections can appear. Therefore, this procedure has to be taken as a simple computing diagnostic from departures from real values of d, which should be complementary to other procedures to correctly specify the series. We believe that as in other standard large-sample testing situations, Wald and LR test statistics against fractional alternatives will have the same null and local limit theory as the LM tests of Robinson (1994). Sowell (1992) employed essentially such a Wald testing procedure but it requires an efficient estimate of d, and while such estimates can be obtained, no closed-form formulae are available and so the LM procedure of Robinson (1994) seems computationally more attractive. In the following section, an empirical application based on this procedure will be performed on the US GNP.

4. An Empirical Application

The time series data used in this application is the log of the quarterly US GNP, seasonally adjusted, for the time period 1955q1 - 1999q4, obtained from the Reserve Federal Bank of St. Louis. We employ throughout the model given by (8) and (2), i.e. testing the null model:

$$y_t = \beta_0 + \beta_1 t + x_t, \quad t = 1, 2, \ldots$$

$$(1 - L)^d x_t = u_t, \quad t = 1, 2, \ldots$$

for values of d = 0, (0.25), 2, with white noise and autoregressive (AR) disturbances.

Across Tables 3 - 5 we report values of the one sided statistic $\hat{r}$ in (4). Thus, significantly positive values of this, ($\hat{r} > z_\alpha$), should be consistent with higher orders of integration, ($\Theta > 0$), and conversely, significantly negative ones ($\hat{r} < -z_\alpha$), will imply smaller values of d ($\Theta < 0$).

Table 3

Testing H_o (3) in (1) and (2) with white noise u_t									
z_t / d	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
$z_t = (1,t)'$ t-value (β_0) t-value (β_1)	36.67 (-9.73) (45.78)	34.27 (-6.00) (39.87)	32.35 (-1.36) (34.10)	29.07 (3.93) (26.08)	20.02 (10.19) (16.50)	6.15 (16.37) (7.29)	-1.92 (19.00) (1.96)	-4.07 (17.64) (0.46)	-4.95 (15.62) (0.30)
$z_t = 1$ t-value (β_0)	30.79 (16.26)	28.51 (11.55)	25.99 (7.68)	26.85 (6.38)	19.55 (11.45)	3.75 (18.68)	-2.30 (21.23)	-4.05 (22.01)	-4.93 (22.40)
$z_t = 0$	30.79	31.20	31.58	28.94	13.47	**0.71***	-3.07	-4.53	-5.32

* and in bold: Non-rejection values at the 5% significance level.

Starting with the case of white noise disturbances, in Table 3, we observe that including an intercept and a linear time trend, the null hypothesis is always rejected. The same happens with an intercept, and the only non rejection value takes place when d = 1.25 if we do not include regressors. However, we also observe across this table that the value of the test statistic changes its sign when d goes from 1.25 through 1.50 for both cases of an intercept and with an intercept and a linear time trend. Thus, in Table 4, we re compute the same statistic as in Table 3, again for white noise disturbances, but this time for a range of values of d constrained between 1 and 1.5 with 0.1 increments. We see now that including a linear time trend, H_o (3) cannot be rejected if d = 1.40 and both coefficients (the intercept and the slope) are significant. Including an intercept, the null hypothesis cannot be rejected for the same value of d (i.e., 1.40), and including no regressors, the null cannot rejected if d = 1.25 and 1.30. However, in view of the significance of the coefficients in case of the inclusion of a linear time trend, we take this model as a potential specification for this series, being preferred to the remaining non-rejected cases.

Table 4

Testing H_o (3) in (1) and (2) with white noise u_t							
z_t / d	1.00	1.10	1.20	1.25	1.30	1.40	1.50
$z_t=(1,t)'$ t-value (β_0) t-value (β_1)	20.02 (10.19) (16.50)	14.51 (12.87) (12.85)	8.78 (15.32) (8.92)	6.15. (16.37) (7.29)	3.82 (17.27) (5.84)	**0.25*** **(18.53)** **(3.52)**	-1.92 (19.00) (1.96)
$z_t=1$ t-value (β_0)	19.55 (11.45)	12.82 (14.79)	6.30 (17.63)	3.75 (18.68)	1.76 (19.50)	**-0.85*** **(20.59)**	-2.30 (21.23)
$z_t=0$	13.47	6.73	2.23	**0.71***	**-0.44***	-2.04	-3.07

* and in bold: Non-rejection values at the 5% significance level.

The significance of the results in Tables 3 and 4 may be in large part due to the un accounted for I(0) autocorrelation in u_t. Table 5 displays the results of the same statistic as in Tables 3 and 4 but based on AR(1) disturbances. Higher AR orders were also tried, and the results were practically the same as those reported here for the AR(1) model. We choose now values of d constrained between 0 and 1 with 0.1 increments, the reason being that the null was rejected for values of d higher than 1. We see that including a linear time trend, H_o (3) cannot be rejected if d ranges between 0.30 and 0.70 and, in all cases, the t-values indicate significant coefficients for the time trend. The lowest statistic in absolute value across d appears when d = 0.40, i.e., one unit less than in the previous case of white noise disturbances. It may result striking that the non rejection d's are now smaller than those in Tables 3 and 4, indicating how the AR model is somewhat confounded with the fractional model, and the delicacy of modelling in this situation. This may be due to the fact that we use Yule Walker estimates, which entail roots which are automatically less than one in absolute value, but that can be arbitrarily close to it. Thus, it might exist some type of competition between the AR and the fractional differencing parameters in describing the nonstationary component of the series. In fact, we also observe in Table 5 that if we include an intercept, the non rejection values take place when d = 0 and 0.10, and the corresponding AR parameters (not reported in the tables) were respectively 0.99 and 0.98. If we do not include regressors, the values of $\hat{r}$ behave erroneously for values of d constrained between 0 and 0.40, in the sense that the test statistic does not decrease monotonically with d as we should expect in view of the fact that they are one sided statistics, again this being probably due to the proximity of the AR parameter to the unit root case.

Table 5

Testing H_o (3) in (1) and (2) with AR (1) u_t									
z_t / d	0	0.10	0.20	0.30	0.40	0.50	0.60	0.70	-0.80
$z_t = (1,t)'$ t-value (β_0) t-value (β_1)	17.57 (-156.7) (733.5)	13.03 (-107.1) (558.2)	4.88 (-57.08) (342.9)	**1.30*** **(-30.82)** **(232.2)**	**-0.02*** **(-16.15)** **(177.8)**	**-0.12*** **(-5.88)** **(146.9)**	**-0.14*** **(2.59)** **(118.7)**	**-1.06*** **(8.71)** **(86.02)**	-2.44 (11.99) (57.08)
$z_t = 1$ t-value (β_0)	**-0.43*** **(61.45)**	**-1.03*** **(48.07)**	-1.72 (36.22)	-2.48 (26.11)	-3.38 (19.60)	-4.15 (15.11)	-4.52 (12.85)	-4.55 (13.53)	-4.66 (13.59)
$z_t = 0$	—	—	—	—	—	3.43	3.15	**0.24***	-2.62

* and in bold: Non-rejection values at the 5% significance level. "—" means that the value of the test statistic does not monotonically decrease with d.

5. Conclusions

In this article we have shown that the tests of Robinson (1994) may have serious problems in distinguishing between I(d) statistical models in the presence of deterministic trends. Several Monte Carlo experiments conducted across the paper showed that the tests perform relatively poor when we include a linear time trend and it should not be included in the model, and also when we misspecify a trend that should be included, the latter case being especially worrisome when testing values of d higher than 1. To solve this problem we propose a simple procedure based on the t - values of the coefficients from the null differenced regression. This procedure was applied to the US GNP series. The results show that if we model the I(0) disturbances as white noise, the time trend must be required, the order of integration of the series being around 1.40. Modelling, however, the disturbances as autoregressions, the time trend also seems to be important but the order of integration reduces considerably, probably due to the competition with the AR parameters in describing the nonstationarity of the series. A natural following up step should be to propose a model selection criterion to determine the correct model specification in relation to the short run components of the series. Work in this direction is now under progress.

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