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Pring Fixed-Strike Lookback Options[1)]

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Abstract

A fixed-strike lookback option is an option whose payoff is determined by the maximum (or minimum) price of the underlying asset within the option's life. Under the Black-Scholes framework, the time-t price of an equity asset follows a geometric Brownian motion. Applying the method of Esscher transforms, this paper will derive explicit pricing formulas for fixed-strike lookback call and put options, respectively. In addition, this paper will show a relationship (duality property) between the pricing formulas of the call and put options. Finally, this paper will derive explicit pricing formulas for the fixed-strike lookback options when their underlying asset pays dividends continuously at a rate proportional to its price.

Keywords : Esscher transforms, fixed-strike lookback option, duality property, Brownian motion

1. Introduction

Suppose an investor believes that a stock will rise substantially in the next three months and buys a plain-vanilla call option with a maturity of three months. After the purchase, the stock rises to a satisfactory level. But the stock drops unexpectedly a few days before maturity. Though the investor has forecasted the stock's overall trend correctly, he receives a lesser payoff than if he had sold the option a few days earlier. As pointed out in Heynen and Kat (1994a), plain-vanilla options have a drawback in that incorrect timing of market exit can seriously affect even option holders with a largely correct view. Heynen and Kat suggested fixed-strike lookback options as alternatives to plain-vanilla options.

A fixed-strike lookback option is an option whose payoff is determined by the maximum (or minimum) price of the underlying asset within the option's life. This option looks like a plain-vanilla option except that the underlying asset price at maturity is replaced with its maximum (or minimum). In other words, the payoff of a fixed-strike lookback call option is the excess of the maximum price over the strike price if the maximum price is greater than the strike price. The payoff of a fixed-strike lookback put option is the excess of the strike price over the minimum price if the minimum price is less than the strike price. When the

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underlying asset drops (or increases) substantially at maturity, these options provide the holders with better payoffs than the corresponding plain-vanilla options.

Conze and Viswanathan (1991) derived explicit pricing formulas for fixed-strike lookback options whose monitoring period is the whole life of these options. A drawback is that these options are expensive. The high premiums of the fixed-strike lookback options prevent them from being widely used. To overcome this drawback, Heynen and Kat (1994a) presented explicit pricing formulas for fixed-strike lookback options whose monitoring period starts at an arbitrary date before maturity and ends at maturity.

However, in their paper, there is no complete proof for the formulas. This paper will derive explicit pricing formulas that are equivalent to formula (13) of Heynen and Kat. Section 2 will describe the method of Esscher transforms. Sections 3 and 4 will cover the fixed-strike lookback call and put options, respectively. In addition, Section 4 will show a relationship between the pricing formulas of the call and put options. If negative one multiplies some parameters of the call option formula, then the call option formula will be the negative of the time-0 value of the corresponding put option, and vice versa. In other words, if one of the two formulas is obtained, the other one can be straightforwardly derived. Finally, Section 5 will derive explicit pricing formulas for the fixed-strike lookback options when their underlying asset pays dividends continuously at a rate proportional to its price. These pricing formulas are generalizations of the pricing formulas in Sections 3 and 4.

2. Esscher Transform and Some Useful Formulas

This section describes the method of Esscher transforms and presents some formulas useful for pricing lookback options. Let $S(t)$ denote the time-t price of an equity asset. Assume that the asset is constructed with all dividends reinvested. Assume that for $t \geq 0$,

$$S(t) = S(0)e^{X(t)} \tag{2.1}$$

where $\{X(t)\}$ is a Brownian motion with drift μ, diffusion coefficient σ and $X(0) = 0$. Thus, the Brownian motion is a stochastic process with independent and stationary increments, and $X(t)$ has a normal distribution with mean μt and variance $\sigma^2 t$.

First, this section briefly summarizes a special case of the method of Esscher transforms developed by Gerber and Shiu (1994, 1996). For a nonzero real number h, the moment generating function of $X(t)$, $E[e^{hX(t)}]$, exists for all $t \geq 0$, because $X(t)$ is the Brownian motion as described above. The stochastic process

$$\{e^{hX(t)}E[e^{hX(1)}]^{-t}\} \tag{2.2}$$

is a positive martingale which can be used to define a new probability measure Q. More precisely, this process is used to define the Radon-Nikodym derivative dQ/dP, where P is the original probability measure. We call Q the Esscher measure of parameter h.

For a random variable Y that is a real-valued function of $\{X(t), 0 \le t \le T\}$, the expectation of Y under the new probability measure Q is calculated as

$$E[Y\frac{e^{hX(T)}}{E[e^{hX(1)}]^T}] \tag{2.3}$$

which will be denoted by $E[Y;h]$. The risk-neutral Esscher measure is the Esscher measure of parameter $h = h^*$ under which the process $\{e^{-rt}S(t)\}$ is a martingale. Here, r denotes continuously compound interest rate. Thus

$$E[e^{-rt}S(t);h^*] = S(0). \tag{2.4}$$

Therefore, h^* is the solution of

$$\mu + h^*\sigma^2 = r - \sigma^2/2. \tag{2.5}$$

For $t \ge 0$, the moment generating function of $X(t)$ under the Esscher measure of parameter h is

$$E[e^{zX(t)};h] = exp\{(\mu + h\sigma^2)tz + \sigma^2tz^2/2\}, \tag{2.6}$$

which implies that $X(t)$ has a normal distribution with mean $(\mu + h\sigma^2)t$ and variance σ^2t under the Esscher measure. It can be shown in (A.1) and (A.2) of the Appendix that the process $\{X(t)\}$ under the Esscher measure has independent and stationary increments. Thus, the process is a Brownian motion with drift $\mu + h\sigma^2$ and diffusion coefficient σ under the Esscher measure of parameter h.

Now, let us consider a special case of the *factorization formula* (Gerber and Shiu, 1994, p 177; 1996, p 188). For a random variable Y that is a real-valued function of $\{X(t), 0 \le t \le T\}$,

$$E[e^{cX(T)}Y;h] = E[e^{cX(T)};h]E[Y;h+c] \tag{2.7}$$

In particular, for an event B whose condition is determined by $\{X(t), 0 \le t \le T\}$, the formula (2.7) can be expressed as follows:

$$E[e^{cX(T)}I(B);h] = E[e^{cX(T)};h]Pr[B;h+c], \tag{2.8}$$

where $I(\cdot)$ denotes the indicator function and $Pr(B;h)$ denotes the probability of the event B under the Esscher measure of parameter h.

Now, let us discuss some basic formulas useful for pricing lookback options. For $0 \le s \le t$, let

$$M(s,t) = max\{X(\tau), s \le \tau \le t\} \tag{2.9}$$

be the maximum of the Brownian motion between time s and time t. Consider a bivariate standard normal distribution. Note that

$$\Phi_2(a,b;\rho) = \Phi_2(b,a;\rho) \tag{2.10}$$

and

$$\Phi(a) - \Phi_2(a,b;\rho) = \Phi_2(a,-b;-\rho), \tag{2.11}$$

where $\Phi_2(a,b;\rho)$ denotes the bivariate standard normal distribution function with correlation

coefficient ρ. In the Appendix, we shall prove that, if $-\rho b+\sqrt{1-\rho^2}c=a$ and $\rho \geq 0$, then

$$\Phi_2(a,b;-\rho)+\Phi_2(-a,c;-\sqrt{1-\rho^2})=\Phi(b)\Phi(c) \tag{2.12}$$

and

$$\Phi_2(a,b;-\rho)+\Phi(-b)\Phi(c)=\Phi_2(a,c;\sqrt{1-\rho^2}), \tag{2.13}$$

which will play a role in the proof of (3.5).

Let a random vector Z = (Z_1, Z_2, Z_3) have a standard trivariate normal distribution with correlation coefficients $\mathrm{Corr}(Z_i, Z_j) = \sigma_{ij}$ $(i, j = 1, 2, 3)$. The distribution function of the random vector Z is

$$\Phi_3(a, b, c\ ;\ \sigma_{12}\ ,\ \sigma_{13},\ \sigma_{23}) = \Pr(Z_1 \leqq a,\ Z_2 \leqq b,\ Z_3 \leqq c).$$

For $0 < s < t < T$, the joint distribution function of $X(T)$ and $M(s,t)$ is

$$\begin{aligned} &Pr(M(s,t) \leq m, X(T) \leq x) \\ &=\Phi_3\left(\frac{x-\mu T}{\sigma\sqrt{T}}, \frac{m-\mu t}{\sigma\sqrt{t}}, \frac{m-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\ &-e^{\frac{2\mu}{\sigma^2}m}\Phi_3\left(\frac{x-2m-\mu T}{\sigma\sqrt{T}}, \frac{-m-\mu t}{\sigma\sqrt{t}}, \frac{m+\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right), \end{aligned} \tag{2.14}$$

which is proved in Lee (2003). For numerical implemetation of multivariate normal distribution functions, see Drezner (1978, 1994). If variable x in (2.14) approaches infinity, the distribution function (2.14) will be

$$\begin{aligned} &\Pr(M(s,\ t) \leqq m) \\ &=\Phi_2\left(\frac{m-\mu t}{\sigma\sqrt{t}}, \frac{m-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{s}{t}}\right)-e^{\frac{2\mu}{\sigma^2}m}\Phi_2\left(\frac{-m-\mu t}{\sigma\sqrt{t}}, \frac{m+\mu s}{\sigma\sqrt{s}}; -\sqrt{\frac{s}{t}}\right). \end{aligned} \tag{2.15}$$

Finally, let us discuss expectations necessary for deriving formulas for fixed-strike lookback options. Let random variable X be normal with mean μ and variance σ^2. Let random variables Z follow the standard normal distribution. We assume that a, b and θ are real numbers, and $\sigma_1 > 0$. We also assume that Z is independent of X. Then,

$$\begin{aligned} \mathrm{E}[\mathrm{I}(X < a)\Phi(\frac{\theta X+b}{\sigma_1})] &= \mathrm{E}[\mathrm{I}(X < a)\mathrm{E}[\mathrm{I}(Z < \frac{\theta X+b}{\sigma_1} \mid X]] \\ &= \mathrm{E}[\mathrm{I}(X < a,\ Z < \frac{\theta X+b}{\sigma_1})] \\ &= \Pr\{X < a,\ \sigma_1 Z - \theta X < b\} \\ &= \Phi_2\left(\frac{a-\mu}{\sigma}, \frac{b+\theta\mu}{\sqrt{\theta^2\sigma^2+\sigma_1^2}}; -\theta\frac{\sigma}{\sqrt{\theta^2\sigma^2+\sigma_1^2}}\right). \end{aligned} \tag{2.16}$$

A generalized version of (2.16) can be calculated as follows:

$$\begin{aligned}
\mathrm{E}[e^{hX}\mathrm{I}(X < a)\Phi(\frac{\theta X+b}{\sigma_1})] &= \mathrm{E}[e^{hX}]\ \mathrm{E}[\frac{e^{hX}}{E[e^{hX}]}\mathrm{I}(X < a)\Phi(\frac{\theta X+b}{\sigma_1})] \\
&= e^{\mu h+\frac{1}{2}\sigma^2 h}\mathrm{E}[\mathrm{I}(X < a)\Phi(\frac{\theta X+b}{\sigma_1});\ h] \\
&= e^{\mu h+\frac{1}{2}\sigma^2 h}\ \Phi_2(\frac{a-\mu_h}{\sigma}, \frac{b+\theta\mu_h}{\sqrt{\theta^2\sigma^2+\sigma_1^2}}; -\theta\frac{\sigma}{\sqrt{\theta^2\sigma^2+\sigma_1^2}}), \quad (2.17)
\end{aligned}$$

where μ_h denotes $\mu+h\sigma^2$. Note that the first and second equalities of (2.17) come from the factorization formula (2.7) and that the last equality of (2.17) holds because of (2.16).

3. Fixed-Strike Lookback Call Option

As mentioned above, the payoff of a fixed-strike lookback call option is the excess of the maximum price over the strike price if the maximum price is greater than the strike price. The fixed-strike lookback call option looks like the plain-vanilla call option except that the underlying asset price at maturity is replaced with the maximum price attained within a partial life of the option. In this section, we shall derive an explicit pricing formula for the fixed-strike lookback call option.

Assume that the strike price is K. Let $k = \log(K/S(0))$. The payoff of this fixed-strike lookback call option is as follows:

$$(S(0)e^{M(t,T)}-K)I(M(t,T)\geq k). \quad (3.1)$$

Applying the fundamental theorem of asset pricing, we obtain the time-0 value of (3.1),

$$e^{-rT}E[(S(0)e^{M(t,T)}-K)I(M(t,T)\geq k);h^*], \quad (3.2)$$

which can be divided into two terms,

$$e^{-rT}[S(0)E[e^{M(t,T)}I(M(t,T)\geq k);h^*]-KPr(M(t,T)\geq k;h^*)]. \quad (3.3)$$

Here, applying (2.15), we can obtain the probability term in (3.3) as follows:

$$\begin{aligned}
&Pr(M(t,T) > k) \\
&= 1-Pr(M(t,T)\leq k) \\
&= 1-\left[\Phi_2(\frac{k-\mu T}{\sigma\sqrt{T}}, \frac{k-\mu t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}) - e^{\frac{2\mu}{\sigma^2}k}\Phi_2(\frac{-k-\mu T}{\sigma\sqrt{T}}, \frac{k+\mu t}{\sigma\sqrt{t}}; -\sqrt{\frac{t}{T}})\right] \\
&=: P_2(\mu,k) \quad (3.4)
\end{aligned}$$

If the parameter μ in formula (3.4) is replaced by $r-\sigma^2/2$, then the probability term in (3.3) is $P_2(r-\sigma^2/2)$.

Now, let us consider the expectation in (3.3). It will be proved in the Appendix that for a nonzero real number c, $c+\xi\neq 0$ and $\xi = 2\mu/\sigma^2$,

$$
\begin{aligned}
&E[e^{cM(t,T)}I(M(t,T)>k)]\\
&=\frac{2c+\xi}{c+\xi}e^{c\mu T+\frac{1}{2}c^2\sigma^2T}\Phi_2\left(\frac{-k+(\mu+c\sigma^2)T}{\sigma\sqrt{t}},\frac{(\mu+c\sigma^2)(T-t)}{\sigma\sqrt{T-t}};-\sqrt{1-\frac{t}{T}}\right)\\
&+\frac{\xi}{c+\xi}e^{(c+\xi)k}\Phi_2\left(\frac{k+\mu t}{\sigma\sqrt{t}},-\frac{k+\mu T}{\sigma\sqrt{T}};-\sqrt{\frac{t}{T}}\right)\\
&+\frac{\xi}{c+\xi}e^{c\mu t+\frac{1}{2}c^2\sigma^2t}\Phi\left(\frac{-k+(\mu+c\sigma^2)t}{\sigma\sqrt{t}}\right)\Phi\left(-\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right)\\
&=:I_1(\mu,c,k).
\end{aligned}
\tag{3.5}
$$

If the parameter μ in formula (3.5) is replaced by $r-\sigma^2/2$ and c is equal to one, then the expectation term in (3.3) is $I_1(r-\sigma^2/2,1,k)$. Applying the probability (3.4) and the expectation formula (3.5), we have the time-0 value of the fixed-strike lookback call option,

$$
e^{-rT}[S(0)I_1(r-\sigma^2/2,1,k)-KP_2(r-\sigma^2/2,k)],
\tag{3.6}
$$

which can be rewritten as

$$
\begin{aligned}
&S(0)\left(\frac{\sigma^2}{2r}+1\right)\Phi_2\left(\frac{-k+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},\frac{(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}};\sqrt{1-\frac{t}{T}}\right)\\
&-S(0)\frac{\sigma^2}{2r}e^{-rT}e^{k\frac{2r}{\sigma^2}}\Phi_2\left(\frac{k+(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}},-\frac{k+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}};-\sqrt{\frac{t}{T}}\right)\\
&+S(0)\left(1-\frac{\sigma^2}{2r}\right)e^{-r(T-t)}\Phi\left(\frac{-k+(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)\Phi\left(-\frac{(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&+e^{-rT}K+e^{-rT}K\Phi_2\left(\frac{k-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}},\frac{k-(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}};\sqrt{\frac{t}{T}}\right)\\
&=:V_{Call}(S(0),K,r,\sigma).
\end{aligned}
\tag{3.7}
$$

Finally, let us show that the probability formula (3.4) is a special case of the expectation formula (3.5). If c is zero, formula (3.5) is

$$
\begin{aligned}
&\Phi_2\left(\frac{-k+\mu T}{\sigma\sqrt{T}},\frac{\mu(T-t)}{\sigma\sqrt{T-t}};\sqrt{1-\frac{t}{T}}\right)\\
&+e^{\frac{2\mu}{\sigma^2}k}\Phi_2\left(\frac{k+\mu t}{\sigma\sqrt{t}},-\frac{k+\mu T}{\sigma\sqrt{T}};\sqrt{\frac{t}{T}}\right)\\
&+\Phi\left(\frac{-k+\mu t}{\sigma\sqrt{t}}\right)\Phi\left(-\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}
\tag{3.8}
$$

The second term in (3.8) is the same as the last term in (3.4). To prove that the probability formula (3.4) is a special case of formula (3.5) when c is zero, it is sufficient to show that the sum of the first and last terms in (3.8) is equivalent to the sum of the first two terms on the right-hand side of (3.5). By applying formula (2.11) to the first and last equalities of

(3.9) and by applying (2.13) to the second equality of (3.9), the sum of the first and last terms in (3.8) can be calculated as follows:

$$\begin{aligned}
&\Phi_2\left(\frac{-k+\mu T}{\sigma\sqrt{T}}, \frac{\mu(T-t)}{\sigma\sqrt{T-t}}; \sqrt{1-\frac{t}{T}}\right)+\Phi\left(\frac{-k+\mu t}{\sigma\sqrt{t}}\right)\Phi\left(-\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right)\\
&=\Phi_2\left(\frac{-k+\mu T}{\sigma\sqrt{T}}, \frac{\mu(T-t)}{\sigma\sqrt{T-t}}; \sqrt{1-\frac{t}{T}}\right)-\Phi\left(\frac{-k+\mu t}{\sigma\sqrt{t}}\right)\Phi\left(\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right)+\Phi\left(\frac{-k+\mu t}{\sigma\sqrt{t}}\right)\\
&=\Phi_2\left(\frac{-k+\mu T}{\sigma\sqrt{T}}, \frac{k-\mu t}{\sigma\sqrt{t}}; -\sqrt{\frac{t}{T}}\right)+\Phi\left(-\frac{k-\mu t}{\sigma\sqrt{t}}\right)\\
&=\Phi_2\left(\frac{-k+\mu T}{\sigma\sqrt{T}}, \frac{k-\mu t}{\sigma\sqrt{t}}; -\sqrt{\frac{t}{T}}\right)+1-\Phi\left(\frac{k-\mu t}{\sigma\sqrt{t}}\right)\\
&=1-\Phi_2\left(\frac{k-\mu T}{\sigma\sqrt{T}}, \frac{k-\mu t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}\right). \qquad (3.9)
\end{aligned}$$

4. Fixed-Strike Lookback Put Option

It will be recalled that the payoff of a fixed-strike lookback put option is the excess of the strike price over the minimum price if the minimum price is less than the strike price. The fixed-strike lookback put option looks like the plain-vanilla put option except that the underlying asset price at maturity is replaced with its minimum attained within a partial life of the option. In this section, we shall derive an explicit pricing formula for the fixed-strike lookback put option.

The payoff of this lookback put option is as follows:

$$(K-S(0)e^{m(t,T)})I(m(t,T)<k), \qquad (4.1)$$

where $m(t, T)$ denotes the mimimum of the Brownian motion between time t and time T. In other words,

$$m(t, T) = min\{X(\tau), t \le \tau \le T\}.$$

Applying the fundamental theorem of asset pricing, we obtain the time-0 value of (4.1),

$$e^{-rT}E[(K-S(0)e^{m(t,T)})I(m(t,T)<k);h^*], \qquad (4.2)$$

which can be divided into two terms,

$$-e^{-rT}\{S(0)E[e^{m(t,T)}I(m(t,T)<k);h^*]-KPr(m(t,T)<k,h^*)\}. \qquad (4.3)$$

The expectation of (4.3) under the original probability measure can be easily calculated as follows:

$$\begin{aligned}
&E[e^{cm(t,T)}I(m(t,T)<k)]\\
&=E[e^{-cM(t,T)}I(M(t,T)>-k);-\xi]\\
&=I_1(-\mu,-c,-k)\\
&=\frac{2c+\xi}{c+\xi}e^{c\mu T+\frac{1}{2}c^2\sigma^2T}\Psi_2\left(\frac{-k+(\mu+c\sigma^2)T}{\sigma\sqrt{T}}, \frac{(\mu+c\sigma^2)(T-t)}{\sigma\sqrt{T-t}}; \sqrt{1-\frac{t}{T}}\right)
\end{aligned}$$

$$+\frac{\xi}{c+\xi}e^{(c+\xi)k}\Psi_2\left(\frac{k+\mu t}{\sigma\sqrt{t}},-\frac{k+\mu T}{\sigma\sqrt{T}};\sqrt{\frac{t}{T}}\right)$$

$$+\frac{\xi}{c+\xi}e^{c\mu t+\frac{1}{2}c^2\sigma^2 t}\Psi\left(\frac{-k+(\mu+c\sigma^2)t}{\sigma\sqrt{t}}\right)\Psi\left(-\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right), \tag{4.4}$$

where $\Psi(x) := \Phi(-x)$ and $\Psi_2(x, y;\rho) := \Phi_2(-x, -y;\rho)$. Applying

$$m(t, T) = -\max\{-X(\tau), s \le \tau \le t\}$$

and the fact that $\{X(t)\}$ is a Brownian motion with drift

$$\mu - \xi\sigma^2 = -\mu$$

and diffusion coefficient σ under the Esscher measure of parameter $-\xi$, we obtain the first equality of (4.4). Note that formula (4.4) is the same as formula (3.8) except that the symbols Φ and Φ_2 are replaced by Ψ and Ψ_2, respectively. In addition, we obtain the probability of (4.3) under the original probability measure,

$$\begin{aligned} Pr(m(t, T) < k) &= Pr(M(t, T) > -k; -\xi) \\ &= P_2(-\mu, -k). \end{aligned} \tag{4.5}$$

Applying the facts which we have done in the first equality of (4.4), the first equality of (4.5) holds. Hence, applying (4.4) and (4.5) to (4.3), we see that the time-0 value of the fixed-strike lookback put option is

$$-e^{-rT}[S(0)I_1(-r+\sigma^2/2, -1, -k) - KP_2(-r+\sigma^2/2, -k)], \tag{4.6}$$

which can be rewritten as

$$-\{S(0)\left(\frac{\sigma^2}{2r}+1\right)\Psi_2\left(\frac{-k+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \frac{(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}; \sqrt{1-\frac{t}{T}}\right)$$

$$-S(0)\frac{\sigma^2}{2r}e^{-rT}e^{k\frac{2r}{\sigma^2}}\Psi_2\left(\frac{k+(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}, -\frac{k+(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}; -\sqrt{\frac{t}{T}}\right)$$

$$+S(0)\left(1-\frac{\sigma^2}{2r}\right)e^{-r(T-t)}\Psi\left(\frac{-k+(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)\Psi\left(-\frac{(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$-e^{-rT}K+e^{-rT}K\Psi_2\left(\frac{k-(r-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \frac{k-(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}; \sqrt{\frac{t}{T}}\right)\}.$$

$$=: V_{Put}(S(0), K, r, \sigma). \tag{4.7}$$

Now, let us show a relationship between the pricing formulas (3.7) and (4.7) of the fixed-strike lookback options. Formulas (3.6) and (4.6) are the same as (3.7) and (4.7), respectively. If -1 is multiplied by some parameters of (3.6), then the expression (3.6) will be the expression (4.6). Note that

$$V_{Put}(S(0), K, r, \sigma) = -V_{Call}(S(0), K, r, -\sigma), \tag{4.8}$$

and

$$V_{Call}(S(0),K,r,\sigma) = -V_{Put}(S(0),K,r,-\sigma). \tag{4.9}$$

In other words, the put option formula (4.7) is the negative of formula (3.7) with its components Φ and Φ_2 replaced by Ψ and Ψ_2, respectively.

5. Continuous Constant-Yield Dividend

Sections 3 and 4 have derived the pricing formulas for the fixed-strike lookback options whose underlying asset pays no dividends. This section will derive explicit pricing formulas for these fixed-strike lookback options when their underlying asset pays dividends continuously at a rate proportional to its price. As was done in Section 2, let $S(t)$ denote the time-t price of an underlying asset. Assume that δ is the constant nonnegative dividend yield rate for the asset such that the asset pay dividends $\delta S(t)dt$ between time t and time $t + dt$. If all dividends of the asset is reinvested in the asset, each share of the asset at time 0 grows to $e^{\delta t}$ shares at time t. In other words, if an investor buys one share of the asset at $S(0)$ and reinvests all dividends in the asset, his fund value invested will be

$$e^{\delta t}S(t) = e^{\delta t}S(t)exp(X(t)) \tag{5.1}$$

at time t. Here, $\{X(t)\}$ is a Brownian motion with drift μ, diffusion coefficient σ and $X(0) = 0$. Thus, the risk-neutral measure is the Esscher measure of parameter $h = h^{**}$ with respect to which the process

$$\{e^{-(r-\delta)t}S(t)\} \tag{5.2}$$

is a martingale. Therefore, h^{**} is the solution of

$$\mu + \sigma^2 h^{**} = r - \delta - \sigma^2/2. \tag{5.3}$$

Note that the process $\{X(t)\}$ is the Brownian motion with drift $\mu + \sigma^2 h^{**}$ and diffusion coefficient σ^2 under the risk-neutral measure. For further discussion, see Section 9 of Gerber and Shiu (1996).

Let us derive a pricing formula for the fixed-strike lookback call option. Applying the fundamental theorem of asset pricing, we obtain the time-0 value of this option

$$e^{-rT}E[(S(0)e^{M(t,T)} - K)I(M(t,T) \geq k);h^{**}], \tag{5.4}$$

whose Esscher parameter h^{**} can be obtained in equation (5.3). The discounted expectation (5.4) can be divided into two terms,

$$e^{-rT}[S(0)E[e^{M(t,T)}I(M(t,T) \geq k);h^{**}] - KPr(M(t,T) \geq k,h^{**})], \tag{5.5}$$

whose expectation and probability are the same as those of (3.6) except that their drift term $r - \sigma^2/2$ is replaced with $r - \delta - \sigma^2/2$. Applying formulas (3.5) and (3.4) to (5.5), we have the time-0 value

$$e^{-rT}[S(0)I_1(r - \delta - \sigma^2/2, 1, k) - KP_2(r - \delta - \sigma^2/2, k)]$$

$$= e^{-\delta T} V_{Call}(S(0), K, r-\delta, \sigma). \tag{5.6}$$

Now, to derive a pricing formula for the fixed-strike lookback put option, we apply the fundamental theorem of asset pricing and obtain the time-0 value of this option,

$$e^{-rT} E[(K - S(0)e^{m(t,T)}) I(m(t,T) < k); h^{**}], \tag{5.7}$$

which can be divided into two terms,

$$-e^{-rT}\{S(0)E[e^{m(t,T)} I(m(t,T) < k); h^{**}] - KPr(m(t,T) < k; h^{**})\}. \tag{5.8}$$

It follows from formulas (4.4) and (4.5) that (5.8) becomes

$$\begin{aligned} &- e^{-rT}[S(0) I_1(-r+\delta+\sigma^2/2, -1, -k) - KP_2(-r+\delta+\sigma^2/2, -k)] \\ &= e^{-\delta T} V_{Put}(S(0), K, r-\delta, \sigma). \end{aligned} \tag{5.9}$$

To conclude this paper, we would point out that there are many books on the mathematics of finance, including Baxter and Rennie (1998), Cochrane (2001), Haug (1998), Jackel (2002), Lamberton and Lapeyre (1996), Lyuu (2002) and Zhang (1998).

Appendix

Proof of independent increments

Assuming that $0 < t_1 < \cdots < t_n \leq T$, $t_0 = 0$ and that $A_1, A_2, \cdots, A_n$ are Borel sets in R,

$$\begin{aligned} &Pr(X(t_1) \in A_1, X(t_2) - X(t_1) \in A_2, \cdots, X(t_n) - X(t_{n-1}) \in A_n; h) \\ &= E[I(X(t_1) \in A_1, X(t_2) - X(t_1) \in A_2, \cdots, X(t_n) - X(t_{n-1}) \in A_n) \frac{e^{hX(t_n)}}{E[e^{hX(t_n)}]}] \\ &= E[\prod_{i=1}^{n} I(X(t_i) - X(t_{i-1}) \in A_i) \frac{e^{h\{X(t_i) - X(t_{i-1})\}}}{E[e^{h(\{X(t_i) - X(t_{i-1})\}}]}] \\ &= \prod_{i=1}^{n} E[I(X(t_i) - X(t_{i-1}) \in A_i) \frac{e^{h\{X(t_i) - X(t_{i-1})\}}}{E[e^{h\{X(t_i) - X(t_{i-1})\}}]}] \\ &= \prod_{i=1}^{n} E[I(X(t_i) - X(t_{i-1}) \in A_i; h] \\ &= \prod_{i=1}^{n} Pr(X(t_i) - X(t_{i-1}) \in A_i; h). \end{aligned} \tag{A.1}$$

Proof of stationary increments

For $t > 0, s > 0$ and $t + s \leq T$,

$$\begin{aligned} E[e^{z\{X(t+s) - X(s)\}}; h] &= E[e^{z\{X(t+s) - X(s)\}} \frac{e^{hX(t+s)}}{E[e^{hX(t+s)}]}] \\ &= E[e^{z\{X(t+s) - X(s)\}} \frac{e^{h\{X(t+s) - X(s)\}}}{E[e^{h\{X(t+s) - X(s)\}}]}] \end{aligned}$$

$$= E[e^{zX(t)}\frac{e^{hX(t)}}{E[e^{hX(t)}]}]$$
$$= E[e^{zX(t)};h]. \tag{A.2}$$

Proof of (2.12) and (2.13)

Let Z_1 and Z_2 follow the standard normal distribution independently. Then the random vectors $(-\rho Z_1+\sqrt{1-\rho^2}Z_2, Z_1)$ and $(\rho Z_1-\sqrt{1-\rho^2}Z_2, Z_2)$ will have standard bivariate normal distributions with correlation coefficients $-\rho$ and $-\sqrt{1-\rho^2}$, respectively. Thus the left-hand side of (2.12) is

$$Pr(-\rho Z_1+\sqrt{1-\rho^2}Z_2 \le a, Z_1 \le b)+Pr(\rho Z_1-\sqrt{1-\rho^2}Z_2 \le -a, Z_2 \le c)$$
$$= Pr(-\rho Z_1+\sqrt{1-\rho^2}Z_2 \le a, Z_1 \le b)+Pr(-\rho Z_1+\sqrt{1-\rho^2}Z_2 \ge a, Z_2 \le c) \tag{A.3}$$

The two events on the right-hand side of (A.3) are disjoint. Hence the two assumptions of (2.12) imply that the union of the two events becomes $\{Z_1 \le b, Z_2 \le c\}$. Thus, applying the independence of Z_1 and Z_2, the right-hand side of (A.3) is

$$Pr(Z_1 \le b, Z_2 \le c) = Pr(Z_1 \le b)Pr(Z_2 \le c) = \Phi(b)\Phi(c).$$

Now, let us prove (2.13). Applying $\Phi(-b) = 1-\Phi(b)$, (2.12) and (2.11),

$$\Phi_2(a,b;-\rho)+\Phi(-b)\Phi(c) = \Phi_2(a,b;-\rho)-\Phi(b)\Phi(c)+\Phi(c)$$
$$= -\Phi_2(-a,c;\sqrt{1-\rho^2})+\Phi(c)$$
$$= \Phi_2(a,c;\sqrt{1-\rho^2}). \tag{A.4}$$

Proof of (3.5)

It follows from (D21) and (D29) of Huang and Shiu (2001) that for a nonzero real number c, $c+\xi \neq 0$ and $k \ge 0$,

$$E[e^{cM(0,t)}I(M(0,t) > k)]$$
$$= \frac{2c+\xi}{c+\xi}e^{c\mu t+\frac{1}{2}c^2\sigma^2 t}\Phi(\frac{-k+(\mu+c\sigma^2)t}{\sigma\sqrt{t}})+\frac{\xi}{c+\xi}e^{(c+\xi)k}\Phi(\frac{-k-\mu t}{\sigma\sqrt{t}}), \tag{A.5}$$

where ξ denotes $2\mu/\sigma^2$. For the expectation in (3.3), let us derive a generalization of the expectation formula (A.5) as follows:

$$E[e^{cM(t,T)}I(M(t,T) > k)]$$
$$= E[e^{cX(t)}E[e^{c[M(t,T)-X(t)]}I(M(t,T)-X(t) > k-X(t)) \mid X(t)]], \tag{A.6}$$

which can be divided into two expectations,

$$E[e^{cX(t)}I(k \le X(t))E[e^{c[M(t,T)-X(t)]}I(M(t,T)-X(t) > 0) \mid X(t)]]$$
$$+ E[e^{cX(t)}I(k > X(t))E[e^{c[M(t,T)-X(t)]}I(M(t,T)-X(t) > k-X(t)) \mid X(t)]]. \tag{A.7}$$

Because the random variables $M(t,T)-X(t)$ and $X(t)$ are independent and the random variable $M(0,T-t)$ has the same distribution as $M(t,T)-X(t)$, the first conditional expectation in (A.7) can be calculated as

$$E[e^{c[M(t,T)-X(t)]}I(M(t,T)-X(t) > 0) \mid X(t)]$$

$$= E[e^{c[M(t,T)-X(t)]}I(M(t,T)-X(t)>0)]$$
$$= E[e^{cM(0,T-t)}I(M(0,T-t)>0)], \tag{A.8}$$

which becomes

$$\frac{2c+\xi}{c+\xi}e^{c\mu(T-t)+\frac{1}{2}c^2\sigma^2(T-t)}\Phi\left(\frac{(\mu+c\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) + \frac{\xi}{\xi+c}\Phi\left(\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right), \tag{A.9}$$

according to formula (A.5) with $t = T\ \ t$ and $k = 0$. Thus the first term in (A.7) is

$$E[e^{cX(t)}I(k \le X(t))(A.9)]$$
$$= E[e^{cX(t)}]E[I(k \le X(t));c](A.9)$$
$$= \frac{2c+\xi}{c+\xi}e^{c\mu T+\frac{1}{2}c^2\sigma^2}\Phi\left(\frac{-k+(\mu+c\sigma^2)t}{\sigma\sqrt{t}}\right)\Phi\left(\frac{(\mu+c\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$+\frac{\xi}{c+\xi}e^{c\mu t+\frac{1}{2}c^2\sigma^2 t}\Phi\left(\frac{-k+(\mu+c\sigma^2)t}{\sigma\sqrt{t}}\right)\Phi\left(-\frac{\mu(T-t)}{\sigma\sqrt{T-t}}\right). \tag{A.10}$$

Consider the second term in (A.7). It follows from formula (A.5) with $k = k - X(t)$ and $t = T-t$ that the second conditional expectation in (A.7) is

$$\frac{2c+\xi}{c+\xi}e^{c\mu(T-t)+\frac{1}{2}c^2\sigma^2(T-t)}\Phi(C_1)+\frac{\xi}{c+\xi}e^{(c+\xi)(k-X(t))}\Phi(C_2), \tag{A.11}$$

where C_1 denotes $\frac{X(t)-k+(\mu+c\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, and C_2 is $\frac{X(t)-k-\mu(T-t)}{\sigma\sqrt{T-t}}$. Replacing the second conditional expectation in (A.7) with (A.11), we have the second term in (A.7),

$$\frac{2c+\xi}{c+\xi}e^{c\mu(T-t)+\frac{1}{2}c^2\sigma^2(T-t)}E[e^{cX(t)}I(k>X(t))\Phi(C_1)]$$
$$+\frac{\xi}{c+\xi}e^{(c+\xi)k}E[e^{-\xi X(t)}I(k>X(t))\Phi(C_2)]. \tag{A.12}$$

By applying formula (2.17) to the two expectations in (A.12), (A.12) becomes

$$\frac{2c+\xi}{c+\xi}e^{c\mu T+\frac{1}{2}c^2\sigma^2 T}\Phi_2\left(\frac{k-(\mu+c\sigma^2)t}{\sigma\sqrt{t}},\frac{-k+(\mu+c\sigma^2)T}{\sigma\sqrt{T}};-\sqrt{\frac{t}{T}}\right)$$
$$+\frac{\xi}{c+\xi}e^{(c+\xi)k}\Phi_2\left(\frac{k+\mu t}{\sigma\sqrt{t}},-\frac{k+\mu T}{\sigma\sqrt{T}};-\sqrt{\frac{t}{T}}\right). \tag{A.13}$$

Therefore, adding (A.13) to (A.10) and applying formula (2.13) to the sum of the first terms from (A.13) and (A.10), we obtain (3.5).

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