J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 11, Number 1 (February 2004), Pages 97–102

RECURRENCE RELATIONS FOR QUOTIENT MOMENTS OF THE PARETO DISTRIBUTION BY RECORD VALUES

MIN YOUNG LEE AND SE KYUNG CHANG

ABSTRACT. In this paper we establish some recurrence relations satisfied by quotient moments of upper record values from the Pareto distribution. Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common continuous distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of $\{X_n, n \geq 1\}$, if $Y_j > Y_{j-1}, j > 1$. The indices at which the upper record values occur are given by the record times $\{u(n)\}, n \geq 1$, where $u(n) = \min\{j | j > u(n-1), X_j > X_{u(n-1)}, n \geq 2\}$ and $u(1) = 1$. Suppose

$$X \in PAR\left(\frac{1}{\beta}, \frac{1}{\beta}\right)$$

then

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) = \frac{1}{s}E\left(\frac{X^r_{u(m)}}{X^s_{u(n-1)}}\right) - \frac{(1+\beta s)}{s}E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}}\right)$$

and

$$E\left(\frac{X^{r+1}_{u(m)}}{X^s_{u(n)}}\right) = \frac{1}{(r+1)\beta}\left[E\left(\frac{X^{r+1}_{u(m)}}{X^s_{u(n-1)}}\right) - E\left(\frac{X^{r+1}_{u(m-1)}}{X^s_{u(n-1)}}\right) - (r+1)E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}}\right)\right]$$

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common cdf $F(x)$ and pdf $f(x)$. Suppose $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper record value of this sequence if $Y_j > Y_{j-1}, j > 1$. We define the record times $u(n)$ by $u(1) = 1$ and

$$u(n) = \min\{j \mid j > u(n-1),\ X_j > X_{u(n-1)},\ n \geq 2\}.$$

The record times of the sequence $\{X_n, n \geq 1\}$ are random variables and are the same as those for the sequence $\{F(X_n), n \geq 1\}$. We know that the distribution of

Received by the editors November 11, 2003 and, in revised form, February 10, 2004.
2000 *Mathematics Subject Classification.* 60E10, 62E10, 62H10.
Key words and phrases. Pareto distribution, record value, recurrence relation, expectation.

$u(n)$ does not depend on $F(x)$. Hence, the distribution of $u(n)$ can be determined by considering the uniform distribution $F(x) = x$. We will call the random variable $X \in PAR\left(\frac{1}{\beta}, \frac{1}{\beta}\right)$ if the corresponding probability cumulative function $F(x)$ of x is of the form

$$F(x) = \begin{cases} 1 - (1+\beta x)^{-\beta^{-1}}, & x > 0, \ \beta > 0, \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

Some characterizations of the Pareto distribution are known. But mainly some recurrence relations satisfied by the single and product moments of record values. Such results have been established by Balakrishnan, Ahsanullah & Chan [5, 6], and Balakrishnan & Ahsanullah [3, 4] for the extreme value, exponential, Pareto and generalized extreme value distributions.

In this paper, we will give some recurrence relations satisfied by the quotient moments of upper record values from the Pareto distribution.

2. Main Results

Theorem 1. *For* $1 \leq m \leq n-2$, $r = 0, 1, 2, \ldots$ *and* $s = 1, 2, \ldots$,

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}}\right) = \frac{1}{s}E\left(\frac{X^r_{u(m)}}{X^s_{u(n-1)}}\right) - \frac{(1+\beta s)}{s}E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}}\right).$$

Proof. First of all, we have that for the Pareto distribution in (1), $(1+\beta x)f(x) = 1 - F(x)$. The joint pdf of $X_{u(m)}$ and $X_{u(n)}$ is

$$f_{m,n}(x, y) = \frac{1}{\Gamma(m)\Gamma(n-m)} R^{m-1}(x) r(x) [R(y) - R(x)]^{n-m-1} f(y).$$

Let us consider for $1 \leq m \leq n-2$, $r = 0, 1, 2, \ldots$ and $s = 1, 2, \ldots$,

$$\begin{aligned} E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}} + \beta\frac{X^r_{u(m)}}{X^s_{u(n)}}\right) &= \iint_{0<x<y<\infty} \left(\frac{x^r}{y^{s+1}} + \frac{\beta x^r}{y^s}\right) f_{m,n}(x,y)\,dy\,dx \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)} \iint_{0<x<y<\infty} \frac{x^r}{y^{s+1}}(1+\beta y)R^{m-1}(x)r(x) \\ &\qquad \times [R(y) - R(x)]^{n-m-1} f(y)\,dy\,dx \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty x^r R^{m-1}(x) r(x) \end{aligned}$$

$$\times \left(\int_x^\infty \frac{1}{y^{s+1}} (1+\beta y)[R(y)-R(x)]^{n-m-1} f(y)dy \right) dx$$

$$= \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty x^r R^{m-1}(x) r(x)$$

$$\times \left(\int_x^\infty \frac{1}{y^{s+1}} [R(y)-R(x)]^{n-m-1} [1-F(y)]dy \right) dx.$$

Using integrating by parts treating $\frac{1}{y^{s+1}}$ for integration and $[R(y)-R(x)]^{n-m-1}$ $[1-F(y)]$ for differentiation on the second integration, we get

$$\int_x^\infty \frac{1}{y^{s+1}} [R(y)-R(x)]^{n-m-1} [1-F(y)]dy$$

$$= \left[-\frac{1}{s} \frac{1}{y^s} [R(y)-R(x)]^{n-m-1} [1-F(y)] \right]_x^\infty$$

$$+ \frac{(n-m-1)}{s} \int_x^\infty \frac{1}{y^s} [R(y)-R(x)]^{n-m-2} f(y)dy$$

$$- \frac{1}{s} \int_x^\infty \frac{1}{y^s} [R(y)-R(x)]^{n-m-1} f(y)dy.$$

Then we have

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}} + \beta \frac{X^r_{u(m)}}{X^s_{u(n)}} \right)$$

$$= \frac{1}{s\ \Gamma(m)\Gamma(n-m-1)} \iint_{0<x<y<\infty} \frac{x^r}{y^s} R^{m-1}(x) r(x) [R(y)-R(x)]^{n-m-2} f(y)dydx$$

$$- \frac{1}{s\ \Gamma(m)\Gamma(n-m)} \iint_{0<x<y<\infty} \frac{x^r}{y^s} R^{m-1}(x) r(x) [R(y)-R(x)]^{n-m-1} f(y)dydx$$

$$= \frac{1}{s} \iint_{0<x<y<\infty} \frac{x^r}{y^s} f_{m,n-1}(x,y)dydx - \frac{1}{s} \iint_{0<x<y<\infty} \frac{x^r}{y^s} f_{m,n}(x,y)dydx$$

$$= \frac{1}{s} E\left(\frac{X^r_{u(m)}}{X^s_{u(n-1)}} \right) - \frac{1}{s} E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}} \right).$$

Hence

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(n)}} \right) = \frac{1}{s} E\left(\frac{X^r_{u(m)}}{X^s_{u(n-1)}} \right) - \frac{(1+\beta s)}{s} E\left(\frac{X^r_{u(m)}}{X^s_{u(n)}} \right).$$

This completes the proof. □

Corollary 2. *For* $m \geq 1$, $r = 0,1,2,\ldots$ *and* $s = 1,2,\ldots,$

$$E\left(\frac{X^r_{u(m)}}{X^{s+1}_{u(m+1)}} \right) = \frac{1}{s}\ E\left(X^{r-s}_{u(m)} \right) - \frac{(1+\beta s)}{s}\ E\left(\frac{X^r_{u(m)}}{X^s_{u(m+1)}} \right).$$

Proof. Upon substituting $n = m + 1$ in Theorem 1 and simplifying, then we have

$$E\left(\frac{X_{u(m)}^{r}}{X_{u(m+1)}^{s+1}}\right) = \frac{1}{s}\, E\left(X_{u(m)}^{r-s}\right) - \frac{(1+\beta s)}{s}\, E\left(\frac{X_{u(m)}^{r}}{X_{u(m+1)}^{s}}\right).$$

□

Theorem 3. *For* $1 \le m \le n-2$ *and* $r, s = 0, 1, 2, \dots,$

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}\right) = \frac{1}{(r+1)\beta}\left[E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^{s}}\right) - E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n-1)}^{s}}\right) - (r+1)E\left(\frac{X_{u(m)}^{r}}{X_{u(n)}^{s}}\right)\right].$$

Proof. In the same manner as Theorem 1, let us consider for $1 \le m \le n-2$ and $r, s = 0, 1, 2, \dots,$

$$\begin{aligned} E\left(\beta\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}} + \frac{X_{u(m)}^{r}}{X_{u(n)}^{s}}\right) &= \iint_{0<x<y<\infty}\left(\frac{\beta x^{r+1}}{y^s} + \frac{x^r}{y^s}\right) f_{m,n}(x,y)\,dx\,dy \\ &= \frac{1}{\Gamma(m)\Gamma(n-m)}\iint_{0<x<y<\infty}\frac{x^r}{y^s}(1+\beta x)R^{m-1}(x)r(x) \\ &\qquad \times [R(y)-R(x)]^{n-m-1} f(y)\,dx\,dy. \end{aligned}$$

Since the Pareto distribution, $r(x) = \dfrac{f(x)}{1-F(x)} = \dfrac{1}{1+\beta x}$.

Upon substituting the above expressions and simplifying the resulting equations, we obtain that

$$\begin{aligned} &E\left(\beta\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}} + \frac{X_{u(m)}^{r}}{X_{u(n)}^{s}}\right) \\ &\quad = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty \frac{1}{y^s} f(y)\left(\int_0^y x^r R^{m-1}(x)[R(y)-R(x)]^{n-m-1}dx\right)dy. \end{aligned}$$

Using integrating by parts treating x^r for integration and $R^{m-1}(x)$ $[R(y) - R(x)]^{n-m-1}$ for differentiation on the second integration, we get

$$\begin{aligned} &\int_0^y x^r R^{m-1}(x)[R(y)-R(x)]^{n-m-1}dx \\ &\quad = \left[\frac{1}{(r+1)}x^{r+1}\, R^{m-1}(x)[R(y)-R(x)]^{n-m-1}\right]_0^y \\ &\qquad + \frac{(n-m-1)}{(r+1)}\int_0^y x^{r+1}R^{m-1}(x)r(x)[R(y)-R(x)]^{n-m-2}dx \\ &\qquad - \frac{(m-1)}{(r+1)}\int_0^y x^{r+1}R^{m-2}(x)r(x)[R(y)-R(x)]^{n-m-1}dx. \end{aligned}$$

Then we have

$$E\left(\beta\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}+\frac{X_{u(m)}^{r}}{X_{u(n)}^{s}}\right)$$

$$=\frac{1}{(r+1)\,\Gamma(m)\Gamma(n-m-1)}\iint_{0<x<y<\infty}\frac{x^{r+1}}{y^{s}}R^{m-1}(x)r(x)\times[R(y)-R(x)]^{n-m-2}f(y)dxdy$$

$$-\frac{1}{(r+1)\,\Gamma(m-1)\Gamma(n-m)}\iint_{0<x<y<\infty}\frac{x^{r+1}}{y^{s}}R^{m-2}(x)r(x)\times[R(y)-R(x)]^{n-m-1}f(y)dxdy$$

$$=\frac{1}{(r+1)}\iint_{0<x<y<\infty}\frac{x^{r+1}}{y^{s}}f_{m,n-1}(x,y)dxdy-\frac{1}{(r+1)}\iint_{0<x<y<\infty}\frac{x^{r+1}}{y^{s}}f_{m-1,n-1}(x,y)dxdy$$

$$=\frac{1}{(r+1)}\,E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^{s}}\right)-\frac{1}{(r+1)}\,E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n-1)}^{s}}\right).$$

Hence

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n)}^{s}}\right)=\frac{1}{(r+1)\beta}\left[E\left(\frac{X_{u(m)}^{r+1}}{X_{u(n-1)}^{s}}\right)-E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(n-1)}^{s}}\right)-(r+1)E\left(\frac{X_{u(m)}^{r}}{X_{u(n)}^{s}}\right)\right].$$

This completes the proof. □

Corollary 4. *For $m\geq 1$ and $r,s=0,1,2,\ldots,$*

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(m+1)}^{s}}\right)=\frac{1}{(r+1)\beta}\left[E(X_{u(m)}^{r-s+1})-E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(m)}^{s}}\right)-(r+1)E\left(\frac{X_{u(m)}^{r}}{X_{u(m+1)}^{s}}\right)\right].$$

Proof. Upon substituting $n=m+1$ in Theorem 3 and simplifying, then we have

$$E\left(\frac{X_{u(m)}^{r+1}}{X_{u(m+1)}^{s}}\right)=\frac{1}{(r+1)\beta}\left[E(X_{u(m)}^{r-s+1})-E\left(\frac{X_{u(m-1)}^{r+1}}{X_{u(m)}^{s}}\right)-(r+1)E\left(\frac{X_{u(m)}^{r}}{X_{u(m+1)}^{s}}\right)\right].$$

□

References

1. M. Ahsanuallah: *Record Statistics.* Nova Science Publishers, Inc, Commack NY, 1995.
2. B. C. Arnold, N. Balakrishnan & H. N. Nagaraja: *A first course in order statistics.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical

Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1992. MR **94a**:62076

3. N. Balakrishnan & M. Ahsanullah: Recurrence relations for single and product moments of record values from generalized Pareto distribution. *Comm. Statist. Theory Methods* **23** (1994), no. 10, 2841–2852. CMP 1294008
4. ______: Relations for single and product moments of record values from exponential distribution. *J. Appl. Statist. Sci.* **2** (1995), no. 1, 73–87. MR **96f**:62030
5. N. Balakrishnan, M. Ahsanullah & P. S. Chan: Relations for single and product moments of record values from Gumbel distribution. *Statist. Probab. Lett.* **15** (1992), no. 3, 223–227. MR **93j**:62143
6. ______: Recurrence relations for moments of record values from generalized extreme value distribution. *Comm. Statist. Theory Methods* **22** (1993), no. 5, 1471–1482. CMP 1225252
7. J. Galambos & S. Kotz: *Characterizations of probability distributions.* A unified approach with an emphasis on exponential and related models. Lecture Notes in Mathematics, 675. Springer, Berlin, 1978. MR **80a**:62022
8. M. Y. Lee: Characterizations of the Pareto distribution by conditional expectations of record values. *Commun. Korean Math. Soc.* **18** (2003), no. 1, 127–131.

(M. Y. LEE) DEPARTMENT OF APPLIED MATHEMATICS, DANKOOK UNIVERSITY, 29 ANSEO-DONG, CHEONAN, CHUNGNAM 330-714, KOREA
Email address: `leemy@dku.edu`

(S. K. CHANG) DEPARTMENT OF APPLIED MATHEMATICS, DANKOOK UNIVERSITY, 29 ANSEO-DONG, CHEONAN, CHUNGNAM 330-714, KOREA
Email address: `skchang@dankook.ac.kr`