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MATRIX PRESENTATIONS OF THE TEICHMÜLLER SPACE OF A PUNCTURED TORUS

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ABSTRACT. A punctured torus $\Sigma(1,1)$ is a building block of oriented surfaces. The goal of this paper is to formulate the matrix presentations of elements of the Teichmüller space of a punctured torus. Let C be a matrix presentation of the boundary component of $\Sigma(1,1)$. In the level of the matrix group $\mathbf{SL}(2,\mathbb{R})$, we shall show that the trace of C is always negative.

INTRODUCTION

The $(\mathbf{PSL}(2,\mathbb{R}),\mathbb{H}^2)$-structures on a connected smooth surface M are called the *hyperbolic* structures on M. If $\chi(M)<0$, then the equivalence classes of hyperbolic structures on M form a deformation space $\mathfrak{T}(M)$ called the *Teichmüller space.*

Let $\pi=\pi_1(M)$ be the fundamental group of M. Given a hyperbolic structure on M, the action of π by deck transformation on the universal covering space $\tilde{M}$ of M determines a homomorphism $\pi\to\mathbf{PSL}(2,\mathbb{R})$ called the *holonomy homomorphism* and it is well-defined up to conjugation in $\mathbf{PSL}(2,\mathbb{R})$. Thus the Teichmüller space $\mathfrak{T}(M)$ has a natural topology which identified with an open subset of the orbit space $\mathrm{Hom}(\pi,\mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$. Since holonomy homomorphisms $\pi\to\mathbf{PSL}(2,\mathbb{R})$ are isomorphic to their images, the generators of π can be presented by the conjugacy classes of matrices in $\mathbf{PSL}(2,\mathbb{R})$.

Let $M=\Sigma(g,n)$ be a compact connected oriented surface with g-genus and n-boundary components. Then M can be decomposed as a disjoint union of g punctured tori $\Sigma(1,1)$ and $2g-2+n$ pairs of pants $\Sigma(0,3)$. Thus a punctured torus and a pair of pants $\Sigma(0,3)$ are building blocks of an oriented surface M. The matrix

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presentations of a pair of pants $\Sigma(0,3)$ are classified in the preceding paper Kim [7]. The purpose of this paper is to formulate the matrix presentations of elements of the Teichmüller space of a punctured torus $\Sigma(1,1)$.

In Section 1, we recall some preliminary definitions and describe the relation between the deformation space $\mathfrak{D}(M)$ of (G,X)-structures on a smooth manifold M and the orbit space $\mathrm{Hom}(\pi,G)/G$. In Section 2, we define the hyperbolic elements of $\mathbf{SL}(2,\mathbb{R})$ and $\mathbf{PSL}(2,\mathbb{R})$ and classify the locations of fixed points and principal lines of hyperbolic elements. In Section 3, we calculate the matrix presentations of elements of the Teichmüller space $\mathfrak{T}(\Sigma(1,1))$. Let C be a matrix presentation of the boundary component of $\Sigma(1,1)$. In the level of the matrix group $\mathbf{SL}(2,\mathbb{R})$, we shall show that the trace of C is always negative.

1. (G,X)-STRUCTURES ON A SMOOTH MANIFOLD M

1.1. An action of a connected Lie group G on a smooth manifold X is called *strongly effective* if $g_1, g_2 \in G$ agree on a nonempty open set of X, then $g_1 = g_2$. Let Ω be an open subset of X. A map $\phi : \Omega \to X$ is called *locally-*(G,X) if for each component $W \subset \Omega$, there exists a (G,X)-transformation $g \in G$ such that $\phi|_W = g|_W$. Since G acts strongly effectively on X, above element g is unique for each component. Clearly a locally-(G,X) map is a local diffeomorphism.

Let M be a connected smooth n-manifold. A (G,X)*-structure* on M is a maximal collection of coordinate charts $\{(U_\alpha, \psi_\alpha)\}$ such that

(1) $\{U_\alpha\}$ is an open covering of M.
(2) For each α, $\psi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image.
(3) The change of coordinates is locally-(G,X); If (U_α, ψ_α) and (U_β, ψ_β) are two coordinate charts with $U_\alpha \cap U_\beta \neq \emptyset$, then the transition function $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ is locally-(G,X).

Now we give an example of a (G,X)-structure.

Example 1.1. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half complex plane. Then $\mathbf{SL}(2,\mathbb{R})$ acts on $\mathbb{H}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}\,. \tag{1.1}$$

Since we have $A \cdot z = (-A) \cdot z$ for any $A \in \mathbf{SL}(2,\mathbb{R})$ and $z \in \mathbb{H}^2$, the Lie group $\mathbf{PSL}(2,\mathbb{R}) = \mathbf{SL}(2,\mathbb{R})/\pm I$ acts strongly effectively on $\mathbb{H}^2$.

Definition 1.2. A $(\mathbf{PSL}(2,\mathbb{R}),\mathbb{H}^2)$-structure on a smooth surface M is called a *hyperbolic structure* on M.

1.2. A manifold M with a (G,X)-structure is called a (G,X)*-manifold.* Let N be a (G,X)-manifold. If $f : M \to N$ is a local diffeomorphism of smooth manifolds, then we can give the induced (G,X)-structure on M via f. In particular every covering space of a (G,X)-manifold has the canonically induced (G,X)-structure.

Let M and N be (G,X)-manifolds and $f : M \to N$ a smooth map. Then f is called a (G,X)*-map* if for each coordinate chart (U,ψ_U) on M and (V,ψ_V) on N, the composition $\psi_V \circ f \circ \psi_U^{-1} : \psi_U(f^{-1}(V)\cap U) \to \psi_V(f(U)\cap V)$ is locally-(G,X).

The following *Development Theorem* is the fundamental fact about (G,X)-structures. For more details (see Thurston [8]).

Theorem 1.3. *Let $p : \tilde{M} \to M$ denote a universal covering map of a (G,X)-manifold M, and π the corresponding group of covering transformations.*

(1) *There exist a (G,X)-map* $\mathbf{dev} : \tilde{M} \to X$ *and homomorphism $h : \pi \to G$ such that for each $\gamma \in \pi$ the following diagram commutes:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ \gamma\downarrow & & \downarrow h(\gamma) \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X \end{array}$$

(2) *Suppose $(\mathbf{dev}',h')$ is another pair satisfying above conditions. Then there exists a (G,X)-transformation $g \in G$ such that $\mathbf{dev}' = g\circ\mathbf{dev}$ and $h' = \iota_g \circ h$ where $\iota_g : G \to G$ denotes the inner automorphism defined by g; that is,*

$$h'(\gamma) = (\iota_g \circ h)(\gamma) = g\circ h(\gamma)\circ g^{-1} :$$

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X & \xrightarrow{g} & X \\ \gamma\downarrow & & \downarrow h(\gamma) & & \downarrow h'(\gamma) \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X & \xrightarrow[g]{} & X \end{array}$$

The (G,X)-map $\mathbf{dev} : \tilde{M} \to X$ called the *developing map* and the homomorphism $h : \pi \to G$ is called the *holonomy homomorphism.* The image $\Gamma = h(\pi) \subset G$ is called the *holonomy group* and the image $\Omega = \mathbf{dev}(\tilde{M}) \subset X$ is called the *developing image.* By Theorem 1.3, the developing pair $(\mathbf{dev},h)$ is unique up to the G-action by composition and conjugation respectively.

Consider a pair (f, N) where N is a (G, X)-manifold and $f : M \to N$ is a diffeomorphism. Then M admits the induced (G, X)-structure via f. The set of all such pairs (f, N) is denoted by $\mathcal{A}(M)$. Then $\mathcal{A}(M)$ is the space of all (G, X)-structures on M. We say two pairs (f, N) and (f', N') in $\mathcal{A}(M)$ are *equivalent* if there exists a (G, X)-diffeomorphism $g : N \to N'$ such that $g \circ f$ is isotopic to f'. The set of equivalence classes $\mathcal{A}(M)/\sim$ will be denoted by $\mathfrak{D}(M)$ and called the *deformation space* of (G, X)-structures on M.

Definition 1.4. Let M be a connected smooth 2-manifold. The deformation space of the hyperbolic structures on M is called the *Teichmüller space* and denoted by $\mathfrak{T}(M)$.

1.3. The deformation space $\mathfrak{D}(M)$ is closely related to $\mathrm{Hom}(\pi, G)/G$ the orbit space of homomorphisms $\phi : \pi \to G$. Suppose $M = \Sigma(g, n)$ is a compact oriented smooth surface with g-genus, n-boundary components and $\chi(M) = 2 - 2g - n < 0$. Then π admits $2g+n$ generators $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n$ with a single relation

$$R = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} C_1 \cdots C_n = I.$$

From the correspondence of the homomorphism $\phi : \pi \to G$ to the image of generators, $\mathrm{Hom}(\pi, G)$ may be identified with the collection of all $(2g + n)$-tuples $(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) \subset G^{2g+n}$ elements of G satisfying

$$R(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) = I.$$

Since $R : G^{2g+n} \to G$ is a polynomial equation and

$$\mathrm{Hom}(\pi, G) = R^{-1}(I) \subset G^{2g+n}, \tag{1.2}$$

if G is an algebraic Lie group, then $\mathrm{Hom}(\pi, G)$ is an *algebraic variety.*

The group G acts on $\mathrm{Hom}(\pi, G)$ by conjugation as follows; For $g \in G$ and $\phi \in \mathrm{Hom}(\pi, G)$, the action $g \cdot \phi$ is defined by

$$(g \cdot \phi)(\gamma) = g \circ \phi(\gamma) \circ g^{-1}$$

where $\gamma \in \pi$. Taking the holonomy homomorphism of a (G, X)-structure defines a map

$$\mathbf{hol} : \mathfrak{D}(M) \longrightarrow \mathrm{Hom}(\pi, G)/G$$

which is a local diffeomorphism. See Goldman [2] and Johnson & Millson [5] for details. For the hyperbolic structures on M, the Teichmüller space $\mathfrak{T}(M)$ embeds into $\mathrm{Hom}\left(\pi, \mathbf{PSL}(2, \mathbb{R})\right) / \mathbf{PSL}(2, \mathbb{R})$. (*cf.* Goldman [3])

Theorem 1.5. *Let M be a compact oriented surface with $\chi(M) = 2 - 2g - n < 0$. Then* **hol** : $\mathfrak{T}(M) \to \operatorname{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$ *is an embedding onto a Hausdorff real analytic manifold of dimension $6g - 6 + 3n$.*

Therefore the Teichmüller space $\mathfrak{T}(M)$ is homeomorphic to $\mathbb{R}^{6g-6+3n}$ and an element of $\mathfrak{T}(M)$ will be identified with a conjugacy class of $\operatorname{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))$. In the next section, we shall explicitly formulate the algebraic presentation of elements of $\mathfrak{T}(M)$ for a punctured torus $M = \Sigma(1,1)$.

2. Matrix presentations of a punctured torus

2.1. An element A of $\mathbf{SL}(2,\mathbb{R})$ is said to be *hyperbolic* if A has two distinct real eigenvalues. Since the characteristic polynomial of A is $f(\lambda) = \lambda^2 - t\lambda + 1$ where $t = \operatorname{tr}(A)$, A is hyperbolic if and only if $\operatorname{tr}(A)^2 > 4$. Thus a hyperbolic element A in $\mathbf{SL}(2,\mathbb{R})$ can be expressed by the diagonal matrix

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \tag{2.1}$$

via an $\mathbf{SL}(2,\mathbb{R})$-conjugation where $\alpha^2 > 1$.

An element A of $\mathbf{PSL}(2,\mathbb{R})$ is said to be *hyperbolic* if A has two distinct fixed points on $\partial\mathbb{H}^2$. Since the absolute value of trace is still defined, A is hyperbolic if and only if $|\operatorname{tr}(A)| > 2$. The following theorem is from Beardon's book [1]. It was certainly known to Fenchel, Nielsen and probably earlier.

Theorem 2.1. *Suppose that M is a compact connected oriented hyperbolic surface. Then every nontrivial element of the holonomy group $\Gamma \subset$* $\mathbf{PSL}(2,\mathbb{R})$ *is hyperbolic.*

Let $M = \Sigma(g,n)$ be a compact connected oriented surface with g-genus and n-boundary components. If $\chi(M) = 2 - 2g - n < 0$, then there exist $2g - 3 + n$ nontrivial homotopically-distinct disjoint simply-closed curves on M such that they decompose M as the disjoint union of g punctured tori $\Sigma(1,1)$ and $g - 2 + n$ pairs of pants $\Sigma(0,3)$. Thus the punctured torus $\Sigma(1,1)$ and a pair of pants $\Sigma(0,3)$ are building blocks of an oriented surface M. For more detail (see Wolpert [9]).

The matrix presentations of a pair of pants $\Sigma(0,3)$ are classified in the preceding paper Kim [7]. Thus the goal of this section is to find expressions of the elements of the Teichmüller space $\mathfrak{T}(\Sigma(1,1))$ of a punctured torus. Since $\mathfrak{T}(\Sigma(1,1))$ embeds into $\operatorname{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$, we should calculate the matrix presentations of the conjugacy classes of $\operatorname{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))$.

2.2. First we consider the positions of fixed points and principal lines of hyperbolic elements in $\mathbf{SL}(2,\mathbb{R})$. The *principal line* of a hyperbolic element $A \in \mathbf{SL}(2,\mathbb{R})$ is the A-invariant unique geodesic in $\mathbb{H}$ and it is the line joining two fixed points of A. Since the principal line has a distinct direction, we call one of fixed point of A is called *repelling* fixed point z_r and the other is called *attracting* fixed point z_a. For more easy understanding (see Beardon [1]) or Figure 1.

Proposition 2.2. *Suppose*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } B = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

are hyperbolic elements of $\mathbf{SL}(2,\mathbb{R})$. *If z is a fixed point of A, then $-z$ is a fixed point of B.*

Proof. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then we can get $PAP^{-1} = B$. Let z be a fixed point of A and $w = Pz$. Since

$$Bw = (PAP^{-1})(Pz) = P(Az) = Pz = w,$$

$w = Pz$ is a fixed point of B. Therefore if z is a fixed point of A, then the point

$$w = Pz = \frac{1 \cdot z + 0}{0 \cdot z - 1} = -z$$

is a fixed point of B. □

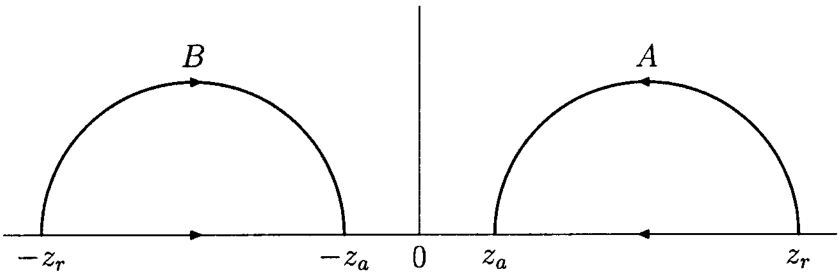

Figure 1. The fixed points of the matrices A and B

Thus the principal lines of A and B are symmetric with respect to the imaginary axis.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R})$$

be a hyperbolic element. We now consider the location of the principal line of A and the relations of entries of A.

Theorem 2.3. *Suppose $A \in \mathbf{SL}(2,\mathbb{R})$ represents a hyperbolic transformation of $\mathbb{H}^2$ and z_r, z_a are the repelling and attracting fixed points of A. Then*

(1) $0 < z_a + z_r < \infty$ *if and only if* $(a-d)\,c > 0$.
(2) $z_a \cdot z_r > 0$ *if and only if* $b\,c < 0$.
(3) $z_a < z_r$ *if and only if* $(a+d)\,c < 0$.

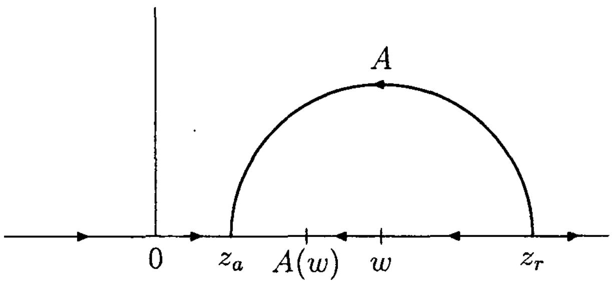

Figure 2. The principal line with $0 < z_a < z_r < \infty$

Proof. Since z_a and z_r are the fixed points of the hyperbolic transformation $A(z) = \frac{az+b}{cz+d}$, they are the roots of the equation

$$cz^2 + (d-a)\,z - b = 0. \tag{2.2}$$

Suppose $0 < z_a + z_r < \infty$ or $z_a \cdot z_r > 0$. Then the fixed points of A are neither infinity nor zero. First we claim that $c \neq 0$. If $c = 0$, then $1 = \det(A) = ad$. Thus $d = a^{-1}$ and $A(z) = a^2 z + ab$. This yields that ∞ is a fixed point of $A(z)$ since $a \neq 0$. It contradicts the assumption. Since $z_a + z_r = \frac{a-d}{c}$ and $z_a \cdot z_r = \frac{-b}{c}$, it proves $0 < z_a + z_r < \infty$ if and only if $(a-d)\,c > 0$ and $z_a \cdot z_r > 0$ if and only if $b\,c < 0$.

Since we have $c \neq 0$, the roots z_a, z_r of the Equation (2.2) can be expressed by

$$z_a,\, z_r = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}. \tag{2.3}$$

Suppose that the attracting fixed point z_a is smaller than the repelling fixed point z_r; *i. e.*, $z_a < z_r$. Let w be the mid point of the fixed points z_a and z_r; *i. e.*, $w = (z_a + z_r)/2 = (a-d)/(2c)$. Then the condition $z_a < z_r$ is equivalent to

$A(w) < w$. From the computation

$$\begin{aligned} A(w) - w &= \frac{a(\frac{a-d}{2c}) + b}{c(\frac{a-d}{2c}) + d} - \left(\frac{a-d}{2c}\right) \\ &= \frac{a(a-d) + 2bc}{(a+d)c} - \left(\frac{a-d}{2c}\right) = \frac{(a+d)^2 - 4}{2(a+d)c}, \end{aligned}$$

and the fact that $(a+d)^2 > 4$, it proves $z_a < z_r$ if and only $(a+d)\,c < 0$. This completes the proof. □

Theorem 2.4. *Let $A \in \mathbf{SL}(2, \mathbb{R})$ represent a hyperbolic transformation of $\mathbb{H}^2$ and z_r, z_a the repelling and attracting fixed points of A. Then $-\infty < z_a < 0 < z_r < \infty$ if and only if $b\,c > 0$, $a\,c < 0$ and $b\,d < 0$.*

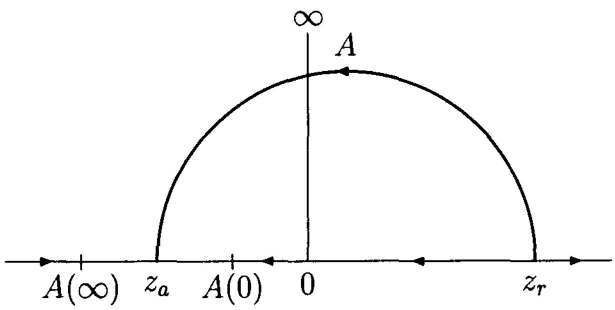

Figure 3. The principal line with $-\infty < z_a < 0 < z_r < \infty$

Proof. From the Theorem 2.3, we can show that $z_a \cdot z_r < 0$ if and only if $b\,c > 0$. Suppose $-\infty < z_a < 0 < z_r < \infty$. The images of the origin and infinity under A should be negative as in the Figure 3. That means $A(0) = b/d < 0$ and $A(\infty) = a/c < 0$. Thus we have $b\,d < 0$ and $a\,c < 0$. Conversely, the relations $b\,c > 0$ and $b\,d < 0$ derive $c\,d < 0$. Thus we get $(a+d)\,c < 0$, equivalently $z_a < z_r$. The fact $b\,c < 0$ implies $z_a \cdot z_r < 0$. Since all entries of A are non-zero, we can conclude $-\infty < z_a < 0 < z_r < \infty$. □

Corollary 2.5. *Let $A \in \mathbf{SL}(2, \mathbb{R})$ represent a hyperbolic transformation of $\mathbb{H}^2$.*

(1) *Suppose $b > 0$. Then $-\infty < z_a < 0 < z_r < \infty$ if and only if $a < 0$, $c > 0$, $d < 0$.*
(2) *Suppose $b < 0$. Then $-\infty < z_a < 0 < z_r < \infty$ if and only if $a > 0$, $c < 0$, $d > 0$.*

Proof. This follows from the result of the Theorem 2.4. □

Theorem 2.6. *Suppose $C \in \mathbf{SL}(2,\mathbb{R})$ is representing a hyperbolic transformation of $\mathbb{H}^2$ with the repelling and attracting fixed points w_r, w_a. Suppose $0 < w_a < w_r < \infty$, then $(a-d)\,c > 0$, $(a+d)\,c < 0$, $b\,c < 0$, $a^2 < d^2$ and $b\,d > 0$.*

Proof. Since $0 < w_a + w_r < \infty$, $w_a \cdot w_r > 0$ and $w_a < w_r$, from Theorem 2.3, we have the relations $(a-d)\,c > 0$, $b\,c < 0$ and $(a+d)\,c < 0$. Thus $(a-d)(a+d)c^2 = (a^2-d^2)c^2 < 0$ implies $a^2 < d^2$. Since $w_a < w_r$, the image of the origin under C should be positive as in the Figure 2. That means $C(0) = b/d > 0$. Thus we have $b\,d > 0$. This also implies $b \neq 0$ and $d \neq 0$. □

Remark 2.7. The image of infinity of C is just less than w_a. Thus it is possible that $C(\infty)$ has positive, zero, or negative signs.

Corollary 2.8. *Suppose $C \in \mathbf{SL}(2,\mathbb{R})$ is representing a hyperbolic transformation of $\mathbb{H}^2$.*

(1) *Suppose that $b > 0$. Then $0 < w_a < w_r < \infty$ if and only if $c < 0$, $d > 0$, $|a| < d$.*
(2) *Suppose that $b < 0$. Then $0 < w_a < w_r < \infty$ if and only if $c > 0$, $d < 0$ $|a| < (-d)$.*

Proof. Suppose $0 < w_a < w_r < \infty$ and $b > 0$. Since we have the relations $b\,c < 0$, $b\,d > 0$ and $a^2 < d^2$, the condition $b > 0$ yields that $c < 0$, $d > 0$, and $|a| < |d| = d$. Conversely, the condition $|a| < d$ derives $(a-d) < 0$, and $(a+d) > 0$. Since $c < 0$ we get $(a-d)\,c > 0$ and $(a+d)\,c < 0$. Since $b\,c < 0$, this induces $0 < w_a < w_r < \infty$. We can similarly prove for the case $b < 0$. □

2.3. Recall that a punctured torus $M = \Sigma(1,1)$ is a torus with a hole. Suppose M is equipped with a hyperbolic structure. Since the holonomy homomorphism is isomorphic to its image, the fundamental group π of M will be identified with

$$\pi = \langle\, A, B, C \in \mathbf{PSL}(2,\mathbb{R}) \mid R = CB^{-1}A^{-1}BA = I \,\rangle.$$

Let A, B, $C \in \mathbf{PSL}(2,\mathbb{R})$ represent elements of the fundamental group of M as in Figure 4. We will find the expression of the generators A, B and C of π in terms of $\mathbf{SL}(2,\mathbb{R})$ instead of $\mathbf{PSL}(2,\mathbb{R})$ because $\mathbf{SL}(2,\mathbb{R})$ is easier to compute than $\mathbf{PSL}(2,\mathbb{R})$. Since the matrices A, B, $C \in \mathbf{SL}(2,\mathbb{R})$ are hyperbolic and represented up to conjugate, without loss of generality, we can assume

$$B = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}$$

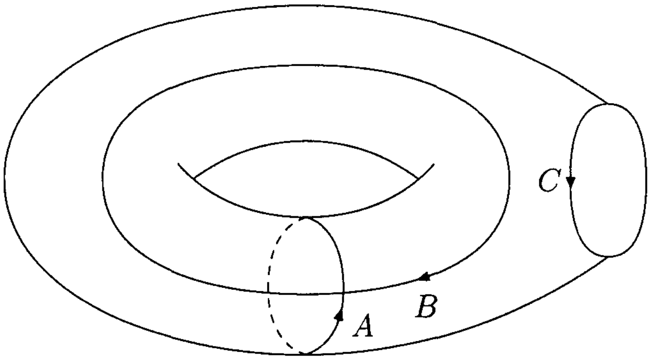

Figure 4. A punctured torus $M = \Sigma(1,1)$

with $\mu^2 > 1$. Since we have

$$B(z) = \frac{\mu^{-1} \cdot z + 0}{0 \cdot z + \mu} = \frac{z}{\mu^2},$$

∞ is the repelling fixed point and 0 is the attracting fixed point of B. By the discreteness of holonomy group, $A(0) \neq 0$. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $b \neq 0$. If $b = 0$, then

$$A(0) = \frac{a \cdot 0 + b}{c \cdot 0 + d} = 0,$$

contradicting for $A(0) \neq 0$. Suppose $\mathrm{tr}(A) = \lambda + \lambda^{-1}$ where $\lambda^2 > 1$. Since $a + d = \mathrm{tr}(A) = \lambda + \lambda^{-1}$, we have $d = -a + \lambda + \lambda^{-1}$. Since $\det(A) = ad - bc = 1$, we obtain

$$bc = ad - 1 = a(-a + \lambda + \lambda^{-1}) - 1 = -(a - \lambda)(a - \lambda^{-1}).$$

Thus we have $c = -(a - \lambda)(a - \lambda^{-1})b^{-1}$ since $b \neq 0$. Therefore

$$A = \begin{pmatrix} a & b \\ -(a - \lambda)(a - \lambda^{-1})b^{-1} & -a + \lambda + \lambda^{-1} \end{pmatrix}.$$

Suppose $b > 0$. Let

$$P = \begin{pmatrix} \sqrt{b^{-1}} & 0 \\ 0 & \sqrt{b} \end{pmatrix},$$

then

$$PAP^{-1} = \begin{pmatrix} a & 1 \\ -(a - \lambda)(a - \lambda^{-1}) & -a + \lambda + \lambda^{-1} \end{pmatrix},$$

$$PBP^{-1} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} = B.$$

Similary if $b < 0$, then there exists

$$Q = \begin{pmatrix} \sqrt{-b^{-1}} & 0 \\ 0 & \sqrt{-b} \end{pmatrix}$$

such that

$$QAQ^{-1} = \begin{pmatrix} a & -1 \\ (a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix},$$
$$QBQ^{-1} = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} = B.$$

Since $R = CB^{-1}A^{-1}BA = I$, we can get $C = A^{-1}B^{-1}AB$. Therefore, the generators A, B and C of π are expressed by

$$A = \begin{pmatrix} a & -1 \\ (a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}, \tag{2.4}$$

$$C = \begin{pmatrix} (a-\lambda)(a-\lambda^{-1})(\mu^{-2}-1)+1 & -(-a+\lambda+\lambda^{-1})(\mu^2-1) \\ -a(a-\lambda)(a-\lambda^{-1})(1-\mu^{-2}) & (a-\lambda)(a-\lambda^{-1})(\mu^2-1)+1 \end{pmatrix} \tag{2.5}$$

or

$$A = \begin{pmatrix} a & 1 \\ -(a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}, \tag{2.6}$$

$$C = \begin{pmatrix} (a-\lambda)(a-\lambda^{-1})(\mu^{-2}-1)+1 & (-a+\lambda+\lambda^{-1})(\mu^2-1) \\ a(a-\lambda)(a-\lambda^{-1})(1-\mu^{-2}) & (a-\lambda)(a-\lambda^{-1})(\mu^2-1)+1 \end{pmatrix} \tag{2.7}$$

up to $\mathbf{SL}(2,\mathbb{R})$-conjugation. As a result, the trace of C is the same in both cases; that is

$$\mathrm{tr}(C) = (a-\lambda)(a-\lambda^{-1})(\mu^2-2+\mu^{-2})+2. \tag{2.8}$$

Suppose $\mathrm{tr}(C) = \nu+\nu^{-1}$ with $\nu^2 > 1$. After some simple computations, we have

$$a = \frac{(\lambda+\lambda^{-1}) \pm \sqrt{(\lambda-\lambda^{-1})^2+4\beta}}{2} \quad \text{where} \quad \beta = \frac{\nu+\nu^{-1}-2}{\mu^2+\mu^{-2}-2}. \tag{2.9}$$

Therefore $\{\lambda, \mu, \nu\}$ is a coordinate for the Teichmüller space $\mathfrak{T}(\Sigma(1,1))$.

Corollary 2.9. *Suppose z_r, z_a are the repelling and attracting fixed points of the hyperbolic matrix A in (2.4) with $\lambda^2 > 1$. Then $-\infty < z_a < 0 < z_r < \infty$ if and only if $0 < \lambda^{-1} < a < \lambda$.*

Proof. Let A_{ij} stand for the (i,j)-th entry of the matrix A. Since $A_{12} < 0$, by Corollary 2.5, we have the relations $A_{21} < 0$, $A_{11} > 0$, and $A_{22} > 0$. I claim that $\lambda > 1$. Suppose $\lambda < -1$. Then $\lambda < \lambda^{-1} < 0$. Since $A_{21} = (a-\lambda)(a-\lambda^{-1}) < 0$, it derives $\lambda < a < \lambda^{-1} < 0$. It contradicts for $A_{11} = a > 0$. Therefore $\lambda > 1$ and

$0 < \lambda^{-1} < a < \lambda$. Conversely, if $0 < \lambda^{-1} < a < \lambda$, then we can easily show that $A_{21} < 0$, $A_{11} > 0$, and $A_{22} > 0$. □

Thus above matrix A in (2.4) has positive valued trace $\lambda + \lambda^{-1}$.

Corollary 2.10. *Suppose z_r, z_a are the repelling and attracting fixed points of the hyperbolic matrix A in (2.6) with $\lambda^2 > 1$. Then $-\infty < z_a < 0 < z_r < \infty$ if and only if $\lambda < a < \lambda^{-1} < 0$.*

Proof. It can be proved in the same way as in the Corollary 2.9. □

Since A, B, C are hyperbolic elements and the holonomy group is discrete, the locations of the principal lines of A,B,C are one of follows. For more details (see Keen [6] or Goldman [4]).

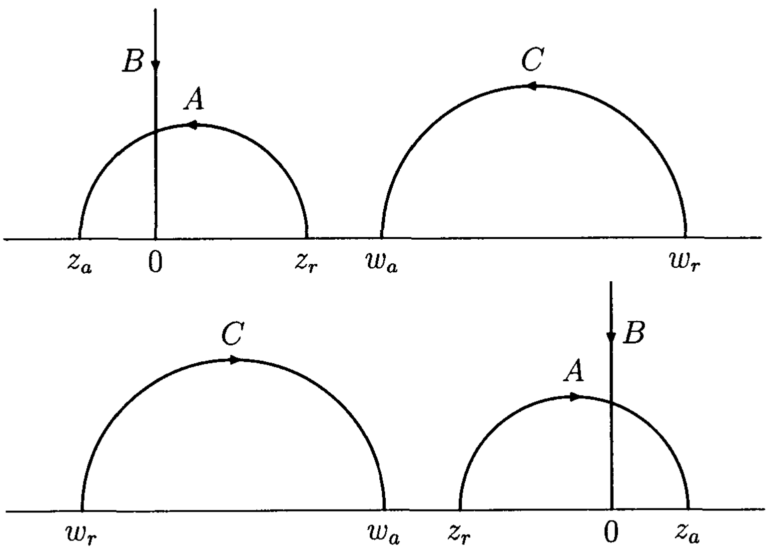

Figure 5. The locations of the principal lines of A, B, C

Relation between two diagrams is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff A = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Thus it is enough to show that the case $z_a < 0 < z_r < w_a < w_r$.

Theorem 2.11. *Suppose z_r, w_r, z_a, w_a are the repelling and attracting fixed points of the hyperbolic matrices A in (2.4) and C in (2.5) respectively and $\mu^2 > 1$. If we have $-\infty < z_a < 0 < z_r < \infty$ and $0 < w_a < w_r < \infty$, then $0 < \lambda^{-1} < a < \lambda$, and $(a-\lambda)(a-\lambda^{-1})(\mu^2 - 2 + \mu^{-2}) < -4$.*

Proof. Let C_{ij} stand for the (i,j)-th entry of the matrix C. Since $-\infty < z_a < 0 < z_r < \infty$, we have $0 < \lambda^{-1} < a < \lambda$. From the assumption $\mu^2 > 1$, we get

$$C_{12} = -(-a + \lambda + \lambda^{-1})(\mu^2 - 1) < 0$$

and

$$C_{21} = -a(a - \lambda)(a - \lambda^{-1})(1 - \mu^{-2}) > 0.$$

By Theorem 2.3, the equivalent conditions for $0 < w_a < w_r < \infty$ are $C_{21}C_{12} < 0$, $(C_{11} + C_{22})C_{21} < 0$ and $(C_{11} - C_{22})C_{21} > 0$. Clearly $C_{12}C_{21} < 0$. And $(C_{11} - C_{22})C_{21} > 0$ because of $C_{21} > 0$ and

$$(C_{11} - C_{22}) = (a - \lambda)(a - \lambda^{-1})(\mu^{-2} - \mu^2) > 0.$$

Since $C_{21} > 0$, $(C_{11} + C_{22})C_{21} < 0$ if and only if $\operatorname{tr}(C) = C_{11} + C_{22} < 0$. Because we know C is hyperbolic matrix with $\operatorname{tr}(C) < 0$, $\operatorname{tr}(C)$ must be less than -2. Thus

$$\operatorname{tr}(C) = (a - \lambda)(a - \lambda^{-1})(\mu^2 - 2 + \mu^{-2}) + 2 < -2. \tag{2.10}$$

It completes the proof. □

Now we consider the position of fixed points of the matrix A and C.

Theorem 2.12. *Suppose that A is the hyperbolic matrix in (2.4) with $-\infty < z_a < 0 < z_r < \infty$. Then the fixed points of A are*

$$z_a = \frac{1}{a - \lambda} \quad \text{and} \quad z_r = \frac{1}{a - \lambda^{-1}}. \tag{2.11}$$

Proof. By the Equation (2.3),

$$\begin{aligned} z_a,\, z_r &= \frac{(2a - \lambda - \lambda^{-1}) \pm \sqrt{(\lambda + \lambda^{-1})^2 - 4}}{2(a - \lambda)(a - \lambda^{-1})} \\ &= \frac{(2a - \lambda - \lambda^{-1}) \pm |\lambda - \lambda^{-1}|}{2(a - \lambda)(a - \lambda^{-1})} \\ &= \frac{(2a - \lambda - \lambda^{-1}) \pm (\lambda - \lambda^{-1})}{2(a - \lambda)(a - \lambda^{-1})} \\ &= \frac{2(a - \lambda^{-1})}{2(a - \lambda)(a - \lambda^{-1})} \text{ or } \frac{2(a - \lambda)}{2(a - \lambda)(a - \lambda^{-1})} \\ &= \frac{1}{(a - \lambda)} \text{ or } \frac{1}{(a - \lambda^{-1})}. \end{aligned}$$

Since $(a - \lambda) < 0$ and $(a - \lambda^{-1}) > 0$, the attracting fixed point z_a of A is $1/(a - \lambda)$ and the repelling fixed point z_r of A is $1/(a - \lambda^{-1})$. □

Theorem 2.13. *Suppose that C is the hyperbolic matrix in (2.5) with $0 < w_a < w_r < \infty$. Then the fixed points of C are*

$$w_a = \frac{E - \sqrt{D}}{F} \quad \text{and} \quad w_r = \frac{E + \sqrt{D}}{F}$$

where $E = (\lambda - a)(a - \lambda^{-1})(\mu^4 - 1)$, $F = 2a(\lambda - a)(a - \lambda^{-1})(\mu^2 - 1)$, *and* $D = \left[(a - \lambda)(a - \lambda^{-1})(\mu^2 - 1)^2 + 2\mu^2\right]^2 - 4\mu^4$.

Proof. By the Equation (2.3), the fixed points w_a, w_r of C are

$$\begin{aligned}
& \frac{(C_{11} - C_{22}) \pm \sqrt{(C_{11} + C_{22})^2 - 4}}{2C_{21}} \\
= & \frac{[(a - \lambda)(a - \lambda^{-1})(\mu^{-2} - \mu^2)] \pm \sqrt{(C_{11} + C_{22})^2 - 4}}{-2a(a - \lambda)(a - \lambda^{-1})(1 - \mu^{-2})} \\
= & \frac{[(a - \lambda)(a - \lambda^{-1})(1 - \mu^4)] \pm \sqrt{[(C_{11} + C_{22})\mu^2]^2 - 4\mu^4}}{-2a(a - \lambda)(a - \lambda^{-1})(\mu^2 - 1)} \\
= & \frac{[(\lambda - a)(a - \lambda^{-1})(\mu^4 - 1)] \pm \sqrt{D}}{2a(\lambda - a)(a - \lambda^{-1})(\mu^2 - 1)}
\end{aligned}$$

where

$$\begin{aligned}
D &= [(C_{11} + C_{22})\mu^2]^2 - 4\mu^4 \\
&= [(a - \lambda)(a - \lambda^{-1})(\mu^4 + 1 - 2\mu^2) + 2\mu^2]^2 - 4\mu^4 \\
&= [(a - \lambda)(a - \lambda^{-1})(\mu^2 - 1)^2 + 2\mu^2]^2 - 4\mu^4.
\end{aligned}$$

Therefore the facts $0 < \lambda^{-1} < a < \lambda$ and $\mu^2 > 1$ prove the theorem. □

Theorem 2.14. *Suppose the matrices A, B, C in (2.4) and (2.5) satisfy $0 < \lambda^{-1} < a < \lambda$, $\mu^2 > 1$ and $(a - \lambda)(a - \lambda^{-1})(\mu^2 - 2 + \mu^{-2}) < -4$. Then $\{A, B, C\}$ form generators of the fundamental group π of a punctured torus $\Sigma(1, 1)$.*

Proof. We should show that $-\infty < z_a < 0 < z_r < w_a < w_r < \infty$. By Theorem 2.11, it is enough to show that $z_r < w_a$. Since $(a - \lambda^{-1}) > 0$, we have to show that $2a(\lambda - a)(\mu^2 - 1) < E - \sqrt{D}$; that is

$$\begin{aligned}
\sqrt{D} &< E - 2a(\lambda - a)(\mu^2 - 1) \\
&= (\lambda - a)(\mu^2 - 1)\left[(a - \lambda^{-1})(\mu^2 + 1) - 2a\right] \\
&= (\lambda - a)(\mu^2 - 1)\left[(a - \lambda^{-1})(\mu^2 - 1) - 2\lambda^{-1}\right].
\end{aligned}$$

Since $(a-\lambda^{-1})(\mu^2-1) > (a-\lambda^{-1})(\mu^2+\mu^{-2}-2) > 4(\lambda-a)^{-1} > 2\lambda^{-1}$, the right-hand-side of the inequality is positive. Hence we will show

$$\begin{aligned} D &= \left[(a-\lambda)(a-\lambda^{-1})(\mu^2-1)^2+2\mu^2\right]^2-4\mu^4 \\ &< (\lambda-a)^2(\mu^2-1)^2\left[(a-\lambda^{-1})(\mu^2-1)-2\lambda^{-1}\right]^2. \end{aligned}$$

After some calculations we can get

$$-(a-\lambda^{-1})\mu^2 < -\lambda^{-1}(\lambda-a)(a-\lambda^{-1})(\mu^2-1)+\lambda^{-2}(\lambda-a).$$

This is equivalent to $\left[(a-\lambda^{-1})\mu^2+(\lambda-a)\right](\lambda^{-1}a) > 0$. The conditions $0 < \lambda^{-1} < a < \lambda$ and $\mu^2 > 1$ prove the theorem. □

Theorem 2.15. *Suppose the matrices A, B, C in (2.6) and (2.7) satisfy $\lambda < a < \lambda^{-1} < 0$, $\mu^2 > 1$ and $(a-\lambda)(a-\lambda^{-1})(\mu^2-2+\mu^{-2}) < -4$. Then $\{A,B,C\}$ form generators of the fundamental group π of a pair of pants.*

Proof. This can be proved by the same way in the Theorem 2.14. □

Finally we consider the relations of traces of A, B and C in $\mathbf{SL}(2,\mathbb{R})$. The matrices A and B can be endowed with positive or negative traces. For each case, we have $(a-\lambda)(a-\lambda^{-1}) < 0$ and $\mu^2 > 1$. Thus the trace of matrix C is always negative. *i. e.*,

$$\operatorname{tr}(C) = (a-\lambda)(a-\lambda^{-1})(\mu^2-2+\mu^{-2})+2 < -2. \tag{2.12}$$

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References

1. A. Beardon: *The geometry of discrete groups. Graduate Texts in Mathematics, 91.* Springer-Verlag, New York, 1983. MR **85d**:22026
2. W. M. Goldman: Geometric structures on manifolds and varieties of representations. *Geometry of group representations (Boulder, CO, 1987),* (pp. 169–198), Contemp. Math., 74, Amer. Math. Soc., Providence, RI, 1988. MR **90i**:57024
3. ______: Convex real projective structures on compact surfaces. *J. Differential Geom.* **31** (1990), no. 3, 791–845. MR **91b**:57001
4. ______: The modular group action on real $SL(2)$-characters of a one-holed torus. *Geometry and Topology* **7** (2003), 443–486.

5. D. Johnson & J. J. Millson: Deformation spaces associated to compact hyperbolic manifolds. In: Roger Howe (Ed.), *Discrete groups in geometry and analysis (New Haven, Conn., 1984)*, (pp. 48–106), Progr. Math., 67, Birkhäuser Boston, Boston, MA, 1987. MR **88j**:22010
6. L. Keen: Canonical polygons for finitely generated Fuchsian groups. *Acta Math.* **115** (1965) 1–16. MR **32**#1349
7. H. C. Kim: Matrix presentations of the Teichmüller space of a pair of pants. *J. Korean Math. Soc.* (2004), 1–23. To Appear.
8. W. Thurston: *Three-dimensional geometry and topology. Vol. 1.* Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. MR **97m**:57016
9. S. Wolpert: On the Weil-Petersson geometry of the moduli space of curves. *Amer. J. Math.* **107** (1985), no. 4, 969–997. MR **87b**:32040

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