Journal of the Korean Statistical Society (2004), **33**: 2, pp 245–254

A JOINT DISTRIBUTION OF TWO-DIMENSIONAL BROWNIAN MOTION WITH AN APPLICATION TO AN OUTSIDE BARRIER OPTION†

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ABSTRACT

This paper derives a distribution function of the terminal value and running maximum of two-dimensional Brownian motion $\{\mathbf{X}(\tau) = (X_1(\tau), X_2(\tau))', \tau > 0\}$. One random variable of the joint distribution is the terminal time value, $X_1(T)$. The other random variable is the maximum of the Brownian motion $\{X_2(\tau),\ \tau > 0\}$ between time s and time t. With this distribution function, this paper also derives an explicit pricing formula for an outside barrier option whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity.

AMS 2000 subject classifications. Primary 60J65; Secondary 91B70.
Keywords. Brownian motion, terminal time value, running maximum, barrier option, trivariate normal distribution.

1. INTRODUCTION

Barrier options have been substantially traded in the over-the-counter market of U.S. since the late 1980s because of their attractiveness to both buyers and sellers. The payoffs of barrier options are the same as those of their corresponding plain-vanilla options if the path of the underlying asset satisfies an activating condition, but will be zero otherwise. A barrier option is cheaper than its corresponding plain-vanilla option because the payoff of the barrier option is less than or equal to that of the plain-vanilla option. In addition, barrier options are useful in designing and pricing equity-linked products. For further discussions, see Heynen and Kat (1994a, b), Zhang (1998) and Lee (2003).

Received November 2003; accepted April 2004.
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However, pricing barrier options is a challenging problem due to the complex payoff structure. A key to pricing barrier options is a joint distribution function of the terminal time value and running maximum of their underlying assets. This paper derives a distribution function of the terminal value and running maximum of two-dimensional Brownian motion $\{\mathbf{X}(\tau) = (X_1(\tau), X_2(\tau))', \tau > 0\}$. One random variable of the joint distribution is the terminal time value, $X_1(T)$. The other random variable is the maximum of the Brownian motion $\{X_2(\tau), \tau > 0\}$ between time s and time t. With this distribution function, this paper also derives an explicit pricing formula for an outside barrier option whose monitoring period of the option starts at an arbitrary date and ends at another arbitrary date before maturity.

2. Two-dimensional Brownian Motion and Its Distributions

This section discusses one-dimensional Brownian motion and its distribution function of the terminal time value and running maximum. Also, this section presents a generalization of the distribution function in the case of two-dimensional Brownian motion.

First of all, consider one-dimensional Brownian motion $\{X(\tau), \tau \geq 0\}$ with drift μ and diffusion coefficient σ. Thus, the Brownian motion is a stochastic process with independent and stationary increments, and $X(\tau)$ has a normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$. Let

$$M(s,t) = \max\{X(\tau),\ s \leq \tau \leq t\} \tag{2.1}$$

be the maximum of the Brownian motion between time s and time t. Let a random vector $\mathbf{Z} = (Z_1, Z_2, Z_3)$ have a standard trivariate normal distribution with correlation coefficients $\mathrm{Corr}(Z_i, Z_j) = \rho_{ij}$ $(i, j = 1, 2, 3)$. The distribution function of the random vector $\mathbf{Z}$ is

$$\Phi_3(a, b, c; \rho_{12}, \rho_{13}, \rho_{23}) = Pr(Z_1 \leq a, Z_2 \leq b, Z_3 \leq c).$$

Note that

$$\Phi_2(b, c; \rho_{23}) - \Phi_3(a. b, c; \rho_{12}, \rho_{13}, \rho_{23}) = \Phi_3(-a, b, c; -\rho_{12}, -\rho_{13}, \rho_{23}), \tag{2.2}$$

where $\Phi_2(a, b; \rho)$ denotes the bivariate standard normal distribution function with correlation coefficient ρ.

For $0 < s < t < T$, the joint probability distribution function of $X(T)$ and $M(s,t)$ is

$$\begin{aligned} &Pr(X(T) \le x, M(s,t) \le m) \\ &= \Phi_3\left(\frac{x-\mu T}{\sigma\sqrt{T}}, \frac{m-\mu t}{\sigma\sqrt{t}}, \frac{m-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) - \exp\left(\frac{2\mu}{\sigma^2}m\right) \\ &\quad \times \Phi_3\left(\frac{x-2m-\mu T}{\sigma\sqrt{T}}, \frac{-m-\mu t}{\sigma\sqrt{t}}, \frac{m+\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right), \end{aligned} \tag{2.3}$$

which is proved by Lee (2003). If variable x in (2.3) approaches infinity, the probability (2.3) becomes

$$\begin{aligned} Pr(M(s,t) \le m) = &\Phi_2\left(\frac{m-\mu t}{\sigma\sqrt{t}}, \frac{m-\mu s}{\sigma\sqrt{s}}; \sqrt{\frac{s}{t}}\right) \\ &- \exp\left(\frac{2\mu}{\sigma^2}m\right)\Phi_2\left(\frac{-m-\mu t}{\sigma\sqrt{t}}, \frac{m+\mu s}{\sigma\sqrt{s}}; -\sqrt{\frac{s}{t}}\right), \end{aligned} \tag{2.4}$$

which will be used in the proof of (2.6).

Next, let us consider a two-dimensional Brownian motion $\{\mathbf{X}(t) = (X_1(t), X_2(t))'\}$ with drift vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, $X_i(0) = 0$ and diffusion matrix equal to

$$\mathbf{V} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Thus, the two-dimensional Brownian motion is a stochastic process with independent and stationary increments, and $(X_1(\tau), X_2(\tau))'$ follows a bivariate normal distribution with mean vector $\tau\boldsymbol{\mu}$ and covariance matrix $\tau\mathbf{V}$. For $0 < s < t$, let

$$M_2(s,t) = \max\{X_2(\tau),\ s \le \tau \le t\} \tag{2.5}$$

be the maximum of the Brownian motion $\{X_2(\tau),\ 0 \le \tau\}$ between time s and time t. In Section 3, we shall prove that for $0 < s < t \le T$, the joint distribution function of $M_2(s,t)$ and $X_1(T)$ is

$$\begin{aligned} &Pr(X_1(T) \le x, M_2(s,t) \le m) \\ &= \Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, \frac{m-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{m-\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) - \exp\left(\frac{2\mu_2}{\sigma_2^2}m\right) \\ &\quad \times\Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho m}{\sigma_2\sqrt{T}}, \frac{-m-\mu_2 t}{\sigma_2\sqrt{t}}, \frac{m+\mu_2 s}{\sigma_2\sqrt{s}}; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right). \end{aligned} \tag{2.6}$$

If $\mu_1 = \mu_2, \sigma_1 = \sigma_2$ and $\rho = 1$, then the random vector $(X_1(T), M_2(s,t))$ has the same distribution as the random vector $(X_1(T), M_1(s,t))$.

3. PROOF OF (2.6)

Let us derive the joint distribution function of $X_1(T)$ and $M_2(s,t)$. First, to simplify calculation, consider the case that the correlation coefficient ρ is nonzero. Let

$$Z(\tau) := \frac{\sigma_2}{\sigma_1}X_1(\tau) - \rho X_2(\tau). \tag{3.1}$$

The random variable $Z(\tau)$ is independent of $X_2(\tau)$ because their covariance is zero. Thus, the stochastic processes $\{Z(\tau)\}$ and $\{X_2(\tau)\}$ are independent, and $Z(\tau)$ is normally distributed with mean $(\sigma_2\mu_1/\sigma_1 - \rho\mu_2)\tau$ and variance $\sigma_2^2(1-\rho^2)\tau$. The joint distribution function of $X_1(T)$ and $M_2(s,t)$ can be calculated as follows:

$$\begin{aligned}
&Pr(X_1(T) \le x,\ M_2(s,t) \le m)\\
&= Pr\left(\frac{\sigma_1}{\sigma_2}\{Z(T) + \rho X_2(T)\} < x,\ M_2(s,t) \le m\right)\\
&= E\left[Pr\left(\rho X_2(T) < \frac{\sigma_2}{\sigma_1}x - Z(T),\ M_2(s,t) \le m \middle| Z(T)\right)\right].
\end{aligned} \tag{3.2}$$

Now, let us calculate the inside conditional probability term in the last line of (3.2). The independence of $Z(T)$ and $(X_2(T), M_2(s,t))$ implies that for a real number z,

$$\begin{aligned}
&Pr\left(\rho X_2(T) < \frac{\sigma_2}{\sigma_1}x - Z(T),\ M_2(s,t) \le m \middle| Z(T) = z\right)\\
&= Pr\left(\rho X_2(T) < \frac{\sigma_2}{\sigma_1}x - z,\ M_2(s,t) \le m\right),
\end{aligned} \tag{3.3}$$

which applying (2.3), can be easily obtained in the case of positive ρ in (3.3). For $\rho < 0$ in (3.3), we need to calculate the following probability

$$\begin{aligned}
&Pr(X_2(T) > x,\ M_2(s,t) < m)\\
&= Pr(M_2(s,t) < m) - Pr(X_2(T) < x,\ M_2(s,t) < m),
\end{aligned}$$

which applying (2.4) and (2.3), becomes

$$\begin{aligned}
&\Phi_2\left(d, e; \sqrt{\frac{s}{t}}\right) - \exp\left(\frac{2\mu_2}{\sigma_2^2}m\right)\Phi_2\left(f, g; -\sqrt{\frac{s}{t}}\right)\\
&- \left\{\Phi_3\left(\frac{x - \mu_2 T}{\sigma_2\sqrt{T}}, d, e; \sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) - \exp\left(\frac{2\mu_2}{\sigma_2^2}m\right)\right.\\
&\left.\times\, \Phi_3\left(\frac{x - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, f, g; \sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right)\right\},
\end{aligned} \tag{3.4}$$

where $d = \dfrac{m-\mu_2 t}{\sigma_2\sqrt{t}}, e = \dfrac{m-\mu_2 s}{\sigma_2\sqrt{s}}, f = \dfrac{-m-\mu_2 t}{\sigma_2\sqrt{t}}$, and $g = \dfrac{m+\mu_2 s}{\sigma_2\sqrt{s}}$.

It follows from equation (2.2) that (3.4) is

$$
\begin{aligned}
&Pr(X_2(T) > x, M_2(s,t) < m) \\
&= \Phi_3\left(-\frac{x-\mu_2 T}{\sigma_2\sqrt{T}}, d, e; -\sqrt{\frac{t}{T}}, -\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) \\
&\quad - \exp\left(\frac{2\mu_2}{\sigma_2^2}m\right)\Phi_3\left(-\frac{x-2m-\mu_2 T}{\sigma_2\sqrt{T}}, f, g; -\sqrt{\frac{t}{T}}, \sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right). \qquad (3.5)
\end{aligned}
$$

Combining the probability formulas (2.3) and (3.5), we can obtain the probability formula,

$$
\begin{aligned}
&Pr(\rho X_2(T) < x, M_2(s,t) < m) \\
&= \Phi_3\left(s(\rho)\frac{x/\rho-\mu_2 T}{\sigma_2\sqrt{T}}, d, e; s(\rho)\sqrt{\frac{t}{T}}, s(\rho)\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right) - \exp\left(\frac{2\mu_2}{\sigma_2^2}m\right) \\
&\quad \times \Phi_3\left(s(\rho)\frac{x/\rho-2m-\mu_2 T}{\sigma_2\sqrt{T}}, f, g; s(\rho)\sqrt{\frac{t}{T}}, -s(\rho)\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right), \qquad (3.6)
\end{aligned}
$$

where $s(\rho)$ is 1 if ρ is greater than zero and $s(\rho)$ is -1 otherwise. Hence, the conditional probability (3.3) is the same as (3.6) with $x = \sigma_2 x/\sigma_1 - z$.

Now, we are ready to compute (3.2). It follows from applying (3.6) to the last line of (3.2) that the joint distribution function of $X_1(T)$ and $M_2(s,t)$ can be rewritten as follows:

$$
\begin{aligned}
&E\left[\Phi_3\left(s(\rho)\frac{\{\sigma_2 x/\sigma_1 - Z(T)\}/\rho - \mu_2 T}{\sigma_2\sqrt{T}}, d, e; s(\rho)\sqrt{\frac{t}{T}}, s(\rho)\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right)\right] \\
&\quad - \exp\left(\frac{2\mu_2 m}{\sigma_2^2}\right) E\left[\Phi_3\left(s(\rho)\frac{\{\sigma_2 x/\sigma_1 - Z(T)\}/\rho - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, f, g;\right.\right. \\
&\quad \left.\left. s(\rho)\sqrt{\frac{t}{T}}, -s(\rho)\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right)\right]. \qquad (3.7)
\end{aligned}
$$

Consider the first expectation in (3.7). Let (U, V, W) be a random vector with trivariate standard normal distribution and correlation coefficients $\mathrm{Corr}(U, V) = s(\rho)\sqrt{t/T}$, $\mathrm{Corr}(U, W) = s(\rho)\sqrt{s/T}$ and $\mathrm{Corr}(V, W) = \sqrt{s/t}$. Assume that the random variable $Z(T)$ is independent of the random vector (U, V, W). Then, the first expectation in (3.7) can be calculated as follows:

$$\begin{aligned}
&E\left[E\left\{I\left(U < s(\rho)\frac{\{\sigma_2 x/\sigma_1 - Z(T)\}/\rho - \mu_2 T}{\sigma_2\sqrt{T}}, V < d, W < e\right)\Big|Z(T)\right\}\right]\\
&= E\left[E\left\{I\left(U < \frac{\{\sigma_2 x/\sigma_1 - Z(T)\} - \rho\mu_2 T}{|\rho|\sigma_2\sqrt{T}}, V < d, W < e\right)\Big|Z(T)\right\}\right]\\
&= E\left[I\left(|\rho|\sigma_2\sqrt{T}U + Z(T) < \frac{\sigma_2}{\sigma_1}x - \rho\mu_2 T, V < d, W < e\right)\right]\\
&= E\left[I\left(U^* < \frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, V < d, W < e\right)\right], \qquad (3.8)
\end{aligned}$$

where U^* denotes the standardized random variable of $|\rho|\sigma_2\sqrt{T}U+Z(T)$ and $I(\cdot)$ denotes the indicator function. Note that random vector (U^*, V, W) follows a trivariate standard normal distribution with correlation coefficients $\text{Corr}(U^*, V) = \rho\sqrt{t/T}$, $\text{Corr}(U^*, W) = \rho\sqrt{s/T}$ and $\text{Corr}(V, W) = \sqrt{s/t}$. Hence, we can obtain the first expectation in (3.7) as follows:

$$\Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}, d, e; \rho\sqrt{\frac{t}{T}}, \rho\sqrt{\frac{s}{T}}, \sqrt{\frac{s}{t}}\right). \qquad (3.9)$$

Next, let us calculate the second expectation in (3.7) in a similar way to the first expectation in (3.7). Assume that the random vector (U, V, W) has a trivariate standard normal distribution with correlation coefficients $\text{Corr}(U, V) = s(\rho)\sqrt{t/T}$, $\text{Corr}(U, W) = -s(\rho)\sqrt{s/T}$ and $\text{Corr}(V, W) = -\sqrt{s/t}$. Also assume that the random variable $Z(T)$ is independent of the random vector (U, V, W). Then, the second expectation in (3.7) will be calculated as follows:

$$\begin{aligned}
&E\left[E\left\{I\left(U < s(\rho)\frac{\{\sigma_2 x/\sigma_1 - Z(T)\}/\rho - 2m - \mu_2 T}{\sigma_2\sqrt{T}}, V < f, W < g\right)\Big|Z(T)\right\}\right]\\
&= E\left[I\left(|\rho|\sigma_2\sqrt{T}U + Z(T) < \frac{\sigma_2}{\sigma_1}x - 2\rho m - \rho\mu_2 T, V < f, W < g\right)\right]\\
&= \Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}} - \frac{2\rho m}{\sigma_2\sqrt{T}}, f, g; \rho\sqrt{\frac{t}{T}}, -\rho\sqrt{\frac{s}{T}}, -\sqrt{\frac{s}{t}}\right). \qquad (3.10)
\end{aligned}$$

Placing (3.9) and (3.10) into (3.7), we have the joint distribution function of $X_1(T)$ and $M_2(s,t)$ when ρ is nonzero.

Finally, it is straightforward to consider the case that the correlation coefficient ρ is zero. Applying the fact that the stochastic processes $\{X_1(\tau)\}$ and $\{X_2(\tau)\}$ are independent, we obtain

$$Pr(X_1(T) \le x, M_2(s,t) \le m) = Pr(X_1(T) \le x)Pr(M_2(s,t) \le m). \qquad (3.11)$$

It follows from equation (2.4) that (3.11) can be calculated as follows.

$$\Phi\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}}\right)\left\{\Phi_2\left(d,e;\sqrt{\frac{s}{t}}\right)-\exp\left(\frac{2\mu_2}{\sigma_2^2}m\right)\Phi_2\left(f,g;-\sqrt{\frac{s}{t}}\right)\right\}$$
$$=\Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}},d,e;0,0,\sqrt{\frac{s}{t}}\right)-\exp\left(\frac{2\mu_2}{\sigma_2^2}m\right)$$
$$\times\,\Phi_3\left(\frac{x-\mu_1 T}{\sigma_1\sqrt{T}},f,g;0,0,-\sqrt{\frac{s}{t}}\right), \tag{3.12}$$

which is the same as formula (2.6) with the correlation coefficient $\rho = 0$. Here $\Phi(\cdot)$ denotes the standard normal distribution function.

4. APPLICATION TO AN OUTSIDE BARRIER OPTION

Merton (1973) and Reiner and Rubinstein (1991) have developed pricing formulas for standard barrier options. The word "standard" means that the monitoring period is the entire option life. Heynen and Kat (1994b) derived pricing formulas for barrier options whose monitoring periods are $[0,t]$ or $[t,T]$ instead of the entire option life, $[0,T]$. Heynen and Kat (1994a) derived pricing formulas for outside barrier options whose monitoring period is $[0,T]$. Bermin (1996) developed explicit pricing formulas for outside barrier options with the monitoring period from time 0 to time $t\,(t<T)$. This section applies the joint distribution function of $X_1(T)$ and $M_2(s,t)$ to derive an explicit pricing formula for an outside barrier option whose monitoring period starts at an arbitrary date and ends at another arbitrary date before maturity. This formula is a generalization of option formulas mentioned above.

The payoff of outside barrier options depends on prices of two underlying assets: one asset, called the payoff asset, is used for determining the payoff, and the other asset, called the barrier asset, for determining whether the options knock in or out. Let $S_1(t)$ and $S_2(t)$ denote the time-t prices of the payoff asset and the barrier asset, respectively. Assume that these assets pay no dividends. Assume that for $t \geq 0$, $i = 1$ and 2,

$$S_i(t) = S_i(0)\exp(X_i(t)),$$

where $\{(X_1(t), X_2(t))'\}$ is a 2-dimensional Brownian motion as mentioned in Section 2. Assume that the strike price is K, and the barrier level is B. Let $b = \log[B/S_2(0)]$ and $k = \log[K/S_1(0)]$. The activating condition of an up-and-out outside barrier option is

$$\{\max\{S_2(\tau), s \le \tau \le t\} < B\} = \{M_2(s,t) < b\}.$$

The payoff of an up-and-out outside barrier put option will be $K - S_1(T)$ if $S_2(\tau)$ is less than B for any τ in $[s,t]$ and $S_1(T)$ is less than K. In other words, the payoff can be expressed as follows:

$$\begin{cases} K - S_1(T), & \text{if } M_2(s,t) < b \text{ and } X_1(T) < k, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Let us calculate the time-0 value of the payoff (4.1). By the fundamental theorem of asset pricing and by the method of Esscher transforms (Gerber and Shiu, 1996), the time-0 value of the payoff (4.1) is

$$\begin{aligned} &e^{-rT}E^*\left[-\{S_1(T) - K\}I(M_2(s,t) < b, X_1(T) < k)\right] \\ &= -e^{-rT}E^*\left[S_1(T)I(M_2(s,t) < b, X_1(T) < k)\right] \\ &\quad + e^{-rT}KPr^*(M_2(s,t) < b, X_1(T) < k) \end{aligned} \tag{4.2}$$

where r is continuously compound interest rate and mark * in (4.2) denotes the risk-neutral measure with respect to which the process $\{e^{-rt}S_i(t)\}(i = 1,2)$ is a martingale. Under this measure, the process $\{(X_1(t), X_2(t))'\}$ is a 2-dimensional Brownian motion with drift vector

$$(\mu_1^*, \mu_2^*) = \left(r - \frac{\sigma_1^2}{2}, r - \frac{\sigma_2^2}{2}\right) \tag{4.3}$$

and diffusion matrix $\mathbf{V}$. By the factorization formula (Gerber and Shiu, 1994 and 1996), the second expectation in (4.2) can be factorized as follows:

$$\begin{aligned} &e^{-rT}E^*\left[S_1(T)I(M_2(s,t) < b, X_1(T) < k)\right] \\ &= e^{-rT}E^*\left[S_1(T)\right]E^*\left[\frac{S_1(T)}{E^*\left[S_1(T)\right]}I(M_2(s,t) < b, X_1(T) < k)\right] \\ &= e^{-rT}E^*[S_1(T)]E^{**}[I(M_2(s,t) < b, X_1(T) < k)], \end{aligned} \tag{4.4}$$

where mark ** in (4.4) denotes shifted measure, under which this process $\{(X_1(t), X_2(t))'\}$ is a two-dimensional Brownian motion with drift vector

$$\begin{aligned} (\mu_1^{**}, \mu_2^{**}) &= (\mu_1^*, \mu_2^*) + (1,0)\mathbf{V} \\ &= \left(r + \frac{\sigma_1^2}{2}, r - \frac{\sigma_2^2}{2} + \rho\sigma_1\sigma_2\right) \end{aligned} \tag{4.5}$$

and diffusion matrix $\mathbf{V}$. Note that

$$\frac{S_1(T)}{E^*[S_1(T)]} = \frac{S_1(T)^1 S_2(T)^0}{E^*[S_1(T)^1 S_2(T)^0]} \tag{4.6}$$

in the second line of (4.4) is used as the Radon-Nikodym derivative and that the exponent parts $(1, 0)$ in the right-hand side of (4.6) is applied to (4.5). For further discussion, see Gerber and Shiu (1996). Applying the fact that $\{e^{-rt}S_1(t)\}$ is a martingale, we have

$$e^{-rT}E^*[S_1(T)] = S_1(0). \tag{4.7}$$

Placing (4.7) into (4.4), we have the time-0 value of (4.1),

$$-S_1(0)Pr^{**}(M_2(s,t) < b, X_1(T) < k) + e^{-rT}KPr^*(M_2(s,t) < b, X_1(T) < k). \tag{4.8}$$

Now, the final step for pricing the outside barrier option is to calculate the probabilities of (4.8). These probabilities are the same as (2.6) except that the drift vectors of the first and second probabilities in (4.8) are (4.5) and (4.3), respectively.

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