論文

탄성지지된 3경간 철근콘크리트 교량의 간단한 진동해석법

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Simple Method of Vibration Analysis of Three Span Continuous Reinforced Concrete Bridge with Elastic Intermediate Supports

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ABSTRACT

A method of calculating the natural frequency corresponding to the first mode of vibration of beams and tower structures, with irregular cross sections and with arbitrary boundary conditions was developed and reported by Kim, D. H. in 1974. In this paper, the result of application of this method to the three span continuous reinforced concrete bridge with elastic intermediate supports is presented. Such bridge represents either concrete or sandwich type three span bridge on polymeric supports for passive control or on actuators for active control. The concrete slab is considered as a special orthotropic plate. Any method may be used to obtain the deflection influence surfaces needed for this vibration analysis. Finite difference method is used for this purpose, in this paper. The influence of the modulus of the foundation and D_{22}, D_{12}, D_{66} stiffnesses on the natural frequency is thoroughly studied.

초 록

불균등 단면이나 임의의 경계조건을 가진 보나 탑 구조물의 고유진동수를 계산하는 방법은 김덕현에 의해 1960년 후반기에 개발 응용되고 1974년의 한 국제회의에서 발표되었다. 이 논문에는 이 방법을 탄성지지된 3경간연속 철근콘크리트 교량에 적용한 결과가 실려있다.

콘크리트 상판은 특별직교 이방성 판으로 취급되었다. 이 진동해석에 필요한 변위의 영향계수는 여러방법으로 구할 수 있으나 이 논문에서는 유한차분법이 사용되었다. 기초의 탄성계수와, D_{22}, D_{12}, D_{66} 강성의 고유진동수에 대한 영향이 철저하게 연구되었다.

Key Words : 진동(vibration), 연속교량(continuous bridge), 콘크리트 상판(concrete slab), 탄성지지(elastic support), 고유진동수 (natural frequency), 유한차분법(finite difference method).

1. Introduction

The problem of deteriorated highway concrete slab is very serious all over the world. Before making any decision on repair work, reliable non-destructive evaluation is necessary. One of the dependable methods is to evaluate the in-situ stiffness of the slab by means of obtaining the natural frequency. By comparing the in-situ stiffness with the one obtained at the design stage, the degree of damage can be estimated rather accurately.

Recently, use of polymeric bridge supports has become quite popular. Unlike the metal hinges and rollers, these

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polymers behave like elastic supports. The actuators for the active control of the bridge behave, at least partially, as the elastic supports. The reinforced concrete slab can be assumed as a special orthotropic plate, as a close approximation, assuming that the influence of B_{16}, B_{26}, D_{16} and D_{26} stiffnesses are negligible. The senior author has reported that some laminate orientations such as $[\alpha/\beta]_r$, $[\alpha/\beta/\alpha]_r$, $[\alpha/\beta/\beta/\alpha/\alpha/\beta]_r$, and $[\alpha/\beta/\beta/\gamma/\alpha/\alpha/\beta]_r$ with $\alpha=-\beta$, and $\gamma=0°$ or $90°$, and with increasing r, have decreasing values of B_{16}, B_{26}, D_{16}, and D_{26} stiffnesses, where α, β, and γ are the fiber orientations in degrees measured from the laminate axes, positive in the counterclockwise direction. r is an integer, and B_{ij} and D_{ij} are the bending-stretching coupling stiffness matrix and the flexural stiffness matrix, respectively. D_{ij} expresses the relation between the stress couples, M_{ij}, and the curvatures, x_{ij}. B_{ij} relates M_{ij} to the mid-surface strains, ε_{0ij} and the in-plane stress resultants, N_{ij} to x_{ij} B_{16} and B_{26} cause bending-shearing and stretching-twisting coupling. D_{16} and D_{26} cause bending twisting coupling. Such laminates given above may be very useful when one tries to apply the advanced composite materials to new constructions such as building slabs, bridge decks, and so on. One can obtain the advantages of the advanced composite materials using simplified equations.

For such laminates, the three partial differential equations for the laminate bending,

$$A_{11}\frac{\partial^2 u}{\partial x^2}+2A_{16}\frac{\partial^2 u}{\partial x\partial y}+A_{66}\frac{\partial^2 u}{\partial y^2}+A_{16}\frac{\partial^2 v}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2 v}{\partial x\partial y}$$
$$+A_{26}\frac{\partial^2 v}{\partial y^2}-B_{11}\frac{\partial^3 w}{\partial x^3}-3B_{16}\frac{\partial^3 w}{\partial x^2\partial y}-(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x\partial y^2}-B_{26}\frac{\partial^3 w}{\partial y^3}=0 \quad (1)$$

$$A_{16}\frac{\partial^2 u}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2 u}{\partial x\partial y}+A_{26}\frac{\partial^2 u}{\partial y^2}+A_{66}\frac{\partial^2 v}{\partial x^2}+2A_{26}\frac{\partial^2 v}{\partial x\partial y}$$
$$+A_{22}\frac{\partial^2 v}{\partial y^2}-B_{16}\frac{\partial^3 w}{\partial x^3}-(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 w}{\partial x\partial y^2}-B_{22}\frac{\partial^3 w}{\partial y^3}=0 \quad (2)$$

$$D_{11}\frac{\partial^4 w}{\partial x^4}+4D_{16}\frac{\partial^4 w}{\partial x^3\partial y}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\partial y^2}+4D_{26}\frac{\partial^4 w}{\partial x\partial y^3}$$
$$+D_{22}\frac{\partial^4 w}{\partial y^4}-B_{11}\frac{\partial^3 u}{\partial x^3}-3B_{16}\frac{\partial^3 u}{\partial x^2\partial y}-(B_{12}+2B_{66})\frac{\partial^3 u}{\partial x\partial y^2} \quad (3)$$
$$-B_{26}\frac{\partial^3 u}{\partial y^3}-B_{16}\frac{\partial^3 v}{\partial x^3}-(B_{12}+2B_{66})\frac{\partial^3 v}{\partial x^2\partial y}-3B_{26}\frac{\partial^3 v}{\partial x\partial y^2}-B_{22}\frac{\partial^3 v}{\partial y^3}=q(x,y)$$

can be reduced to one equation, for the special orthotropic plate,

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}=q(x,y). \quad (4)$$

where $D_1=D_{11}, D_2=D_{22}, D_3=(D_{12}+2D_{66})$.

However such plates will have different stress distribution through each ply of the laminate, quite different from the "real" special orthotropic plates.

Several materials should be tested to find out the best type of materials for the future bridge decks, especially advanced composite bridge decks. Such plates are subject to the concentrated mass/masses in the form of traffic loads, or the test equipments such as accelerator in addition to their own masses. Analysis of such problems is usually very difficult. Most of the civil and architectural structures are large in sizes and the number of laminae is large, even though the thickness to length ratio is small enough to allow to neglect the transverse shear deformation effect in stress analysis. For such plates, there are enough number of fiber orientations for which theories for special orthotropic plates can be applied, [1,2] and simple formulae developed by the senior author can be used [3,4].

In case of a laminated composite plate with boundary conditions other than Navier or Levy solution types, or with irregular cross section, or with nonuniform mass including point masses, analytical solution is very difficult to obtain. Numerical method for eigenvalue problems are also very much involved in seeking such a solution.

The basic concept of the Rayleigh method, the most popular analytical method for vibration analysis of a single degree of freedom system, is the principle of conservation of energy; the energy in a free vibrating system must remain constant if no damping forces act to absorb it. In case of a beam, which has an infinite number of degrees of freedom, it is necessary to assume a shape function in order to reduce the beam to a single degree of freedom system [5]. The frequency of vibration can be found by equating the maximum strain energy developed during the motion to the maximum kinetic energy. This method, however, yields the solution either equal to or larger than the real one. Recall that Rayleigh's quotient ≥ 1 [4, pp. 189~191]. For a complex beam, assuming a correct shape function is not possible. In such cases, the solution obtained is larger than the real one.

A simple but exact method of calculating the natural frequency corresponding to the first mode of vibration of beam and tower structures with irregular cross-sections and attached mass/masses was developed and was reported by

Kim, D. H. in 1974. This method consists of determining the deflected mode shape of the member due to the inertia force under resonance condition. Beginning with initially "guessed" mode shape, "exact" mode shape is obtained by the process similar to iteration. Recently, this method was extended to two dimensional problems including composite laminates, and has been applied to composite plates with various boundary conditions with/without shear deformation effects and reported at several international conferences including the Eighth Structures Congress(1990) and Fourth Materials Congress(1996) of American Society of Civil Engineers.

In this paper, the result of application of this method to the subject problem is presented.

2. Method of Analysis

2.1 Vibration Analysis

In this paper, the result of application of this method to the subject problem is presented.

Since the method of analysis used for this paper is given, in detail, in the senior author's book [4], it is not repeated here.

2.2 Finite difference method

The method used in this paper requires the deflection influence surfaces. Since no reliable analytical method is available for the subject problem, F.D.M. is applied to the governing equation of the special orthotropic plates,

$$D_1\frac{\partial^4 w}{\partial x^4}+2D_3\frac{\partial^4 w}{\partial x^2\partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}$$

$$=q(x,y)-kw+Nx\frac{\partial^2 w}{\partial x^2}+Ny\frac{\partial^2 w}{\partial y^2}+2Nxy\frac{\partial^2 w}{\partial xy} \qquad (5)$$

where $D_1=D_{11}, D_2=D_{22}, D_3=(D_{12}+2D_{66})$.

The number of the pivotal points required for the forth order derivatives in the case of the order of error Δ^2, where Δ is the mesh size, is five for the central differences. This makes the procedure at the boundaries complicated. In order to solve such problem, the three simultaneous partial differential equations of equilibrium with three dependent variables, w, Mx, and My, are used instead of equation (5) with $N_x=N_y=N_{xy}=0$ [6,7].

$$D_{11}\frac{\partial^2 Mx}{\partial x^2}-4D_{66}\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^2 My}{\partial y^2}=q(x,y)+kw(x,y) \qquad (6)$$

$$Mx=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2} \qquad (7)$$

$$My=-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2} \qquad (8)$$

If F.D.M. is applied to these equations, the resulting matrix equation is very large in sizes, but the tridiagonal matrix calculation scheme used by Kim, D. H [6-9] is very efficient to solve such equations.

In order to confirm the accuracy of the F.D.M., $[A/B/A]_r$ type laminate with aspect ratio of a/b=1m/1m=1 is considered. The material properties are :

E_1 = 67.36 GPa　　E_2 = 8.12 GPa
ν_{12} = 0.272,　　ν_{21} = 0.0328,
G_{12} = 3.0217 GPa,

The thickness of a ply is 0.005m. As the r increases, B_{16}, B_{26}, D_{16}, and D_{26} decrease and the equations for the special orthotropic plates can be used. For simplicity, it is assumed that A=0°, B=90°, and r=1. Then $D_{11}=18492.902$ Nm.

Since one of the few efficient analytical solutions of the special orthotropic plate is Navier solution, and this is good for the case of the four simply supported edges, F.D.M. is used to solve this problem and the result is compared with the Navier solution. The mesh size is $\Delta x=a/10=0.1$m, $\Delta y=b/10=0.1$m. The deflection at (x,y), under the uniform load of $100\,N/m^2$, the origin of the coordinates being the corner of the plate, is obtained, and the ratio of the Navier solution to the F.D.M. solution is given in Table 1.

Table 1 Deflection ratio of Navier solution to F.D.M. solution

	Navier / F.D.M				
y(m) \ x(m)	0.1	0.3	0.5	0.7	0.9
0.1	0.1005946E+01	0.1004916E+01	0.1004713E+01	0.1004916E+01	0.1005946E+01
0.3	0.1001279E+01	0.1000028E+01	0.9996814E+01	0.1000028E+01	0.1001279E+01
0.5	0.1000134E+01	0.9989528E+01	0.9985780E+01	0.9989530E+01	0.1000134E+01
0.7	0.1001279E+01	0.1000028E+01	0.9996815E+01	0.1000028E+01	0.1001279E+01
0.9	0.1005946E+01	0.1004916E+01	0.1004714E+01	0.1004916E+01	0.1005946E+01

Calculation is carried out with different mesh sizes and the maximum errors at the center of the plate are as follows:

10×10 case : 0.14%   20×20 case : 0.035%
40×40 case : 0.009%

The error is less than 1%. This is smaller than the predicted errors;

$(\Delta_{10})^2=(0.1)^2=0.01=1\%$

$(\Delta_{20})^2=(0.05)^2=0.0025=0.25\%$

$(\Delta_{40})^2=(0.025)^2=0.000625=0.0625\%$

3. Numerical Examination

3.1 Structure under Consideration

3.1.1 Bridge Outline and the Loading

The bridge considered is as shown in Fig. 1.

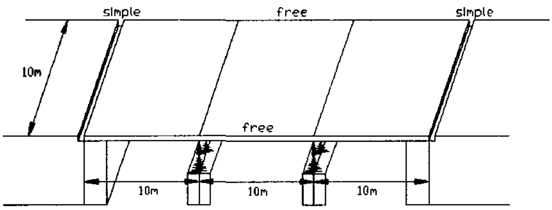

Fig. 1 Three span continuous slab bridge.

The location of the truck loading is as shown in Fig. 2.

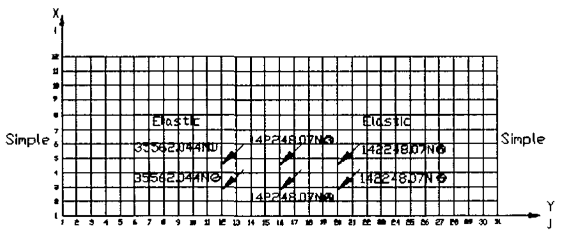

Fig. 2 Location of truck loading.

3.1.2 Reinforced Concrete Slab

Fig. 3 shows the cross section of the slab with unit width.

$\sigma_{ck}=210kg/cm^2 = 20.5942926\ MPa$ and $E_c=15000\sqrt{\sigma_{ck}} = 21.317118060\ GPa$.

Poissons ratio $\upsilon_{12}=\upsilon_{21}=0.18$ for concrete.

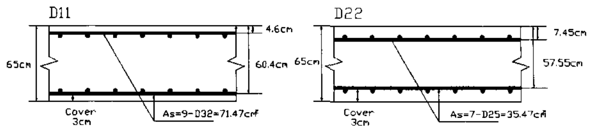

Fig. 3 Cross section of the slab with unit width.

Three different concepts are adopted for obtaining the stiffnesses, D_{ij}. For all cases, the effect of the bending extension coupling stiffness, B_{ij}, is assumed as negligible.

Case 1. Balanced design using the transformed area for steel in calculating the moment of inertia of the cross-section.

Case 2. With $E_c = 15000\sqrt{\sigma_{ck}} = 21.317118060\ GPa$ and $E_s = 199.92\ GPa$, and with concrete $Q_{11}=E_c/(1-\upsilon_{12}^2)$ and steel $Q_{11}=E_s$, the typical formulas for D_{ij} are used.

Case 3. Using the cracked section concept by the maximum moment, the moment of inertia of the cross section is obtained to calculate D_{ij}.

Table 2 shows the flexural stiffnesses of three cases.

Table 2 Flexural stiffnesses of three cases (N·m)

Stiffness \ Case	Case 1	Case 2	Case 3
D_{11}	351761502.8	323428383.7	323416426.7
D_{22}	155665708.1	151828300.8	151827047.8
D_{12}	90690632.4	90690632.4	90690632.4
D_{66}	206573097.2	206573097.2	206573097.2

For all cases, the uncracked section is used to obtain D_{66} and the concrete self-weight is $2.5t/m^3 \times 0.65m = 15925\ Pa$. The deflections at the wheel load points for three cases, when the molulus of foundation, k =14,505 x 10^6 N/m^2, are given in Table 3.

Table 3 The deflections at wheel loading points for three-cases (unit:m)

Load Point \ Case	Case 1	Case 2	Case 3
1	0.2786E-03	0.2955E-03	0.2955E-03
2	0.2314E-03	0.2458E-03	0.2458E-03
3	0.2132E-02	0.2300E-02	0.2300E-02
4	0.1901E-02	0.2054E-02	0.2045E-02
5	0.3900E-03	0.4155E-03	0.4155E-03
6	0.3288E-03	0.3504E-03	0.3504E-03

Table 4 shows the natural frequencies of three-cases, under the same value of k, k=14,505 x 10^6 N/m^2.

Table 4 The natural frequencies fort hree-cases (unit:rad/sec)

Case	Natural Frequency(rad/sec)
Case 1	0.1292903E+02
Case 2	0.1233828E+02
Case 3	0.1233805E+02

Table 5 The stiffnesses of three sub-cases, for Case 2 (unit:N·m)

Case / Stiffness	Case 2-1	Case 2-2	Case 2-3
D_{11}	323428383.7	323428383.7	323428383.7
D_{22}	151828300.8	266228356.0	323428383.7
D_{12}	90690632.4	90690632.4	0.
D_{66}	206573097.2	206573097.2	0.

In order to study the influence of D_{22}, D_{12}, and D_{66} stiffnesses, three sub-cases for Case-2 are considered as Table 5.

3.2 Numerical Result

3.2.1 Influence of D_{22}, D_{12}, D_{66} Stiffnesses

The applied load is the concrete self-weight plus the wheel loads as shown in Fig. 2. The deflections at the wheel load points for three sub-Case 2, when the modulus of foundation, k=14,505 x 10^6 N/m^2, are given in Table 6.

Table 6 The deflections at wheel loading points for three sub-cases of Case 2 (unit:m)

Case / Load Point	Case 2-1	Case 2-2	Case 2-3
1	0.2955E-03	0.2894E-03	0.4409E-03
2	0.2458E-03	0.2411E-03	0.3177E-03
3	0.2300E-02	0.2240E-02	0.2778E-02
4	0.2054E-02	0.2010E-02	0.2319E-02
5	0.4155E-03	0.4030E-03	0.5739E-03
6	0.3504E-03	0.3410E-03	0.4426E-03

Table 7 shows the natural frequencies of three sub-cases, under the same value of k, k=14,505 x 10^6 N/m^2.

Table 7 The natural frequencies for three sub-cases of Case 2 (unit:rad/sec)

Case	Natural Frequency(rad/sec)
Case 2-1	0.1233830E+02
Case 2-2	0.1257711E+02
Case 2-3	0.1101777E+02

3.2.2 Influence of the Modulus of Foundation

The influence of the modulus of foundation, k, is studied by changing k values from 14,505 x 10^3 N/m^2 to 14,505 x 10^7 N/m^2.

Table 8 shows the deflections at the wheel load points for Case 2-1, under changing values of k.

Table 8 Deflection at loading points for Case 2-1 (m)

k(N/m^2) / Load Point	14,505x10^3	14,505x10^5	14,505x10^7
1	0.1573E-01	0.4571E-03	0.2774E-03
2	0.1502E-01	0.3951E-03	0.2296E-03
3	0.1969E-01	0.2499E-02	0.2272E-02
4	0.1866E-01	0.2242E-02	0.2029E-02
5	0.1695E-01	0.6005E-03	0.3944E-03
6	0.1608E-01	0.5224E-03	0.3311E-03

Table 9 shows the natural frequencies for Case 2-1, under changing values of k . Fig. 4 is the graphical presentation of Table 8, and Fig. 5 is that of Table 9.

Table 9 The natural frequency for Case 2-1 (uuit:rad/sec)

k (N/m^2)	Natural Frequency(rad/sec)
14,505x10^3	0.8068337E+01
14,505x10^5	0.1232987E+02
14,505x10^7	0.1233943E+02

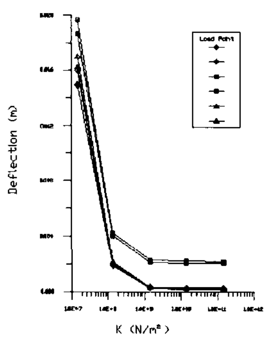

Fig. 4 The deflection at loading points for Case 2-1.

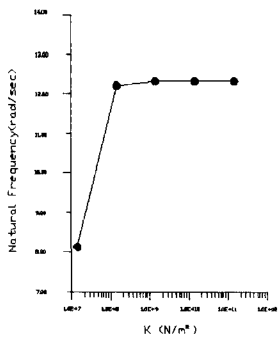

Fig. 5 The natural frequency for Case 2-1.

4. Conclusion

In this paper, the simple and accurate method of vibration analysis developed by Kim, D. H. is presented. The presented method is simple to use but extremely accurate. The boundary condition can be arbitrary. Both stiffness and mass of the element can be variable. One can use any method to obtain the deflection influence coefficients. The accuracy of the solution is dependent on only that of the influence coefficients needed for this method. One should recall that obtaining the deflection influence coefficients is the first step in design and analysis of a structure. The merit of the presented method is that it uses such influence coefficient values, used already for calculating deflection, slope, moment, and shear to obtain the natural frequency of the structure. When the plate has concentrated mass or masses, one can simply add these masses to the plate mass and use the same deflection influence surfaces to obtain the natural frequency. This method is applied to the three span continuous reinforced concrete bridge with elastic intermediate supports.

Recently, use of polymeric bridge support has become quite popular. Unlike the metal hinges and rollers, these polymers behave like elastic support. The actuators for the active control of the bridge, behave, at least partially, as the elastic support.

The finite difference method (FDM) is used to obtain the deflection influence surfaces in this paper. In order to reduce the required number of pivotal points, the three simultaneous partial differential equations of equilibrium with three dependent variables, w, M_x and M_y, are used instead of the fourth order partial differential equation for the special orthotropic plate. If FDM is applied to these equations, the resulting matrix equation is huge in size, but the tridiagonal matrix calculation scheme used by Kim, D. H. is very efficient to solve such problems. [8]

The effect of D_{22}, D_{12} and D_{66} and stiffnesses, and the modulus of foundation, on the natural frequency is thoroughly studied and the results are given in tables to provide a guideline to the practicing engineers.

Acknowledgements

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