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Teaching Linear Algebra to High School Students[1,2]

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University teachers of linear algebra often feel annoyed and disarmed when faced with the inability of their students to cope with concepts that they consider to be very simple. Usually, they lay the blame on the impossibility for the students to use geometrical intuition or the lack of practice in basic logic and set theory. J.-L. Dorier [(2002): Teaching Linear Algebra at University. In: T. Li (Ed.), *Proceedings of the International Congress of Mathematicians* (Beijing: August 20–28, 2002), Vol. III: Invited Lectures (pp. 875–884). Beijing: Higher Education Press] mentioned that the situation could not be improved substantially with the teaching of Cartesian geometry or/and logic and set theory prior to the linear algebra.

In East Asian countries, science-orientated mathematics curricula of the high schools consist of calculus with many other materials. To understand differential and integral calculus efficiently or for other reasons, students have to learn a lot of content (and concepts) in linear algebra, such as ordered pairs, n-tuple numbers, planar and spatial coordinates, vectors, polynomials, matrices, etc., from an early age. The content of linear algebra is spread out from grades 7 to 12. When the high school teachers teach the content of linear algebra, however, they do not concern much about the concepts of content. With small effort, teachers can help the students to build concepts of vocabularies and languages of linear algebra.

Keywords: epistemological analysis, semiotic representation, n-tuple number, Cartesian coordinates, vector, vector space, matrix, linear transformation, system of linear algebra, Gauss · Jordan elimination method, elementary matrix.
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INTRODUCTION

In most countries, calculus and linear algebra represents the two main subjects, as basic core courses, of mathematics taught for science and technology oriented students (usually taken during the first two years) at universities. For a period of more than one hundred years calculus was the only subject of mathematics taken as one of basic core courses at universities.

After Wassily Leontief awarded the 1973 Nobel Prize in Economic Science for his work on mathematical modeling of the U. S. economy using systems of linear equations, more and more universities every year accepted linear algebra as one of basic core courses (cf. Lay 2003, pp. 1–2). Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate subject. In the past 30 years, many studies have been carried out about the teaching of linear algebra (*cf.* Artigue, Chartier & Dorier 2000; Dorier 1995, 2000, 2002; Dorier, Robert, Robinet & Rogalski 2000a, 2000b; Harel 2000; Harel, Hillel, Rogalski, Robinet, Sierpinska & Dorier 1997; Hillel (2000); Hillel & Sierpinska 1994; Pavlopoulou 1994a, 1994b, 1994c; Rogalski 1991, 1994, 1996, 2000)

Dealing with the content of linear algebra such as ordered pairs, n-tuple numbers, systems of linear equations, the planar coordinates, the spatial coordinates, vectors, scalar multiple, inner product of two vectors, linear transformations, matrices, etc. starts with simple ideas and notions, however it become very difficult to calculate if numbers of components become a little bigger. For an example, even though Crammer's rule (or determinant) is a very efficient tool to solve of the systems of linear equations, it involves too much calculation if the system has just four unknowns. If the teacher attempt to explain the notion of calculation of matrix transformation (or decomposition of matrices) to students with example problems, it is almost impossible to make simple examples, which are not routine. Hence, the teaching of linear algebra is universally recognized as a difficult task.

About 1,800–2,200 years ago in China, at the era of Chin-Han dynasties (BC 221–AD 220), some people already used the planar coordinates, systems of linear equations, augmented matrices and the elimination method, which was very similar to Gauss-Jordan elimination method, to solve the systems. Matrices of modern days were invented much later in the mid of the nineteenth century and the last phase of the genesis of the theory of vector spaces, of which roots can be found in the late nineteenth century, really started after 1930 (Dorier 1995, 2000, 2002).

In East Asian countries, science-orientated mathematics curricula of the high schools consist of calculus with many other materials. To understand differential and integral calculus efficiently or for other reasons, students have to learn a lot of content (and concepts) in linear algebra, such as ordered pairs, n-tuple numbers, planar and spatial coordinates, vectors, polynomials, matrices, etc., from an early age. The content of linear algebra in mathematics curricula spreads out from grades 7 to 12. When the high school teachers teach the content of linear algebra, however, they do not concern much about the concepts of content. With small effort, teachers can help the students very much to build concepts of vocabularies and languages of linear algebra.

CURRENT CURRICULUM IN SOUTH KOREA

In 1997, South Korean Ministry of Education announced the first curriculum for the new millennium (so-called the "seventh amendment" of curriculum). The most important issues in the seventh amendment were to encourage the students' activity of learning.

The essential distinctions of the new curriculum compared with the old one are as follows:

1. The implementation of a "differentiated curriculum" for grades 1–10.
2. 30% reduction of mathematical contents and the reconciliation of contents.
3. Elective subjects for mathematics for grades 11 and 12.
4. More uses of technology in mathematics teaching.

The curriculum of twelve-year education of the primary and secondary schools has been divided into two periods:

- "National Common Basic Educational Period" (grades 1–10): 10 steps and each step with 2 stages (or sub-steps), and
- "Subject Selection Educational Period" (grades 11–12).

The curriculum of the National Common Basic Education Period is carried out in a stage-by-stage manner according to students' development of cognitive domain (ability of understanding and application). The contents of each stage were selected according to learning hierarchy and levels of sophistication. There are 20 stages (each step with 2 stages for 10steps) in mathematics courses including 1A, 1B, 2A, 2B, … , 10A and 10B.

The curriculum for the Selective Education Period (grades 11–12) is named as "Subject Selection Differentiated Curriculum". Students in this curriculum period can choose their subject by themselves according to their needs, capacity, preparation, interest and other attributes. These subjects are

Practical Mathematics,
Mathematics I,

Mathematics II,
Calculus,
Probability/Statistics, and
Discrete Mathematics.

Compared to the previous curriculum, the last three subjects are newly added. Among six subjects, Mathematics I, Mathematics II and Calculus are hierarchically ordered and can be taken after finishing the tenth step of National Common Basic Educational Period. Students in grades 11–12 may take Practical Mathematics, Probability/Statistics and Discrete Mathematics regardless of completion of the tenth step of National Common Basic Educational Period.

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Table 1. Layout of Mathematics Courses in Schools

Schools	Grades	Ages	Mathematics Courses
High School	12 11	18 17	**Subject Selection Educational Period**: 6 elective courses
	10	16	**National Common Basic Educational Period**: 10×2 steps of requisite courses (1A, 1B, 2A, …, 9B, 10A, 10B)
Middle School	9 8 7	15 14 13	
Primary School	6 5 4 3 2 1	12 11 10 9 8 7	

Curriculum for the National Common Basic Educational Period does not have much content of linear algebra. However, some notions such as such as ordered pairs, n-tuple numbers, planar and spatial coordinates, vectors, the first degree polynomials and the second degree polynomials spread out from grade 7 to grade 10. Mathematics teachers of this period do not concern much about the concepts of content of linear algebra because they think there are not much of content of linear algebra. Actually, with small effort, teachers can help the students to build concepts of vocabularies and languages of linear algebra.

The elective subjects such as "Mathematics I" and "Mathematics II" for the Selective Education Period (grades 11–12) have some more content of linear algebra. "Mathematics I" contains $n \times m$ matrices just for mention, but it deals with 2×2 matrices for finding the inverse matrices only by the method of cofactors. "Mathematics II" deals much content of linear algebra such as spatial coordinates, the 3-dimenstional space, projections, planar vectors, spatial vectors, addition of two vectors, scalar multiple, etc, but only for view of analytic geometry. Another elective subject "Practical Mathematics" deals much less content of linear algebra than "Mathematics I".

PREVIOUS CURRICULA IN SOUTH KOREA

The author has participated in writing the textbook for "Mathematics III" (*cf.* Choe 1997, pp. 100–102), which was an elective course for the science high school students according to so-called the "fifth amendment" of the national curriculum. Content of linear algebra was as follows:

Chapter I Matrices and Determinants

§1. Matrices and Operations on Matrices
§2. Determinants
§3. Systems of Linear Equations and Matrices
  (1) The Inverse Matrix
  (2) Systems of Linear Equations

Chapter VII Coordinates in the Plane

Chapter VIII Coordinates of the Space

The author also has participated in survey and research for the "Mathematics III" curriculum (Choe 1992, 1997) of so-called the "sixth amendment" of curriculum and gives a suggestion for content of linear algebra the Mathematics III textbook as follows:

Linear Algebra and Analytic Geometry

§1. Matrices and Operations
§2. Inverse Matrices
§3. Determinants
§4. Systems of Linear Equations (Including Underdetermined Systems)
§5. Vector Spaces
§6. Linear Transformations
§7. Vector-Valued Functions

The aim of the study was to find what contents should be in Mathematics III to maximize the efficiency of teaching and learning according to students' interests, aptitudes and abilities as well as their needs.

CONCLUSIONS

Mathematics education research cannot give a miraculous solution to overcome all difficulties in learning and teaching linear algebra (Dorier 2002, p. 882). In South Korea, the new curriculum reduced the content of linear algebra as well as overall content of mathematics. However reducing the content of linear algebra is somewhat crucial for future of science and technology in Korea. We may suggest that the elective subject "Calculus" can be extended to include linear algebra to the level of "Mathematics III" of previous curricula.

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