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A NOTE ON THE CHOQUET BOUNDARY OF TENSOR PRODUCTS

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ABSTRACT. We show that the Choquet boundary of the tensor product of two real function algebras is the product of their Choquet boundaries.

1. INTRODUCTION

The fact that the Choquet boundary of the tensor product of two complex function algebras is the product of their Choquet boundaries was proved several decades ago (see, for instance, Leibowitz [10, Theorem 14, p. 220]). But this result has not yet been proved for real function algebras. In this note, we give a proof of this result.

To begin with, we define a real function algebra as follows: Let X be a compact Hausdorff space. We denote by $C(X)$ the complex Banach algebra of all continuous complex-valued functions on X with the supremum norm. Let $\tau : X \to X$ be a *(homeomorphic) involution on* X, that is, $\tau^2 = \tau \circ \tau =$ the identity map on X. Define

$$C(X, \tau) = \{f \in C(X) : f \circ \tau = \overline{f}\},$$

where $\overline{f}(x) = \overline{f(x)}$ is the complex conjugate of $f(x)$ for $x \in X$. Then $C(X, \tau)$ is a 'real' commutative Banach algebra with the identity 1.

We say that a subset A of $C(X, \tau)$ *separates the points of* X if for any distinct points x and y in X, there exists a function f in A such that $f(x) \neq f(y)$. It is not difficult to show that $C(X, \tau)$ separates the points of X.

Definition 1 (Kulkarni & Limaye [5]). Let X be a compact Hausdorff space and τ a homeomorphic involution on X. A *real function algebra* A on (X, τ) is a uniformly

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closed (real) subalgebra of $C(X, \tau)$ which separates the points of X and contains the real constants.

The first systematic exposition of the theory of real Banach algebras was given by Ingelstam [3] in 1964. In 1981, Kulkarni & Limaye [5] introduced real function algebras and studied Gleason parts of real function algebras along the lines of complex function algebras, and then many authors such as Mehta & Vasavada [11], Kulkarni & Arundhathi [4], Kulkarni & Srinivasan [7, 8, 9], and Hwang [1, 2] developed several properties of real function algebras by contrast with complex function algebras. Kulkarni and Limaye collected most articles about real function algebras (*cf.* Kulkarni & Limaye [6]).

2. MAIN RESULTS

For a real function algebra A on (X, τ), the set of all non-zero real-linear homomorphisms of A into the complex number field is called the *carrier space* of A and denoted by Φ_A. The following definitions and lemmas can be found in Hwang [1].

Let A be a real function algebra on (X, τ), and let $\phi \in \Phi_A$. A *real-part representing measure* (*r.p.r. measure*) for ϕ is a positive regular Borel measure μ on X such that

$$\operatorname{Re}\phi(f) = \int_X \operatorname{Re} f \; d\mu$$

for all $f \in A$ and that $\mu(E) = \mu(\tau(E))$ for all Borel sets E of X.

Definition 2 (See Kulkarni & Limaye [6]). Let A be a real function algebra on (X, τ). The *Choquet boundary* $\operatorname{Ch}(A)$ of A is the set of all $x \in X$ such that the evaluation homomorphim ϕ_x admits a unique r.p.r. measure.

If $x \in \operatorname{Ch}(A)$, then the unique r.p.r. measure for ϕ_x is

$$\frac{1}{2}\{\delta_x + \delta_{\tau(x)}\},$$

where δ_x is the point mass at x. And so $x \in \operatorname{Ch}(A)$ if and only if $\tau(x) \in \operatorname{Ch}(A)$.

Definition 3 (See Kulkarni & Limaye [6]). Let A be a real function algebra on (X, τ). A (closed) subset K of X is a *peak set* for A if there exists $f \in A$ such that

$$\|f\|_X = 1, \;\; f|_K = 1, \;\text{ and } |f(x)| < 1 \quad \text{for} \quad x \in X \setminus K.$$

In this case, we say that f *peaks on* K.

A (closed) subset K of X is a *p-set* or a *weak peak set* for A if K is the non-empty intersection of a collection of peak sets for A.

A point $x \in X$ is called a *peak point* (respectively, *p-point* or *weak peak point*) if the set $\{x\}$ is a peak set (respectively, p-set) for A.

The following lemmas characterize the relation between p-sets and peak sets, and the relation between the Choquet boundaries and p-sets. Proofs of these lemmas can be seen in Kulkarni & Limaye [6].

Lemma 1. *Let A be a real function algebra on (X, τ), and let F be a closed subset of X. Then the following are equivalent:*

(a) *F is a p-set for A.*

(b) *For any open set U containing F with $\tau(U) = U$, there exists a peak set K such that $F \subset K \subset U$.*

Lemma 2. *Let A be a real function algebra on (X, τ). Then the following are equivalent:*

(a) $x \in \mathrm{Ch}(A)$.

(b) *For any (open) neighborhood U of x with $\tau(U) = U$, there exists $f \in A$ such that*

$$\|f\|_X \leq 1, \ |f(x)| > \frac{3}{4}, \ \text{and} \ |f(y)| < \frac{1}{4} \ \text{for all} \ y \in X \setminus U.$$

(c) *$\{x, \tau(x)\}$ is a p-set for A.*

Let X and Y be compact Hausdorff spaces, τ and κ homeomorphic involutions on X and Y, respectively, and let A and B be real function algebras on (X, τ) and (Y, κ), respectively. Denote by $A \otimes B$ the real subalgebra of $C(X \times Y)$ spanned by $f \otimes g$ for $f \in A$ and $g \in B$, where $(f \otimes g)(x, y) = f(x)g(y)$ for $x \in X$ and $y \in Y$. Then $A \otimes B$ is a real subalgebra of $C(X \times Y, \tau \times \kappa)$ which separates the points of $X \times Y$ and contains the real constants.

Let $A \widehat{\otimes} B$ be the uniform closure of $A \otimes B$ in $C(X \times Y, \tau \times \kappa)$. Then we have the following theorem.

Theorem 3. $\mathrm{Ch}(A \widehat{\otimes} B) = \mathrm{Ch}(A) \times \mathrm{Ch}(B)$.

Proof. Note that $x \in \mathrm{Ch}(A)$ if and only if $\{x, \tau(x)\}$ is a p-set for A by Lemma 2.

Let $x_0 \in \mathrm{Ch}(A)$, $y_0 \in \mathrm{Ch}(B)$. Let W be any neighborhood of (x_0, y_0) with $(\tau \times \kappa)(W) = W$. Then there exist neighborhoods U of x_0 and V of y_0 such that $\tau(U) = U$ and $\kappa(V) = V$, and that $(\tau \times \kappa)(U \times V) = U \times V \subset W$. Indeed, let U_1 and V_1 be open sets so that (x_0, y_0) and $(\tau(x_0), \kappa(y_0)) \in U_1 \times V_1 \subset W$. Since $(\tau \times \kappa)(W) = W$, $\tau(U_1) \times \kappa(V_1) = (\tau \times \kappa)(U_1 \times V_1) \subset (\tau \times \kappa)(W) = W$. Take $U = U_1 \cap \tau(U_1)$, $V = V_1 \cap \kappa(V_1)$. Then $\tau(U) = U$ and $\kappa(V) = V$, and $(x_0, y_0) \in U \times V = [U_1 \cap \tau(U_1)] \times [V_1 \cap \kappa(V_1)] = [U_1 \times V_1] \cap [\tau(U_1) \times \kappa(V_1)] \subset W$.

Hence, by Lemma 1 and Lemma 2 there exists $f \in A$ which peaks on K such that $\{x_0, \tau(x_0)\} \subset K \subset U$ and there exists $g \in B$ which peaks on L such that $\{y_0, \kappa(y_0)\} \subset L \subset V$. Then $f \otimes g \in A \widehat{\otimes} B$ peaks on $K \times L$ such that

$$\{(x_0, y_0), (\tau \times \kappa)(x_0, y_0)\} \subset K \times L \subset U \times V \subset W$$

since $\|f \otimes g\| = \|f\|\|g\| = 1$, $(f \otimes g)(x, y) = f(x)g(y) = 1$ for $(x, y) \in K \times L$ and $|(f \otimes g)(x, y)| = |f(x)||g(y)| < 1$ for $(x, y) \neq K \times L$. Thus $\{(x_0, y_0), (\tau \times \kappa)(x_0, y_0)\}$ is a p-set for $A \widehat{\otimes} B$, and therefore $(x_0, y_0) \in \mathrm{Ch}(A \widehat{\otimes} B)$.

Conversely, let $(x_0, y_0) \in \mathrm{Ch}(A \widehat{\otimes} B)$. Without loss of generality, it suffices to show $x_0 \in \mathrm{Ch}(A)$. Let U be any neighborhood of x_0 with $\tau(U) = U$. Then $U \times Y$ is a neighborhood of (x_0, y_0) with $(\tau \times \kappa)(U \times Y) = U \times Y$, so there exists $F \in A \widehat{\otimes} B$ which peaks on M such that $\{(x_0, y_0), (\tau \times \kappa)(x_0, y_0)\} \subset M \subset U \times Y$. Put

$$f(x) = \frac{1}{2}[F(x, y_0) + F(x, \kappa(y_0))]$$

and $K = \{x \in X : (x, y_0) \in M\}$.

First, we will show $f \in A$. Note that $f \in C(X, \tau)$ because

$$\begin{aligned} f(\tau(x)) &= \frac{1}{2}[F(\tau(x), y_0) + F(\tau(x), \kappa(y_0))] \\ &= \frac{1}{2}[F(\tau(x), \kappa(\kappa(y_0))) + F(\tau(x), \kappa(y_0))] \\ &= \frac{1}{2}[\overline{F(x, \kappa(y_0))} + \overline{F(x, y_0)}] = \frac{1}{2}\overline{[F(x, y_0) + F(x, \kappa(y_0))]} \\ &= \overline{f(x)}. \end{aligned}$$

Since $F \in A \widehat{\otimes} B$, F is the uniform limit of a sequence $\{F_n\}$, where $F_n = \sum_{i=1}^{m(n)} f_{ni} \otimes g_{ni} \in A \otimes B$. Put $f_n(x) = \frac{1}{2}[F_n(x, y_0) + F_n(x, \kappa(y_0))]$. Then

$$\begin{aligned} f_n(x) &= \frac{1}{2}\left[\sum_{i=1}^{m(n)} f_{ni}(x) g_{ni}(y_0) + \sum_{i=1}^{m(n)} f_{ni}(x) g_{ni}(\kappa(y_0))\right] \\ &= \frac{1}{2}\sum_{i=1}^{m(n)} f_{ni}(x)[g_{ni}(y_0) + \overline{g_{ni}(y_0)}] = \sum_{i=1}^{m(n)} (\mathrm{Re}\, g_{ni}(y_0))\, f_{ni}(x), \end{aligned}$$

and so $f_n \in A$ for all $n = 1, 2, \ldots$. Now, for each $x \in X$

$$\begin{aligned}|f(x) - f_n(x)| &= \frac{1}{2}\big|F(x, y_0) + F(x, \kappa(y_0)) - F_n(x, y_0) - F_n(x, \kappa(y_0))\big| \\ &\leq \frac{1}{2}\big|F(x, y_0) - F_n(x, y_0)\big| + \frac{1}{2}\big|F(x, \kappa(y_0)) - F_n(x, \kappa(y_0))\big| \\ &\leq \|F - F_n\| \to 0 \text{ as } n \to \infty,\end{aligned}$$

and so $\|f - f_n\| = \sup |f(x) - f_n(x)| \to 0$ as $n \to \infty$. Since A is uniformly closed, $f \in A$.

Now, we will show that f peaks on K and $\{x_0, \tau(x_0)\} \subset K \subset U$ as follows: At first, $|f(x)| = \frac{1}{2}|F(x, y_0) + F(x, \kappa(y_0))| \leq \|F\| = 1$, and thus $\|f\| \leq 1$. Also, $f(x) = \frac{1}{2}[F(x, y_0) + F(x, \kappa(y_0))] = \frac{1}{2}(1 + 1) = 1$ for $x \in K$ and

$$|f(x)| \leq \frac{1}{2}\{|F(x, y_0)| + |F(x, \kappa(y_0))|\} < \frac{1}{2}(1 + 1) = 1$$

for $x \notin K$. So K is a peak set for A. Therefore, $\{x_0, \tau(x_0)\}$ is a p-set for A, and hence by Lemma 2 $x_0 \in \mathrm{Ch}(A)$. $\square$

Remark. As mentioned by a referee, one can prove this theorem using the complexification technique as follows: Denote by

$$\widetilde{E} = E + iE = \{f + ig : f, g \in E\}$$

the complexification of a real function algebra E. Then it is easy to check that $\widetilde{A}\widehat{\otimes}\widetilde{B} = \widetilde{A\widehat{\otimes}B}$. Since the Choquet boundary of the tensor product of complex function algebras is the product of their tensor products (for instance, see Leibowitz [10, Theorem 14, p. 220]), we have $\mathrm{Ch}(\widetilde{\mathrm{A}}\widehat{\otimes}\widetilde{\mathrm{B}}) = \mathrm{Ch}(\widetilde{\mathrm{A}}) \times \mathrm{Ch}(\widetilde{\mathrm{B}})$. And then apply the fact that the Choquet boundary of a real function algebra coincides with the Choquet boundary of its complexifiction (See, for instance, Kulkarni & Limaye [6, Theorem 4.3.7]).

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