J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 11, Number 2 (May 2004), Pages 117–125

WEAKLY KRULL AND RELATED PULLBACK DOMAINS

GYU WHAN CHANG

ABSTRACT. Let T be an integral domain, M a nonzero maximal ideal of T, $K = T/M$, $\varphi : T \to K$ the canonical map, D a proper subring of K, and $R = \varphi^{-1}(D)$ the pullback domain. Assume that for each $x \in T$, there is a $u \in T$ such that u is a unit in T and $ux \in R$. In this paper, we show that R is a weakly Krull domain (*resp.*, GWFD, AWFD, WFD) if and only if $\text{ht}M = 1$, D is a field, and T is a weakly Krull domain (*resp.*, GWFD, AWFD, WFD).

1. INTRODUCTION

Recall that an integral domain D is called a *weakly Krull domain* if

$$D = \cap_{P \in X^1(D)} D_P$$

and this intersection has finite character, that D is a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a primary element, that D is an *almost weakly factorial domain* (AWFD) if for each nonzero nonunit element x of D, there is a positive integer $n = n(x)$ such that x^n can be written as a product of primary elements, and that D is a *weakly factorial domain* (WFD) if each nonzero nonunit element of D is a product of primary elements. Clearly, a WFD is an AWFD and an AWFD is a GWFD. It is also known that a GWFD is a weakly Krull domain Anderson, Chang & Park [4, Corollary 2.3].

Let T be an integral domain, M a nonzero maximal ideal of T, $K = T/M$, $\varphi : T \to K$ the canonical map, D a proper subring of K, and $R = \varphi^{-1}(D)$ the

Received by the editors March 18, 2003 and, in revised form, March 27, 2004.
2000 *Mathematics Subject Classification.* 13F15, 14M05.
Key words and phrases. weakly Krull domain, GWFD, AWFD, WFD, pullback domain.

pullback domain.

$$(\square) \qquad \begin{array}{ccc} R = \varphi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & K = T/M \end{array}$$

We shall refer to R as a pullback domain of type ($\square$) and as a pullback domain of type ($\square^*$) if for each $0 \neq x \in T$, there is a $u \in U(T)$ such that $ux \in R$. If R is a pullback domain of type ($\square$), then M is a divisorial ideal (and hence a t-ideal) of R, M has the same height in both R and T, and for any prime ideal $P(\not\supseteq M)$ of R, $T \subseteq R_P$ and $R_P = T_{PR_P \cap T}$ (*cf.* Fontana & Gabelli [9, p. 805]). One can show that if $T = K + M$ (and hence $R = D + M$), then R is a pullback domain of type ($\square^*$).

In Anderson, Chang & Park [5, Section 2], we showed that if $T = K + M$ (and hence $R = D + M$), then R is a weakly Krull domain (*resp.*, GWFD, AWFD, WFD) if and only if ht$M = 1$, D is a field, and T is a weakly Krull domain (*resp.*, GWFD, AWFD, WFD). The purpose of this paper is to generalize these results to a pullback domain of type ($\square$) or ($\square^*$). That is, we show that if R is a pullback domain of type ($\square$), then R is a weakly Krull domain if and only if ht$M = 1$, D is a field, and T is a weakly Krull domain; and that if R is of type ($\square^*$), then R is a GWFD (*resp.*, AWFD, WFD) if and only if ht$M = 1$, D is a field, and T is a GWFD (*resp.*, AWFD, WFD).

Let D be an integral domain with quotient field K and I a nonzero fractional ideal of D. Then

$$I^{-1} = \{x \in K | xI \subseteq D\}, I_v = (I^{-1})^{-1}, \quad \text{and}$$
$$I_t = \cup\{J_v | (0) \neq J \subseteq I \text{ is finitely generated}\}.$$

If $I_v = I$ (*resp.*, $I_t = I$, $I = (x_1, \ldots, x_n)_v$ for some $(0) \neq (x_1, \ldots, x_n) \subseteq I$), then I is said to be a *divisorial ideal* (*resp.*, *t-ideal*, *finite type t-ideal*). An ideal of D maximal among proper integral t-ideals is called a *maximal t-ideal*. A fractional ideal I is *t-invertible* if $(II^{-1})_t = D$. It is well known that a maximal t-ideal is a prime ideal, every proper integral t-ideal is contained in a maximal t-ideal, a t-invertible t-ideal is of finite type, and a t-invertible prime t-ideal is a maximal t-ideal. We say that D has t-dimension one, denoted by t-dim$(D) = 1$, if each maximal t-ideal of D has height-one.

Let $\mathcal{T}(D)$ be the set of t-invertible fractional t-ideals of an integral domain D. Then $\mathcal{T}(D)$ is an abelian group under the t-product $I * J = (IJ)_t$, and hence the quotient group $Cl(D) = \mathcal{T}(D)/Prin(D)$, called the *class group* of D, is also an abelian group, where $Prin(D)$ is the subgroup of $\mathcal{T}(D)$ of nonzero principal fractional ideals of D. If D is a Krull domain, then $Cl(D)$ is the usual divisor class group, and if D is a Prüfer domain, then $Cl(D)$ is the ideal class group of invertible ideals (or Picard group) of D.

All rings considered in this paper are commutative integral domains with identity and for an integral domain D, $U(D)$ denotes the set of unit elements of D and $X^1(D)$ is the set of height-one prime ideals of D. A nonzero nonunit element a of D is said to be *primary* if aD is a primary ideal. It is known that if aD is primary, then $\sqrt{aD}$ is a maximal t-ideal. The reader is referred to Gilmer [12, § 32 and § 34] and Zafrullah [16] for the t-operation; to Anderson & Zafrullah [1], Anderson, Mott & Zafrullah [2], Anderson, Chang & Park [4, 5] for weakly Krull and related domains; to Brewer & Rutter [8], Fontana & Gabelli [9], Gabelli & Houstongh [11], Lucas [15] for pullback domains; to Anderson [3], Bouvier [6], Bouvier & Zafrullah [7], Fontana & Gabelli [9], Fossum [10] for the class group; and to Fossum [10], Gilmer [12], Kaplansky [14] for standard notations and definitions.

We first study when the pullback domain R is weakly Krull. Recall that a weakly Krull domain has t-dimension one Anderson, Mott & Zafrullah [2, Lemma 2.1].

Theorem 1 (*cf.* Anderson, Chang & Park [5, Theorem 2.3]). *Let R be a pullback domain of type (□). Then R is a weakly Krull domain if and only if $htM = 1$, D is a field, and T is a weakly Krull domain.*

Proof. (⇒) Assume that R is a weakly Krull domain. Then $t\text{-dim}(R) = 1$, and since M is a t-ideal of R, M is a height-one maximal t-ideal of R. If $a \in D \setminus \{0\}$, then $\varphi^{-1}(aD)$ is an invertible ideal of R such that $M \subsetneq \varphi^{-1}(aD) \subseteq R$ (*cf.* Fontana & Gabelli [9, Corollary 1.7]). Hence M being a maximal t-ideal of R implies that D is a field.

We next show that T is weakly Krull. Let $Q(\neq M)$ be a maximal ideal of T, and let $P = Q \cap R$. Then $T_Q = R_P$, and since R_P is weakly Krull Anderson, Chang & Park [5, Lemma 2.1(2)],

$$T_Q = \cap\{T_{Q'} | Q' \in X^1(T) \text{ and } Q' \subseteq Q\}$$

so

$$T = \cap_{Q \in \mathrm{Max}(T)} T_Q = \cap_{Q' \in X^1(T)} T_{Q'}.$$

Note that for each $Q' \in X^1(T) \setminus \{M\}$, $T_{Q'} = R_{Q' \cap R}$ (and hence $\mathrm{ht}(Q' \cap R) = 1$) and T is an overring of R. Hence the intersection $T = \cap_{Q' \in X^1(T)} T_{Q'}$ has finite character, and thus T is weakly Krull.

($\Leftarrow$) Assume that $\mathrm{ht}M = 1$, D is a field, and T is weakly Krull. Let M_1 be a maximal ideal of R such that $M_1 \neq M$. Then $R_{M_1} = T_Q$ for some prime ideal Q of T. Note that T_Q is weakly Krull Anderson, Chang & Park [5, Lemma 2.1(2)]; so

$$T_Q = R_{M_1} = \cap \{R_P | P \in X^1(R) \text{ and } P \subseteq M_1\}.$$

Since $R = \cap\{R_{M'} | M' \text{ is a maximal ideal of } R\}$ and $\mathrm{ht}M = 1$, we have $R = \cap_{P \in X^1(R)} R_P$. Moreover, since $R \subseteq T$ and for each $P \in X^1(R) \setminus \{M\}$, $R_P = T_{Q'}$ for some $Q' \in X^1(T)$, the intersection $R = \cap_{P \in X^1(R)} R_P$ has finite character, and thus R is weakly Krull. □

Our next corollary, which was observed in the proof of Theorem 1 above, will be very useful in the subsequent arguments.

Corollary 2. *Let R be a pullback domain of type ($\square$). If R is a weakly Krull domain, then $X^1(R) = \{Q \cap R | Q \in X^1(T)\}$ and for each $Q \in X^1(T) \setminus \{M\}$, $R_{Q \cap R} = T_Q$.*

Theorem 3 (*cf.* Anderson, Chang & Park [5, Theorem 2.4]). *If R is a pullback domain of type ($\square^*$), then R is a GWFD if and only if $\mathrm{ht}M = 1$, D is a field, and T is a GWFD.*

Proof. ($\Rightarrow$) Assume that R is a GWFD. Then since a GWFD is weakly Krull Anderson, Chang & Park [4, Corollary 2.3], by Theorem 1 above, $\mathrm{ht}M = 1$, D is a field, and T is weakly Krull (and hence $t\text{-}\dim(T) = 1$).

Let $Q \in X^1(T)$ and $P = Q \cap R$. Then $\mathrm{ht}P = 1$ (Corollary 2), and so $P = \sqrt{aR}$ for some $a \in R$ (*cf.* Anderson, Chang & Park [4, Theorem 2.2]). Thus $Q = \sqrt{aT}$ since Q is the unique prime ideal of T lying over P and $t\text{-}\dim(T) = 1$.

($\Leftarrow$) Assume that $\mathrm{ht}M = 1$, D is a field, and T is a GWFD. Then as a GWFD is weakly Krull, R is weakly Krull by Theorem 1. Let $P \in X^1(R) \setminus \{M\}$ and $Q \in X^1(T)$ such that $R_P = T_Q$ (Corollary 2). Then there is an $x \in R$ such that $Q = \sqrt{xT}$ (*cf.* Anderson, Chang & Park [4, Theorem 2.2]) since T is a GWFD and R is of type ($\square^*$). If P' is a minimal prime ideal of xR, then P' is a t-ideal of R, and hence $\mathrm{ht}P' = 1$ (note that $t\text{-}\dim(R) = 1$); so $P' = Q' \cap R$ for some $Q' \in X^1(T)$.

Hence $x \in Q'$, and so $Q = Q'$ and $P = P'$. Therefore, $P = \sqrt{xR}$, and thus R is a GWFD Anderson, Chang & Park [4, Theorem 2.2]. □

The proof of Theorem 3 shows that the " $\Rightarrow$ " implication in Theorem 3 holds for a pullback domain of type (□). Recall that an integral domain D is an AWFD if and only if D is a weakly Krull domain and $Cl(D)$ is torsion Anderson, Mott & Zafrullah [2, Theorem 3.4].

Theorem 4 (*cf.* Anderson, Chang & Park [5, Theorem 2.5]). *If R is a pullback domain of type (□*), then R is an AWFD if and only if $htM = 1$, D is a field, and T is an AWFD.*

Proof. ($\Rightarrow$) Assume that R is an AWFD. Then R is weakly Krull Anderson, Mott & Zafrullah [2, Theorem 3.4]; so by Theorem 1 and Anderson, Mott & Zafrullah [2, Theorem 3.4], it suffices to show that if J is a t-invertible t-ideal of T, then $(J^n)_t$ is principal for some integer $n \geq 1$. Since M is a t-ideal of T (note that $\text{ht}M = 1$) and J is t-invertible, $JJ^{-1} \nsubseteq M$. Thus there is a $u \in J^{-1}$ such that $uJ \nsubseteq M$. Replacing J with uJ, we may assume that $J \nsubseteq M$. Since J is t-invertible and R is of type (□*), there are some $x_1, \ldots, x_n \in R$ such that $J = ((x_1, \ldots, x_n)T)_v = (IT)_t$, where $I = (x_1, \ldots, x_n)R$.

Clearly, $I \nsubseteq M$, and hence $IR_M = R_M$. For $P \in X^1(R) \setminus \{M\}$, let $Q \in X^1(T)$ such that $Q \cap R = P$ and $R_P = T_Q$ (Corollary 2). Then since JT_Q is principal Kang [13, Corollary 2.7], $(IR_P)_t = (IT_Q)_t = ((IT)_t T_Q)_t = (IT)_t T_Q = JT_Q$ is principal Kang [13, Lemma 3.4] (note that Q is a prime t-ideal of T and J is t-invertible). So I is t-locally principal, and hence I is t-invertible Kang [13, Corollary 2.7]. Thus as $Cl(R)$ is torsion, $(I^n)_t = aR$ for some $a \in R$ and integer $n \geq 1$ Anderson, Mott & Zafrullah [2, Theorem 3.4].

We claim that $(J^n)_t = aT$. Let $Q \in X^1(T) \setminus \{M\}$ and $P = Q \cap R$. Then $T_Q = R_P$ (Corollary 2), and since $(J^n)_t$ is a t-invertible t-ideal of T, $(J^n)_t T_Q = ((J^n)_t T_Q)_t$, and hence (*cf.* Kang [13, Lemma 3.4])

$$\begin{aligned}(J^n)_t T_Q &= \left(((IT)_t)^n\right)_t T_Q = \left(((IT)^n)_t T_Q\right)_t = ((IT)^n T_Q)_t = ((IT_Q)^n)_t \\ &= ((IR_P)^n)_t = (I^n R_P)_t = ((I^n)_t R_P)_t = (aR_P)_t = aR_P = aT_Q.\end{aligned}$$

Also, since $I \nsubseteq M$, $aT \nsubseteq M$, and hence $(J^n)_t T_M = T_M = (aT)T_M$. Thus $(J^n)_t = \cap_{Q \in X^1(T)} (J^n)_t T_Q = \cap_{Q \in X^1(T)} (aT)T_Q = aT$ (*cf.* Kang [13, Proposition 2.8]).

($\Leftarrow$) Assume that ht$M$ = 1, D is a field, and T is an AWFD. Let I be a t-invertible t-ideal of R. As in the beginning of the above proof, we may assume that $I \not\subseteq M$. Since I is t-invertible, $II^{-1} \not\subseteq P$ for all $P \in X^1(R)$, and hence $II^{-1} \not\subseteq Q$ for all $Q \in X^1(T)$ by Corollary 2. Hence IT is a t-invertible ideal of T. Also, since T is an AWFD and R is of type ($\square^*$), there are an integer $n \geq 1$ and $a \in R$ such that $(((IT)_t)^n)_t = (I^nT)_t = aT$. Note that $(I^n)_t$ is a t-ideal of R, and that for each $P \in X^1(R) \setminus \{M\}$ and $Q \in X^1(T)$ with $Q \cap R = P$ (Corollary 2), $(I^nR_P)_t = (I^nT_Q)_t = ((I^nT)_tT_Q)_t$ Kang [13, Lemma 3.4]. So by Kang [13, Proposition 2.8], we have

$$\begin{aligned} (I^n)_t &= \cap_{P\in X^1(R)}(I^n)_tR_P \\ &= (I^n)_tR_M \cap \left(\cap \{(I^n)_tR_P | P \in X^1(R) \text{ and } P \neq M\}\right) \\ &= R_M \cap \left(\cap \{(I^n)_tT_Q | Q \in X^1(T) \text{ and } Q \neq M\}\right) \\ &= aR_M \cap \left(\cap \{aT_Q | Q \in X^1(T) \text{ and } Q \neq M\}\right) \\ &= \cap_{P\in X^1(R)} aR_P = aR. \end{aligned}$$

Hence R is an AWFD Anderson, Mott & Zafrullah [2, Theorem 3.4]. □

The proof of Theorem 4 yields the following theorem as a special case for $n = 1$ since R is a WFD if and only if R is weakly Krull and $Cl(R) = 0$ (*cf.* Anderson & Zafrullah [1, Theorem]).

Theorem 5 (*cf.* Anderson, Chang & Park [5, Theorem 2.6]). *If R is a pullback domain of type ($\square^*$), then R is a WFD if and only if ht$M$ = 1, D is a field, and T is a WFD.*

Remark 1. Although some parts of the proofs of Theorems 1, 3, and 4 are the same as those of their counterparts in Anderson, Chang & Park [5], we give them here for the completeness.

We end this paper with an example which shows that Theorems 3, 4, and 5 do not hold without the assumption that R is of type ($\square^*$). However, we do not know if R being of type ($\square^*$) is best possible for Theorems 3, 4 and 5.

Example 6. Let K be a field of characteristic 0, X an indeterminate over K, and Y an indeterminate over the field $K(X)$. Let $\varphi : K(X^2)[Y] \to K(X)$ be the ring homomorphism determined by $Y \mapsto X$, and let $M = ker(\varphi)$. See the following pullback diagram.

$$\begin{array}{ccc} R = \varphi^{-1}(K) & \longrightarrow & K \\ \downarrow & & \downarrow \\ T = K(X^2)[Y] & \xrightarrow{\varphi} & T/M = K(X) \end{array}$$

(1) M is a height-one maximal ideal of T such that $T/M = K(X)$.
(2) R is not of type $(\square^*)$.
(3) The map $\psi : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$, given by $Q \mapsto Q \cap R$, is bijective.
(4) $\dim(R) = \dim(T) = 1$.
(5) R is not a GWFD, while T is a PID, $\mathrm{ht}M = 1$, and K is a field.

Proof. (1) Since $Y^2 - X^2 \in M$, $M \neq (0)$, and hence M is a height-one maximal ideal of T because T is a PID. In particular, $\varphi(T) = T/M$ is a subfield of $K(X)$ containing $K(X^2)$ and X, and thus $T/M = K(X)$.

(2) Note that $U(T) = K(X^2) \setminus \{0\}$; so $Yu \notin R$ for all $u \in U(T)$. For if $Yu \in R$, then $\varphi(Yu) = Xu = a \in K$, and thus $u = \frac{a}{X} \notin K(X^2)$, a contradiction.

(3) and (4) Since K is a field, M is a maximal ideal of R. Hence if P is a prime ideal of R such that $P \neq M$, then there is a unique prime ideal Q of T such that $Q \cap R = P$ and $T_Q = R_P$ (*cf.* Fontana & Gabelli [9, p. 805]). This implies that ψ is bijective and that $\dim(R) = \dim(T) = 1$ by (1) and the fact that T is a PID.

(5) Let $Q = (Y - X^2)T$ and $P = Q \cap R$. Then $Y - X^2$ is a prime element of T, and hence Q is a prime ideal of T. Assume that $P = \sqrt{fR}$ for some $f \in R$. Then Q is a unique prime ideal of T containing f by (3) and (4); so $Q = \sqrt{fT}$. Since T is a PID, there is a positive integer n and $u \in U(T) = K(X^2) \setminus \{0\}$ such that $f = (Y - X^2)^n u$. Moreover, since $f \in R$, we have $\varphi(f) = (X - X^2)^n u = a \in K$, and hence

$$u = \frac{a}{(X - X^2)^n}.$$

However, since the characteristic of K is 0, $(X - X^2)^n \notin K[X^2]$, and thus

$$u = \frac{a}{(X - X^2)^n} \notin K(X^2),$$

a contradiction. Hence P is not the radical of a principal ideal. Therefore R is not a GWFD Anderson, Chang & Park [4, Theorem 2.2] because $\dim(R) = 1$. $\square$

ACKNOWLEDGEMENT

Example 6 was suggested by one of the referees. The author would like to thank the referees for their helpful comments and suggestions.

REFERENCES

1. D. D. Anderson & M. Zafrullah: Weakly factorial domains and groups of divisibility. *Proc. Amer. Math. Soc.* **109** (1990), no. 4, 907–913. MR **90k:**13015
2. D. D. Anderson, J. Mott & M. Zafrullah: Finite character representations for integral domains. *Boll. Un. Mat. Ital. B (7)* **6** (1992), no. 3, 613–630. MR **93k:**13001
3. D. F. Anderson: The class group and local class group of an integral domain. In: Scott T. Chapman & Sarah Glaz (Eds.), *Non-Noetherian commutative ring theory* (pp. 33–55). Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000. MR **2003b:**13015
4. D. F. Anderson, G. W. Chang & J. Park: Generalized weakly factorial domains. *Houston J. Math.* **29** (2003), no. 1, 1–13. MR **2004e:**13003
5. ______: Weakly Krull and related domains of the form $D + M$, $A + XB[X]$, and $A + X^2B[X]$. *Rocky Mountain Math. J.* To Appear.
6. A. Bouvier: Le groupe des classes d'un anneau intégré. *107ème Congrès National des Sociétés Savantes, Brest, fasc.* **IV** (1982), 85–92.
7. A. Bouvier & M. Zafrullah: On some class groups of an integral domain. *Bull. Soc. Math. Grece (N.S.)* **29** (1988), 45–59. MR **91g:**13015
8. J. Brewer & E. Rutter: $D + M$ constructions with general overrings. *Michigan Math. J.* **23** (1976), no. 1, 33–42. MR **53#**5571
9. M. Fontana & S. Gabelli: On the class group and the local class group of a pullback. *J. Algebra* **181** (1996), no. 3, 803–835. MR **97h:**13011
10. R. Fossum: *The divisor class group of a Krull domain.* Springer-Verlag, New York-Heidelberg, 1973. MR **52#**3139
11. S. Gabelli & E. Houston: Ideal theory in pullbacks. In: Scott T. Chapman & Sarah Glaz (Eds.), *Non-Noetherian commutative ring theory* (pp. 199–227). Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000. MR **2003a:**13001
12. R. Gilmer: *Multiplicative ideal theory.* Pure and Applied Mathematics, No. 12. Marcel Dekker, Inc., New York, 1972. MR **55#**323
13. B. G. Kang: Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. *J. Algebra* **123** (1989), no. 1, 151–170. MR **90e:**13017
14. I. Kaplansky: *Commutative rings.* Revised edition. The University of Chicago Press, Chicago, Ill.-London, 1974. MR **49#**10674

15. T. G. Lucas: Examples built with $D + M$, $A + XB[X]$ and other pullback constructions. In: Scott T. Chapman & Sarah Glaz (Eds.), *Non-Noetherian commutative ring theory* (pp. 341–368). Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000. MR **2002g:**13007
16. M. Zafrullah: Putting t-invertibility to use. In: Scott T. Chapman & Sarah Glaz (Eds.), *Non-Noetherian commutative ring theory* (pp. 429–457). Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000. MR **2002g:**13009

Department of Mathematics, University of Inchon, 177 Dohwa-dong, Nam-gu, Incheon 402-749, Korea
Email address: `whan@incheon.ac.kr`