Robust Optimal Control of Robot Manipulators with a Weighting Matrix Determination Algorithm

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ABSTRACT

A robust optimal control design is proposed in this study for rigid robotic systems under the unknown loads and the other uncertainties. The uncertainties are reflected in the performance index, where the uncertainties are bounded for the quadratic square of the states with a positive definite weighting matrix. An iterative algorithm is presented for the determination of the weighting matrix required for necessary robustness. Computer simulations have been done for a weight-lifting operation of a two-link manipulator and the simulation results shows that the proposed algorithm is very effective for a robust control of robotic systems.

Key Words : optimal control, uncertainties, weighting matrix, iterative method, robust control

1. Introduction

The motion control of a robot manipulator has received a great deal of attention in the past decade. Many approaches have been introduced to treat this control problem[1]. Because of the unknown load placed on the manipulator and the other uncertainties in the manipulator dynamics, adaptive control approaches and robust control approaches have been proposed to attenuate these uncertainties. Johansson[2] proposed explicit solutions to the Hamilton-Jacobi equation for optimal control of a rigid body motion, and designed adaptive control for self-optimization to solve the case of unknown or uncertain system parameters. Chen[3] proposed a mixed H_2/H_∞ control design for tracking of rigid robotic systems under parameter perturbations and external disturbances. And Lin[4] translated the robust control problem into an optimal control problem, where the uncertainties were reflected in the performance index.

Manuscript received: January 8, 2004 ; Accepted: March 24, 2004

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This study has been done based on Lin's work. The dynamics of a robot manipulator is to be written as the state space description with the state variables, control inputs, and uncertainties. By expressing the uncertainties as the function of the state variables, the robust control problem can be translated into the optimal control problem. Here, the uncertainties are defined to be bounded for the quadratic square of the states with a positive definite weighting matrix. Once a suitable weighting matrix could be selected, the solution of Riccati equation for the cost function of Eq. (11) could guarantee the required control performance, which will be explained in detail in Section 3. And then, we focus on how to select the suitable (or best) weighting matrix. The positive definite matrix could be defined as the product of a scalar and an identity matrix and this problem is simplified to select a scalar value. For this purpose, an iterative algorithm is devised in sense that the scalar value should be minimum value (converged) satisfying the required control accuracy, but it should be searched as a largest value for the required robustness. It will be explained in detail in Section 4.

The proposed algorithm has been used for the computer simulation of a weight-lifting operation of a two-link manipulator. The result of the simulation shows that the algorithm is very effective as desired for a robust control of robotic systems.

2. Manipulator Dynamics

The dynamics of a robot manipulator is well understood and is given by

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) \quad (1)$$

The position coordinates q with associated velocities $\dot{q}$ and accelerations $\ddot{q}$ are controlled by the driving forces τ. The moment of inertia $M(q)$, the Coriolis, centripetal, and frictional forces $C(q,\dot{q})\dot{q}$, and the gravitational forces $G(q)$ all vary along the trajectories. For simplicity, we denote

$$N(q,\dot{q}) = C(q,\dot{q})\dot{q} + G(q). \quad (2)$$

There are uncertainties in $M(q)$ and $N(q,\dot{q})$ due to unknown load on the manipulator and unmodeled frictions. The following bounds on the uncertainties could be assumed as follows:

1) There exists $M_o(q)$ such that $M(q) \le M_o(q)$.
2) There exists $N_o(q,\dot{q})$ such that

$$\|N(q,\dot{q})\| \le \|N_o(q,\dot{q})\|.$$

The dynamics of a robot manipulator could be reformulated as

$$\begin{aligned}\ddot{q} &= M^{-1}(\tau - N) \\ &= M^{-1}(\tau - N) - M^{-1}N_o + M^{-1}N_o \\ &= M^{-1}(\tau - N_o) + M^{-1}(N_o - N) \\ &= M^{-1}M_oM_o^{-1}(\tau - N_o) + M^{-1}M_oM_o^{-1}(N_o - N) \quad (3)\end{aligned}$$

where M and N are the shorter notation of $M(q)$ and $N(q,\dot{q})$, and M_o and N_o are the shorter notation of $M_o(q)$ and $N_o(q,\dot{q})$, respectively.

Let us define the control input u and the uncertainty w as

$$u = M_o^{-1}(\tau - N_o), \; w = M_o^{-1}(N_o - N). \quad (4)$$

The joint accelerations $\ddot{q}$ are given by

$$\ddot{q} = M^{-1}M_o u + M^{-1}M_o w. \quad (5)$$

The state variables are

$$x = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}. \quad (6)$$

Then, the state equation is given by

$$\dot{x} = Ax + Bu + Bw. \quad (7)$$

where

$$A = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \; B = M^{-1}M_o \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (8)$$

3. Optimal Control Approach

The robust control problem can be translated into the optimal control problem. If the solution to the translated optimal control problem exists, it is a solution to the robust control problem[4]. With this concept we reformulate this methodology as follows.

Our goal is to solve the following robust control problem.

1) Robust Control Problem: Find a feedback control law such that the closed-loop system as $\dot{x} = Ax + Bu + Bw$ is globally asymptotically stable for all uncertainties w satisfying the condition that there exists a nonnegative function w_o such that $\|w\| \le w_o$.

This robust control problem is translated into the following optimal control problem.

2) Optimal Control Problem: For the following system as $\dot{x} = Ax + Bu$, find a feedback control law that minimizes the following cost function:

$$J = \frac{1}{2}\int_0^\infty \left(w_o^T w_o + x^T Qx + u^T Ru\right)dt. \quad (9)$$

In order to translate the robust control problem into the optimal control problem, we need to assume that the uncertainty w satisfies the following condition:

$$w^T Rw < x^T Q_w x \quad (10)$$

for some positive definite matrix Q_w.

Then the optimal control problem reduces to the following linear quadratic regulator (LQR) problem: For the system as $\dot{x} = Ax + Bu$, find a feedback control law that minimizes the following cost function:

$$J = \frac{1}{2}\int_0^\infty \left(x^T Q_w x + x^T Qx + u^T Ru\right)dt. \quad (11)$$

The Hamiltonian is

$$H = \frac{1}{2}\left(x^T Q_w x + x^T Qx + u^T Ru\right) + J_x^{*T}(Ax + Bu). \quad (12)$$

where the minimum cost function J^* is

$$J^* = \min_u \left\{ \frac{1}{2} \int_0^\infty \left(x^T Q_w x + x^T Qx + u^T Ru\right) dt \right\}. \quad (13)$$

The $u = u^*$ for which H has its minimum value is obtained from the partial derivatives with respect to u .

$$\frac{\partial H}{\partial u} = 0, \quad J_x^{*T} B = -\left(u^*\right)^T R. \quad (14)$$

The $Hamilton - Jacobi$ equation gives us such that

$$H^* = 0, \quad J_x^{*T}\left(Ax + Bu^*\right) = -\frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right). \quad (15)$$

Let us define the Lyapunov function candidate V as the minimum cost function J^* :

$$V = J^* = \min_u \left\{ \frac{1}{2} \int_0^\infty \left(x^T Q_w x + x^T Qx + u^T Ru\right) dt \right\}. \quad (16)$$

From Eq.(14), Eq.(15) and Eq.(16), we obtain the following Equations:

$$V_x^T B = -\left(u^*\right)^T R , \quad (17)$$

$$V_x^T\left(Ax + Bu^*\right) = -\frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right). \quad (18)$$

In order to show $\dot{V} = dV / dt < 0$, we have

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x}\frac{dx}{dt} \\ &= V_x^T\left(Ax + Bu^* + Bw\right) \\ &= V_x^T\left(Ax + Bu^*\right) + V_x^T Bw \\ &= -\frac{1}{2}\left(x^T Q_w x + x^T Qx + \left(u^*\right)^T R\left(u^*\right)\right) - \left(u^*\right)^T Rw \\ &= -\frac{1}{2}\left(x^T Q_w x - w^T Rw\right) - \frac{1}{2}x^T Qx - \frac{1}{2}\left(u^* + w\right)^T R\left(u^* + w\right). \end{aligned} \quad (19)$$

With the condition $x^T Q_w x - w^T Rw > 0$ given in Eq. (10), the Lyapunov function derivative is negative definite. Thus, the condition of the Lyapunov global asymptotic stability theorem is satisfied. The optimal control can be obtained by solving the following algebraic Riccati equation:

$$0 = Q_w + Q - PBR^{-1}B^T P + PA + A^T P \quad (20)$$

The optimal control is given by

$$u^* = -R^{-1}B^T Px. \quad (21)$$

4. Determination of the Weighting Matrices

We need to choose the weighting matrices Q, R and Q_w to find the optimal control u^* . According to the selection of Q , R and Q_w , the control performance becomes quite different in result. So it is very important to select the Q , R and Q_w . If the matrix Q is determined that $Q = I$, the problem to select the Q, R and Q_w is reduced to the problem to select the Q_w and R . Let us define $R = rI$ and $Q_w = q_w I$. Then the condition $w^T Rw < x^T Q_w x$ can be expressed as

$$\frac{\|w\|^2}{\|x\|^2} < \frac{q_w}{r} = \gamma . \quad (22)$$

The problem to select Q_w and R is simplified to the problem to determine the scalar value γ . For this purpose, an iterative algorithm is devised in sense that the scalar value should be minimum value (converged) satisfying the required control accuracy, but it should be searched as a largest value for the required robustness. The iterative algorithm is summarized as follows:

1) Given r , initial scalar value γ is assumed and the optimal control simulation is performed.
2) The scalar value γ at each time instance is computed and its largest value is selected.
3) With the new cost function including the selected largest weighting value γ , the optimal control simulation is re-performed.
4) Processes 2) and 3) are repeated until the convergence criterion is satisfied.

To get the weighting value γ on the finite time interval $[0, N]$, let us define the state variables x and the uncertainties w as follows:

$$x_k = (x_k(0), x_k(1), ..., x_k(N)) \quad (23)$$

$$w_k = (w_k(0), w_k(1), ..., w_k(N)) \quad (24)$$

where the subscript k denotes the k th trial.

The weighting value γ_k in the k th trial is selected as its largest value on the finite time interval $[0, N]$. It is defined as the following $l_\infty - norm$:

$$\gamma_k = \|\Gamma_k\|_\infty = \max_{0 \le i \le N} |\Gamma_k(i)| \quad (25)$$

where

$$\Gamma_k = (\Gamma_k(0), \Gamma_k(1), ..., \Gamma_k(N)), \quad (26)$$

$$\Gamma_k(i) = \frac{\|w_k(i)\|^2}{\|x_k(i)\|^2}, \quad i = 0, ..., N. \quad (27)$$

To avoid $\|x_k(i)\|^2 = 0$, a dummy variable δ is used as follows:

$$\Gamma_k'(i) = \frac{\|w_k(i)\|^2}{\|x_k(i)\|^2 + \delta}, \quad \delta \approx 0. \quad (28)$$

Now, we have

$$\Gamma_k(i) = \frac{\|w_k(i)\|^2 \Gamma_k'}{\|w_k(i)\|^2 - \delta \Gamma_k'}. \quad (29)$$

The processes to determine the weighting value γ are repeated until the following convergence criterion is satisfied such as

$$\| \gamma_k - \gamma_{k-1} \| < \varepsilon. \quad (30)$$

where ε is a given error requirement.

5. Example

Fig. 1 illustrates the proposed optimal control approach using an example of a two-link robot manipulator with point masses m_1, m_2 (*kg*), lengths l_1, l_2 (*m*), angular positions q_1, q_2 (*rad*), and torques τ_1, τ_2 (*Nm*). The parameters for the equation of motion are

$$M(q) = \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{pmatrix},$$

$$C(q, \dot{q}) = m_2 l_1 l_2 (c_1 s_2 - s_1 c_2) \begin{pmatrix} 0 & -\dot{q}_2 \\ +\dot{q}_1 & 0 \end{pmatrix},$$

$$G(q) = \begin{pmatrix} -(m_1 + m_2) l_1 g s_1 \\ -m_2 l_2 g s_2 \end{pmatrix}.$$

The short-hand notations $c_1 = \cos(q_1)$, $s_1 = \sin(q_1)$, $c_2 = \cos(q_2)$ and $s_2 = \sin(q_2)$ are used.

For the convenience of simulation, the nominal parameters of the robotic system are given as $m_1 = 1(kg)$, $m_2 = 10(kg)$, $l_1 = 1(m)$, $l_2 = 1(m)$ and the initial values $q_1 = q_2 = \pi/2$ (rad), $\dot{q}_1 = \dot{q}_2 = 0$. The reference values are $q_r = 0$ and $\dot{q}_r = 0$. $M(q)$ and $N(q, \dot{q})$ are the function of q and $\dot{q}$.

The values of $M(q)$ and $N(q, \dot{q})$ keep changing according to the motion of the manipulator and it is not easy to get their boundaries. But in the regulator problem of weight-lifting operation, if it is assumed that the dynamics of the manipulator includes the uncertainty of the mass of the unknown load only, it is not difficult to calculate $M(q)$ and $N(q, \dot{q})$ when the mass of the unknown load is the maximum value. They are selected as $M_o(q)$ and $N_o(q, \dot{q})$ and used in simulations. If the maximum value is 10(*kg*), M_o, C_o, G_o are given by

$$M_o(q) = \begin{pmatrix} 21 & 20(s_1 s_2 + c_1 c_2) \\ 20(s_1 s_2 + c_1 c_2) & 20 \end{pmatrix},$$

$$C_o(q, \dot{q}) = 20(c_1 s_2 - s_1 c_2) \begin{pmatrix} 0 & -\dot{q}_2 \\ +\dot{q}_1 & 0 \end{pmatrix},$$

$$G_o(q) = \begin{pmatrix} -21 g s_1 \\ -20 g s_2 \end{pmatrix}.$$

Iteration method is initialized by the weighting matrices $Q = I$, $R = I$, and the weighting value $\gamma = 0$.

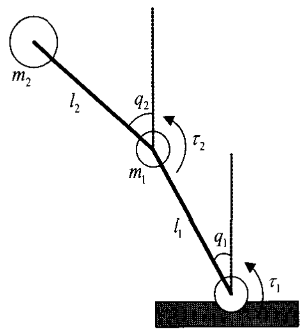

Fig. 1 The two-link manipulator.

The result of iterative method for the mass of the load $m_L = 0$ is shown in Table 1. In the k th trial, the largest weighting value γ_k (denoted as *gamma* in Table 1) on the finite time interval, and γ'_k (*gamma'*), the very moment t, the squared value of the state variable on the final time interval $\|x_k(N)\|^2$ (*xx(5001)*), the sum of the squared value of the state variable on the finite time interval $\sum_{i=0}^{N} \|x_k(i)\|^2$ (*xxSUM*), and the cost function J (*JSUM*) are listed in Table 1.

Table 1 The result of Iterative Method for $m_L = 0$, $\gamma'_k = (\gamma'_{k-1} + \gamma_k)/2$.

k	*t*	*gamma*	*gamma'*	*xx(5001)*	*xxSUM*	*JSUM*
000	0.000	00000.000000	00000.000000	00.000000	009836.282035	000025090.563501
001	2.894	35214.557504	17607.278752	00.000000	002475.407387	043729280.111337
002	0.000	00004.865442	08806.072097	00.000000	002478.113223	021920072.912788
003	0.000	00004.865442	04405.468770	00.000000	002482.352965	011005327.965142
004	0.000	00004.865442	02205.167106	00.000000	002489.043725	005540294.228730
005	0.000	00004.865442	01105.016274	00.000000	002499.909716	002802513.246557
006	0.000	00004.865442	00554.940858	00.000000	002518.140607	001430139.688169
007	0.058	00005.000576	00279.970717	00.000000	002549.343326	000741751.333341
008	0.334	00006.210467	00143.090592	00.000000	002602.273032	000397327.098876
009	0.331	00007.913236	00075.501914	00.000000	002687.932188	000225942.496634
010	0.341	00009.662743	00042.582328	00.000000	002812.639578	000141548.269939
011	0.500	00011.318062	00026.950195	00.000000	002961.734013	000100917.064449
012	0.517	00027.498686	00027.224440	00.000000	002957.832099	000101635.663050
013	0.516	00027.005438	00027.114939	00.000000	002959.381718	000101348.772558
014	0.516	00027.199120	00027.157030	00.000000	002958.784768	000101459.053461
015	0.516	00027.124580	00027.140805	00.000000	002959.014684	000101416.543630
016	0.516	00027.153300	00027.147052	00.000000	002958.926124	000101432.912580
017	0.516	00027.142239	00027.144646	00.000000	002958.960235	000101426.606936
018	0.516	00027.146500	00027.145573	00.000000	002958.947096	000101429.035614
019	0.516	00027.144859	00027.145216	00.000000	002958.952157	000101428.100129
020	0.516	00027.145491	00027.145353	00.000000	002958.950208	000101428.460454
021	0.516	00027.145247	00027.145300	00.000000	002958.950958	000101428.321665
022	0.516	00027.145341	00027.145321	00.000000	002958.950669	000101428.375123
023	0.516	00027.145305	00027.145313	00.000000	002958.950781	000101428.354532
024	0.516	00027.145319	00027.145316	00.000000	002958.950738	000101428.362463
025	0.516	00027.145313	00027.145315	00.000000	002958.950754	000101428.359408
026	0.516	00027.145315	00027.145315	00.000000	002958.950748	000101428.360585
027	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360132
028	0.516	00027.145315	00027.145315	00.000000	002958.950749	000101428.360306
029	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360239
030	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360265
031	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360255
032	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360259
033	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360257
034	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360258
035	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360258
036	0.516	00027.145315	00027.145315	00.000000	002958.950750	000101428.360258

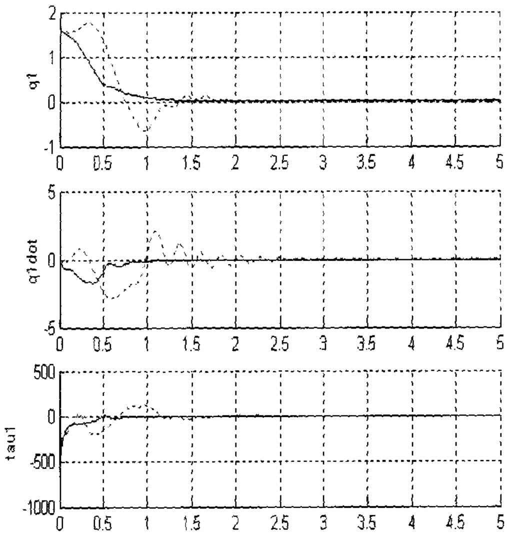

Fig. 2 Response for the mass of the load m_L=0(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
All graphs versus time(s).

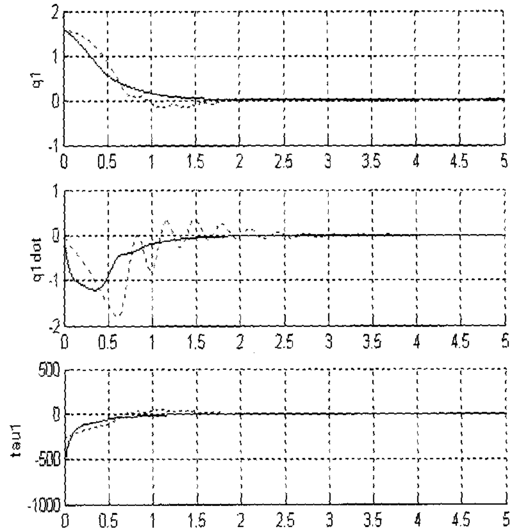

Fig. 4 Response for the mass of the load m_L=5(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
All graphs versus time(s).

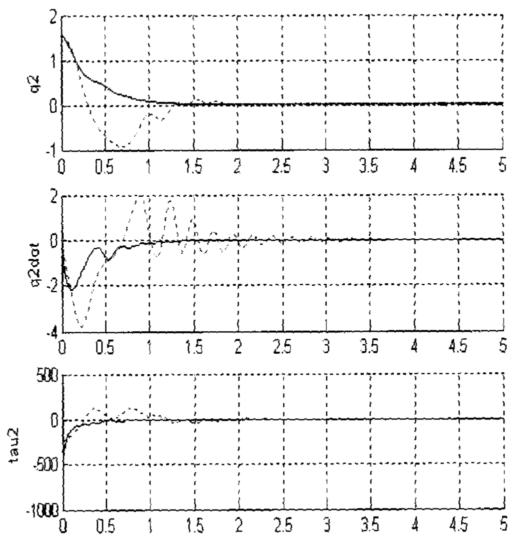

Fig. 3 Response for the mass of the load m_L=0(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
All graphs versus time(s).

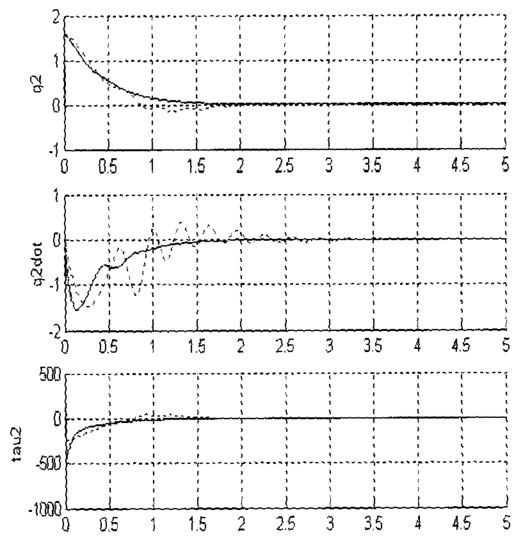

Fig. 5 Response for the mass of the load m_L=5(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
All graphs versus time(s).

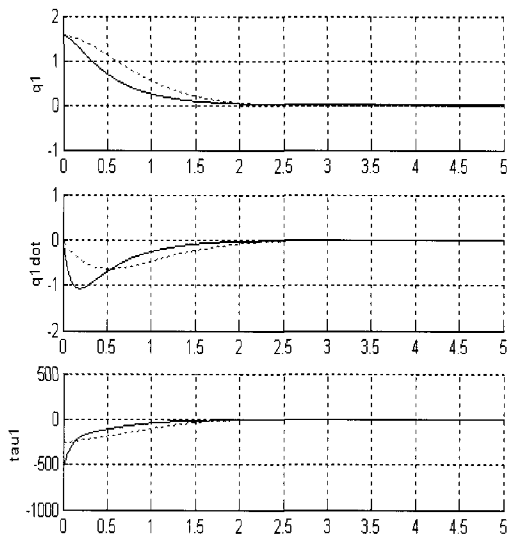

Fig. 6 Response for the mass of the load m_L=10(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
All graphs versus time(s).

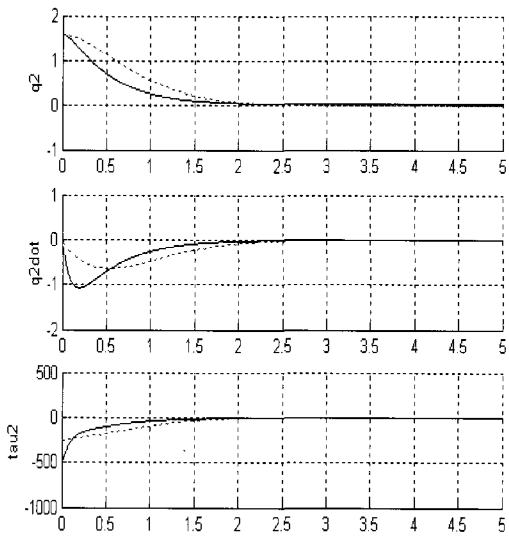

Fig. 7 Response for the mass of the load m_L=10(kg).
Dotted line is the response for γ=0.
Solid line is the response for γ=27.145315.
All graphs versus time(s).

In the first trial, we get the big weighting value *gamma*. By applying the *gamma* to the system, the sum of the squared value of the state variable on the finite time interval *xxSUM* decreases but the cost function *JSUM* increases. We used $\gamma'_k = (\gamma'_{k-1} + \gamma_k)/2$ instead of γ_k for the k th trial and the weighting value γ has converged. Since the 34th trial, the largest weighting value γ and the very moment t converged to $\gamma = 27.145315$, $t = 0.516$. And *xxSUM* and *JSUM* keep the value such as $xxSUM = 2958.95075$, $JSUM = 101428.360258$ which represent that the system is in the steady state.

The weighting matrix selected from the proposed iterative method $Q_w = 27.145315I$ and the weighting matrices $Q = I$, $R = I$ are used to solve the algebraic Riccati equation. The simulation results for γ=0 and γ=27.145315 are shown in Figs. 2-7. In Fig. 2, upper graph shows the joint position q1, middle graph shows the joint velocity q1dot and lower graph shows the applied torque τ1 for m_L = 0, respectively. In Fig. 3, upper graph shows the joint position q2, middle graph shows the joint velocity q2dot and lower graph shows the applied torque τ2 for m_L = 0, , respectively. Figs. 4-5 are the results for m_L = 5 and Figs. 6-7 are the results for m_L = 10. From the figures we can see that the control is very robust with respect to the change in the load.

6. Conclusion

We presented a robust optimal control of robot manipulators with the algorithm to determine the weighting matrix. The dynamics of a robot manipulator has been written as the state space description including uncertainties. By expressing uncertainties as the function of the state variables the robust control problem was translated into the optimal control problem. In order to guarantee the required robustness the right selection of Q_w and R is necessary. The selection problem was simplified to determine γ-value. For this, we proposed an algorithm that searches the largest value of the uncertainties on the finite time interval iteratively. The weighting matrix selected by the proposed algorithm has been used in our simulations. Simulations have been done for a weight-lifting operation of a two-link manipulator and the result of the simulation shows that the proposed algorithm is very effective for a robust control of robotic systems.

Acknowledgement

This work was supported in part by University of Ulsan and also by KOSEF through the Network–based Automation Research Center(NARC) at University of Ulsan.

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