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# THE PL FIBRATORS AMONG GEOMETRIC 4-MANIFOLDS

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ABSTRACT. Fibrators are closed manifolds which afford instant recognition of approximate fibrations. In this note we determine which 4-manifolds with geometric structure are fibrators.

## 1. Introduction

Approximate fibrations form a useful class of maps, in part, because they provide computable relationships involving the domain, image and homotopy fiber.

Fix a closed, connected $n$-manifold $N$. A proper PL map $p : M \to B$ from an $(n+k)$-manifold $M$ into a polyhedron $B$ is *$N$-like* if each fiber collapses to an $n$-complex homotopy equivalent to $N$; $N$ is a *codimension-$k$ (orientable) PL fibrator* if, for every $N$-like map $p : M \to B$, where $M$ is an (respectively, orientable) PL $(n+k)$-manifold, $p$ is an approximate fibration. Codimension-$k$ PL fibrators are necessarily codimension-$k-1$ PL fibrators as well. Moreover, $N$ is a PL (orientable) fibrator if it is a codimension-$k$ (orientable) PL fibrator for all $k > 0$.

Closed surfaces with geometric structure of $S^2$ or $H^2$ are known to be codimension-2 PL fibrators, and there are only two closed surfaces with geometric structure of $E^2$, namely the torus $S^1 \times S^1$ and the Klein bottle $K$, which are known to be the only non-codimension-2 PL fibrators among closed surfaces [2, 5, 28, 29]. Moreover, every closed orientable surface of genus at least 2 is a PL fibrator [4, Theorem 5.9]. On the

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other hand, there are eight 3-dimensional geometries:

$$E^3,\ H^3,\ S^3,\ S^2 \times E^1,\ H^2 \times E^1,\ SL_2(R),\ Nil, \text{ and } Sol.$$

All closed 3-manifolds with geometric structure of $SL_2(R)$ or $Sol$ are codimension-2 PL fibrators, as they have hyperhopfian fundamental groups ([3, Lemma 5.1] and [3, Theorem 7.2]). And having only four manifolds such as $S^2 \times S^1, S^2 \tilde{\times} S^1, RP^2 \times S^1$ and $RP^3 \# RP^3$, all closed 3-manifolds with geometric structure modelled on $S^2 \times E^1$ fail to be codimension-2 PL fibrators. Moreover, every closed hyperbolic 3-manifold is a codimension-2 PL fibrator, as they have hyperhopfian fundamental groups ([11, Corollary 6]).

In a similar vein it is natural to ask

"*What can we say about geometric* $4-$*manifolds?*"

The classification of 4-dimensional geometries in the sense of Thurston was first achieved by R. Filipkiewicz [16]. It turned out there are 19 maximal 4-dimensional geometries. The geometry is determined by the homotopy type, and indeed by the fundamental group ([30] and [32]). Recently Hillman [20, 21, 22, 23, 24, 25] extensively examined the properties of geometric 4-manifolds. In this note as a part of the answer to the question above, combining his results with recent results on fibrators, we determine which 4-manifolds with geometric structure are codimension-$k$ PL fibrators.

## 2. Preliminaries

A group $\Gamma$ is *hopfian* if every epimorphism $\Gamma \to \Gamma$ is an automorphism. Finitely presented groups need not be hopfian, as evidenced by the famous Baumslag-Solitar groups such as $\Gamma = \langle a, b \mid ab^2a^{-1} = b^3 \rangle$; like any finitely presented group, $\Gamma$ is the fundamental group of some closed 4-manifold. As far as we know, fundamental groups of closed 3-manifolds are hopfian [19].

A group $\Gamma$ is said to be *residually finite* if for any non-trivial element $x$ of $\Gamma$ there is a homomorphism $f$ from $\Gamma$ onto a finite group $K$ such that $f(x) \neq 1_K$. Every finitely generated residually finite group is hopfian.

A group $\Gamma$ is *hyperhopfian* if every endomorphism $\psi : \Gamma \to \Gamma$ with normal image and cyclic cokernel is necessarily an automorphism. Clearly groups $\Gamma$ having a cyclic direct factor fail to be hyperhopfian; however, most knot groups [31], fundamental groups of compact hyperbolic manifolds [11, Corollary 6], as well as all nontrivial free products of hopfian groups except $\mathbb{Z}_2 * \mathbb{Z}_2$ [5, 15, 17], are hyperhopfian.

So far the best well-known fact for codimension-2 PL fibrators can be described as follows;

PROPOSITION 2.1. [2, 5, 18, 26, 28, 29] *Let $N$ be a closed $n$-manifold ($n \leq 4$). If either $\pi_1(N)$ is hopfian and $\chi(N) \neq 0$ or $\pi_1(N)$ is hyperhopfian, then $N$ is a codimension-2 PL fibrator.*

Any closed manifold that cyclically cover itself (nontrivially) fails to be a codimension-2 PL fibrator, for example, $S^1$ and $RP^n \# RP^n$ [2, Theorem 4.2]. Hence the mapping torus of a periodic self homeomorphism of any closed manifold fails to be a codimension-2 PL fibrator.

The following proposition is useful when we determine whether a product of two manifolds is not a codimension-$k$ PL fibrator.

PROPOSITION 2.2. *Let $N_1^n$ and $N_2^m$ be closed manifolds. If $N_1$ is not a codimension-$k$ PL fibrator, $N_1 \times N_2$ is not a codimension-$k$ PL fibrator.*

PROOF. Take an $N_1$–like map $p : M^{n+k} \to B^k$ which fails to be an approximate fibration. Then the composition map

$$M^{n+k} \times N_2^m \xrightarrow{projection} M^{n+k} \xrightarrow{p} B^k$$

fails to be an approximate fibration. □

## 3. Geometric 4-manifolds as PL fibrators

The classification of 4-dimensional geometries in the sense of Thurston was first achieved by R. Filipkiewicz [16]. It turned out there are 19 maximal 4-dimensional geometries: and one ($F^4$) of them is not realizable by any closed manifold. So we only consider eighteen 4-dimensional geometries.

Now let us determine which 4-manifolds with geometric structure are codimension-$k$ PL fibrators. As the first step, we will determine 4-manifolds with geometric structure are codimension-2 PL fibrators, for codimension-$k$ PL fibrators are necessarily codimension-($k-1$) PL fibrators as well. While every closed surface has a residually finite fundamental group (so is hopfian), some closed 4-manifold may have a nonhopfian fundamental group, since any finitely presented group can be realized by a closed orientable 4-manifold. If a closed manifold has a non-hopfian fundamental group, we don't have any tool to show that it is a codimension-2 PL fibrator. We will check whether given closed 4-manifolds with given geometric structure have hopfian fundamental groups.

**The Geometry of Compact models : $S^4$, $CP^2 = U(3)/U(2)$ and $S^2 \times S^2$.**

There are only eleven closed 4-manifolds of this type [21]. Having finite fundamental group, all closed 4-manifolds with compact geometric structure have non-zero Euler characteristics, and so they are codimension-2 PL fibrators by Proposition 2.1.

Moreover, there are only two $S^4$-manifolds, namely $RP^4$ and $S^4$, which are known to be codimension-4 PL fibrators but fail to be codimension-5 PL fibrators ([9, Proposition 1] and [2, Theorem 2.1]). Consequently, all $S^4$-manifolds are codimension-4 PL fibrators but fail to be codimension-5 PL fibrators.

There is only one $CP^2$-manifolds, namely $CP^2$. While $CP^n$ is known to be a codimension-$(2n+2)$ PL fibrator, $CP^2$ is known to be a codimension-7 PL fibrator [8, Theorem 7], but we still don't know whether it is a PL fibrator.

Finally, there are eight $S^2 \times S^2$-manifolds. Among them, $S^2 \times S^2$, $RP^2 \times S^2$ and $RP^2 \times RP^2$ are codimension-2 PL fibrators but fail to be codimension-3 PL fibrators. And $S^2 \tilde{\times} S^2 \cong CP^2 \# - CP^2$ is a codimension-5 PL fibrator [12, Theorem 5.2]. There is nontrivial $RP^2$-bundle over $S^2$, which is homeomorphic to $RP^4 \# CP^2$. It is a codimension-5 PL fibrator [12, Theorem 5.2].

**The Geometry of Modelled on $S^2 \times R^2$ or $S^3 \times R$.**

(1) **The Geometry of $S^2 \times H^2$.**

Having a finite covering $S^2 \times B$, where $B$ is a closed orientable hyperbolic surface ([20, 21, 22]), closed 4-manifolds with geometric structure of $S^2 \times H^2$ have negative Euler characteristics and residually finite fundamental groups so that they are codimension-2 PL fibrators.

(2) **The Geometry of $S^2 \times E^2$.**

Having a finite covering $S^2 \times S^1 \times S^1$ [21], closed 4-manifolds with geometric structure of $S^2 \times E^2$ have zero Euler characteristics and their fundamental groups are virtually $\mathbb{Z} \times \mathbb{Z}$, which are residually finite. $S^2 \times S^1 \times S^1$ is obviously not a codimension-2 PL fibrator by Proposition 2.2.

An $S^1 \times S^1$-bundle over $RP^2$ which does not fiber over $S^1$ or an $RP^2\#RP^2$-bundle over $RP^2$ which does not fiber over $S^1$ is codimension-2 PL fibrator, for each have hyperhopfian fundamental group (all possible groups are $(\mathbb{Z}_2 * \mathbb{Z}_2) * (\mathbb{Z}_2 * \mathbb{Z}_2)$ and $(\mathbb{Z}_2 * \mathbb{Z}_2) * (\mathbb{Z} \oplus \mathbb{Z}_2), \mathbb{Z} * (\mathbb{Z}_2 * \mathbb{Z}_2)$, respectively [22]).

(3) **The Geometry of $S^3 \times E^1$.**

Closed 4-manifolds $N$ with geometric structure $S^3 \times E^1$ have a finite covering $S^3 \times S^1$ [21]. So $\chi(N) = 0$ and $\pi_1(N)$ is residually finite. $S^3 \times S^1$ is obviously not a codimension-2 PL fibrator by Proposition 2.2.

**Aspherical geometries:**

(1) **The Geometry of Type A.**

There are six geometries: $E^4$, $Nil^4$, $Nil^3 \times E^1$, $Sol^4_{m,n}$, $Sol^4_0$ and $Sol^4_1$, which are realized by infrasolvmanifols. Every closed 4-manifold $N$ with geometric structure of Type A has a residually finite $\pi_1(N)$ and $\chi(N) = 0$.

Every orientable closed 4-manifold which admits one of geometries $Nil^4$-manifold or $Sol^4_1$ is the mapping torus of a self homeomorphism of a $Nil^3$-manifold [21, p. 95]. But not every $Nil^4$-manifold need to be mapping tori. Hence some orientable closed 4-manifolds which admits one of geometries $Nil^4$-manifold or $Sol^4_1$ fail to be a codimension-2 PL fibrator.

(2) **The Geometry of Type B.**

There are six geometries: $H^3 \times E^1$, $SL_2(R) \times E^1$, $H^2 \times E^2$, $H^2 \times H^2, H^4$ and $H^2(C)$.

Every closed 4-manifold $N$ with geometric structure of either $H^2 \times H^2$, $H^4$ or $H^2(C) = SU(2,1)/S(U(2) \times U(1))$ is a codimension-2 PL fibrator, as $\chi(N) > 0$ ([30, Lemma 1], [32]).

Good characterizations of the possible fundamental groups of Type B are unknown so far.

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