

ON POINTS OF ZERO CHARACTERISTIC PROPERTIES IN DYNAMICAL SYSTEMS

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ABSTRACT. In this paper, we study relationships between zero characteristic properties and minimality of orbit closures or limit sets of points. Also, we characterize the set of points of zero characteristic properties. We show that the set of points of positive zero characteristic property in a compact spaces X is the intersection of negatively invariant open subsets of X .

1. Introduction and definitions

Classification and characterization of flows topologically in terms of the characteristic properties of points and dynamically related sets play an important role in stability theory of dynamical systems theory.

There are many stabilities of points in stability theory of dynamical systems such as Lyapunov stability and orbital stability. Zero characteristic properties we will deal with in this paper are fairly general and weak notions among various stabilities of points.

Continuous flows with zero characteristic properties were introduced by Ahmad in 1970 and, since then, many mathematicians study flows and continuous maps with zero characteristic properties.

The purpose of this paper is to study the dynamical properties of points of various zero characteristic properties.

First, we consider dynamical properties of limit sets of points of zero characteristic properties and relationships between zero characteristic concepts and minimality of orbit closures or limit sets of points and also study zero characteristic properties of orientation preserving conjugate

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flows. Further, we characterize the set of points of positive zero characteristic property. In Theorem 3.3, we show that the set of points of positive zero characteristic property in a compact space X is the intersection of negatively invariant open subsets of X .

Throughout this paper, we assume that X is a metric space with a metric function d and $\pi : X \times R \rightarrow X$ is a continuous real flow on X . Here, the image $\pi(x, t)$ of a point (x, t) in $X \times R$ will be written simply as xt .

For x in X , $O^+(x)$ and $O(x)$ denote the *positive orbit* and *orbit* of x , respectively, and let $\omega(x)$ and $\alpha(x)$ denote the *positive* and *negative limit set* of x , respectively. Also, the sets $D(x)$, $D^+(x)$, $J(x)$ and $J^+(x)$ denote, respectively, the *prolongation*, *positive prolongation*, *prolongational limit set* and *positive prolongational limit set* of x in X .

A point x in X is of characteristic 0^+ or 0 if $D^+(x) = \overline{O^+(x)}$ or $D(x) = \overline{O(x)}$.

In particular, for a subset M of X , we call M is of characteristic 0^+ or 0 if all points in M are of characteristic 0^+ or 0 . A flow (X, π) is of characteristic 0^+ or 0 if every point in X is of characteristic 0^+ or 0 .

A subset M of X is called *minimal* if orbit of every point in M is dense in M .

Let $B(x, \varepsilon)$ denote $\{y \in X : d(x, y) < \varepsilon\}$ and \overline{M} denote the closure of $M \subset X$.

For the basic definitions and properties of dynamical systems used in this paper we refer to [2] and [3].

2. Points of zero characteristic properties

In this section, we consider dynamical properties of limit sets of points of zero characteristic properties and relationships between minimality of orbit closures and limit sets and zero characteristic properties of points. Also, we consider zero characteristic properties of orientation preserving conjugate flows.

PROPOSITION 2.1. *The following are equivalent for a point x in X .*

- (1) x is of characteristic 0^+ ,
- (2) $\omega(x) = \overline{J^+(x)}$,
- (3) $J^+(x) \setminus \overline{O^+(x)} = \emptyset$.

PROOF. We first show that (1) implies (2). Let x be of characteristic 0^+ . If $J^+(x) = \emptyset$, then, clearly, $J^+(x) = \omega(x) = \emptyset$ holds. So assume

that $J^+(x) \neq \emptyset$. If $x \notin J^+(x)$, then, clearly, $\omega(x) = J^+(x)$ holds. Next, suppose that $x \in J^+(x)$. Here, we claim that x must be in $\omega(x)$. To see this, assume on the contrary that $x \notin \omega(x)$. Then we have $O^-(x) \subset O^+(x)$ since $O^-(x) \subset J^+(x) \subset D^+(x) = O^+(x) \cup \omega(x)$ and $O(x) \cap \omega(x) = \emptyset$. This implies that x is a periodic point and, thus, we get $O(x) = \omega(x)$. This contradicts the fact that $x \notin \omega(x)$. Hence x must be in $\omega(x)$. Therefore, we get $J^+(x) \subset D^+(x) = \overline{O^+(x)} \subset \omega(x) \subset J^+(x)$ and so $J^+(x) = \omega(x)$ holds.

Next, we show that (2) implies (3). $\omega(x) = J^+(x)$ implies $\omega(x) = J^+(x) \subset \overline{O^+(x)}$. So, we get $J^+(x) \setminus \overline{O^+(x)} = \emptyset$.

Finally, assume that $J^+(x) \setminus \overline{O^+(x)} = \emptyset$. Then $D^+(x) = \overline{O^+(x)} \cup J^+(x) \subset \overline{O^+(x)}$ implies x is of characteristic 0^+ and this completes the proof. \square

We now give an example which shows that the similar result to Proposition 2.1 does not hold for points of characteristics 0.

EXAMPLE 2.2. Let X be a subset of R^2 such that

$$X = \{(r, \theta) : 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi\}$$

and consider a continuous real flow $\pi(x, t)$ whose phase portrait is as follows; The unit circle contains a rest point $p = (1, 0)$ and an orbit γ such that for each point $q \in \gamma$ we have $\omega(q) = \alpha(q) = \{p\}$. Any orbit of point (r_0, θ_0) in the interior of the unit circle is a periodic orbit $r = r_0$. Then the point $x_0 = (1, \pi/2)$ is of characteristic 0. However, $J(x_0) = S^1$ and $\omega(x_0) = \alpha(x_0) = \{p\}$ and so $\omega(x_0) \cup \alpha(x_0) = J(x_0)$ does not hold.

Here, we consider dynamical properties of limit sets of points of zero characteristic properties and relationships between minimality of orbit closures and limit sets and zero characteristic properties of points. In [4], we can find several results connected with this topic in the case of Lyapunov stability. A point x in X is Lyapunov stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(xt, yt) < \varepsilon$ for all $t \in R$. The Lyapunov stability is an wellknown stability among the various stabilities of points and, trivially, Lyapuniv stable points have zero characteristic properties.

First, we need the following lemma.

LEMMA 2.3 [3]. If $y \in \omega(x)$, then $J^+(x) \subset J^+(y)$.

PROPOSITION 2.4. *Let y_0 be of characteristic 0^+ in $\omega(x)$. Then x is of characteristic 0^+ and $\omega(x) = \omega(y_0)$.*

PROOF. By Proposition 2.1 and Lemma 2.3, we get

$$J^+(x) \subset J^+(y_0) = \omega(y_0) \subset \omega(x) \subset J^+(x).$$

This shows that x is of characteristic 0^+ and $\omega(x) = \omega(y_0)$. \square

PROPOSITION 2.5. *Let $z_0 \in \alpha(x)$ be of characteristic 0^+ . Then $x \in \alpha(x)$ and $\alpha(x) = \omega(z_0) = \overline{O^+(z_0)} = \overline{O(z_0)}$ holds.*

PROOF. $z_0 \in \alpha(x) \subset J^-(x)$ implies $x \in J^+(z_0) = \omega(z_0) \subset \alpha(x)$. Also, for any $y \in \alpha(x)$, $y \in D^+(z_0)$ as $y, z_0 \in \alpha(x)$. Thus $\alpha(x) = \overline{O^+(z_0)} = \overline{O(z_0)}$ since $\alpha(x) \subset D^+(z_0) = \overline{O^+(z_0)} \subset \overline{O(z_0)} \subset \alpha(x)$. Further, $z_0 \in J^-(x)$ implies $x \in J^+(z_0) = \omega(z_0)$. Thus, we get $\alpha(x) = \omega(z_0)$ as $\overline{O(z_0)} = \alpha(x) \subset \omega(z_0) \subset \overline{O(z_0)}$. Hence, the result follows. \square

THEOREM 2.6. *Let x be in X . If $\omega(x)$ is of characteristic 0^+ then $\omega(x)$ is a minimal set. Also, if $\alpha(x)$ is of characteristic 0^+ then $\alpha(x)$ is a minimal set and, further, $\alpha(x) = \omega(x) = \overline{O(x)}$ holds.*

PROOF. Let $y \in \omega(x)$. Then, according to Proposition 2.4, we have $\overline{O(y)} \subset \omega(x) = \omega(y) \subset \overline{O(y)}$. This shows that $\omega(x)$ is a minimal set as $\overline{O(y)} = \omega(x)$.

Next, assume that $\alpha(x)$ is of characteristic 0^+ and let $y \in \alpha(x)$. Then by Proposition 2.5, $\alpha(x) = \overline{O(y)}$ and so $\alpha(x)$ is a minimal set. Further, whenever $\alpha(x)$ is of characteristic 0^+ , $x \in \alpha(x)$ by Proposition 2.5 and, thus, $\alpha(x) = \omega(x) = \overline{O(x)}$ holds as $\omega(x)$ is a nonempty set. \square

In view of Theorem 2.6, we now give a characterization of minimality of limit sets and orbit closures.

COROLLARY 2.7. *Let $x \in X$ and the set $\overline{O(x)}$ be of characteristic 0^+ . Then one of the following holds.*

- (1) $\omega(x) = \alpha(x) = \emptyset$ and $\overline{O(x)}$ is a minimal set.
- (2) $\alpha(x) = \omega(x) = \overline{O(x)}$ is a minimal set.
- (3) $\alpha(x) = \emptyset$ and $\omega(x) \neq \emptyset$ is a minimal set.

PROOF. If $\omega(x) = \alpha(x) = \emptyset$ then clearly, $\overline{O(x)}$ is a minimal set. If $\alpha(x) \neq \emptyset$ then $\omega(x)$ is also nonempty and $\alpha(x) = \omega(x) = \overline{O(x)}$ holds by Proposition 2.5. Proposition 2.3 yields (3) as $\overline{O(y)} \subset \omega(x) = \omega(y) \subset \overline{O(y)}$ for any $y \in \omega(x)$. \square

We say that two continuous flows φ on X and ξ on Y are *orientation preserving conjugate* if there exist a homeomorphism $\eta : X \rightarrow Y$ and a continuous function $\omega : X \times R \rightarrow R$ satisfying that, for all x in X , the following two conditions hold.

- (1) $\omega(x, 0) = \omega_x(0) = 0$ and ω_x is a strictly increasing function, and
- (2) $\eta(\varphi(x, t)) = \xi(\eta(x), \omega_x(t))$.

If η is simply continuous then φ is said to be *homomorphic* to ξ .

We will denote D, O on (X, φ) and (Y, ξ) by D_X, O_X, D_Y and O_Y , respectively.

THEOREM 2.8. *Let a flow φ on X be orientation preserving conjugate to a flow ξ on Y . Then $x \in X$ and $\eta(x) \in Y$ have same zero characteristic properties.*

PROOF. We carry out the proof for points of characteristic 0^+ . A similar argument applies to points of characteristic 0 .

Assume that $x \in X$ is of characteristic 0^+ and let us prove that $\eta(x) = y$ is also of characteristic 0^+ . Let $z \in D_Y^+(y)$ and $\eta^{-1}(z) = b \in X$. It is sufficient to prove that $z \in \overline{O_Y^+(y)}$. By the definition of prolongation, there exist sequences of points $\{y_n\} \subset Y$ and $\{t_n\} \subset R^+$ such that $y_n \rightarrow y$ and $\xi(y_n, t_n) \rightarrow z$. Here, for each n , let

$$\eta^{-1}(y_n) = x_n, \quad \text{and} \quad \omega_{x_n}^{-1}(t_n) = r_n.$$

Since ω_{x_n} is a strictly increasing function, we get $\omega_{x_n}^{-1}(t_n) \in R^+$ for each n . Obviously, $x_n \rightarrow x$ and

$$\varphi(x_n, r_n) = \eta^{-1}(\xi(\eta(x_n), \omega_{x_n}^{-1}(t_n))) = \eta^{-1}(\xi(y_n, t_n)).$$

Hence, we get $\varphi(x_n, r_n) \rightarrow \eta^{-1}(z) = b$, and this shows that $b \in D_X^+(x)$. Since x is of characteristic 0^+ , b is in $O_X^+(x)$. Therefore, there exists a sequence $\{s_n\}$ in R^+ such that $\varphi(x, s_n) \rightarrow b$. Then we get $\eta(\varphi(x, s_n)) = \xi(\eta(x), \omega_x(s_n)) \rightarrow \eta(b) = z$. and, therefore, $z \in \overline{O_Y^+(y)}$. we conclude that $y \in Y$ is also of characteristic 0^+ . \square

The following example shows that if two flows are simply homomorphic, then homomorphic function does not preserve zero characteristic properties.

EXAMPLE 2.9. Let $X = \{-1\} \cup \{x \mid 0 < x < 2\pi\} \subset R$ and a flow (X, π) is defined as the point -1 is a rest point and $(0, 2\pi)$ is a single orbit. Also, let $Y = \{(x, y) \mid x^2 + y^2 = 1\} \subset R^2$ and define a flow (Y, φ) on Y as a point $y_0 \in Y$ is a rest point and the set $Y - \{y_0\}$ as single orbit. A homomorphic function $f : X \rightarrow Y$ is defined by

$$f(-1) = y_0 \quad \text{and} \quad f(0, 2\pi) = Y - \{y_0\}.$$

Then the point $-1 \in X$ is of characteristic 0^+ , but $f(-1) = y_0$ is not of characteristic 0^+ .

3. Set of points of zero characteristic properties

In this section, we consider the topological properties of set of points of positive zero characteristic property.

Let $Q^+(X)$ be the set of points of characteristic 0^+ in X and $N^+(X) = X \setminus Q^+(X)$. The following example shows that, in general, the set $Q^+(X)$ is not an open set.

EXAMPLE 3.1. One may consider a simple real continuous flow $\pi(x, t)$ defined in the set $X = \{(x, y) \in R^2 : 0 \leq x, y \leq 1\}$ whose phase portrait is as follows; All points in the boundary of X are rest points and all other orbits are parallel to the x -axis.

Here, we can easily see that $N^+(x) = \{(x, y) : x = 0, 0 \leq y \leq 1\} \cup \{(x, y) : 0 < x < 1, y = 1\} \cup \{(x, y) : 0 < x < 1, y = 0\}$. This shows that $N^+(X)$ is not closed and, thus, $Q^+(X)$ is not open.

LEMMA 3.2. *Let X be a compact space. Then the set $N^+(X)$ is the union of closed positively invariant subsets of X .*

PROOF. Let \mathfrak{K} be a family of subsets of $X \times X$ such that

$$\mathfrak{K} = \{(U, W) \in X \times X : U \text{ and } W \text{ are open subsets of } X \text{ and } \overline{U} \subset W\}.$$

Also, for any element $(U, W) \in \mathfrak{K}$, consider a set $\mathfrak{C}(U, W)$ defined by

$$\mathfrak{C}(U, W) = \{y \in U : \overline{O^+(y)} \subset \overline{U}\}$$

and, for all neighborhood V of y , $O^+(V) \not\subset \overline{W}$.

First, we will show that

$$(*) \quad N^+(X) = \bigcup_{(U, W) \in \mathfrak{K}} \mathfrak{C}(U, W).$$

To see this, let $x \in N^+(X)$. By Proposition 2.1, there is a point y in $J^+(x) \setminus \overline{O^+(x)}$. Since X is compact there are open sets G and H such that $\overline{O^+(x)} \subset G$, $\overline{G} \subset H$ and $y \notin \overline{H}$. Since $y \in J^+(x)$, for any open neighborhood O of x , we can find a point $z \in O$ and a number $s > 0$ such that $zs \notin \overline{H}$. This shows that $x \in \mathfrak{C}(G, H)$ with $(G, H) \in \mathfrak{K}$.

Conversely, let $x \in \mathfrak{C}(U, W)$ for some $(U, W) \in \mathfrak{K}$. By the definition of $\mathfrak{C}(U, W)$, $\overline{O^+(x)} \subset \overline{U}$ and for any open neighborhood V of x , $O^+(V) \not\subset \overline{W}$. Hence there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ of real numbers with $x_n \rightarrow x$ and $x_n t_n \notin \overline{W}$. The continuity of f yields $\{t_n\}$ is unbounded and so we may assume, without loss of generality, that $\{x_n t_n\}$ converges to y in W^c with $t_n \rightarrow \infty$. This shows that $y \in J^+(x) \setminus \overline{O^+(x)}$ and so $x \in N^+(X)$. Therefore we conclude that the equality (*) holds.

Now, it is sufficient to prove that for any $(U, W) \in \mathfrak{K}$, $\mathfrak{C}(U, W)$ is a positively invariant closed set.

First, we claim that $\mathfrak{C}(U, W)$ is a closed set. To see this, let $\{x_n\}$ be a sequence of points in $\mathfrak{C}(U, W)$ with $x_n \rightarrow x$. Assume that $\overline{O^+(x)} \not\subset \overline{U}$. Then there is a number $s > 0$ such that $xs \notin \overline{U}$. By the continuity of f , it follows that there is a point x_k in $\{x_n\}$ with $x_k s \notin \overline{U}$. But this contradicts the fact that x_k is in $\mathfrak{C}(U, W)$. The fact that, for any open neighborhood G of x , $O^+(G) \not\subset \overline{W}$ is obvious. Therefore, the set $\mathfrak{C}(U, W)$ is closed.

Finally, we claim that $\mathfrak{C}(U, W)$ is positively invariant. To see this, assume on the contrary that $ys \notin \mathfrak{C}(U, W)$ for some $y \in \mathfrak{C}(U, W)$ and $s > 0$. Since $\overline{O^+(ys)} \subset \overline{O^+(y)} \subset \overline{U}$, there must be an open neighborhood H of ys such that $O^+(H) \subset \overline{W}$. The continuity of f yields there is an open neighborhood H' of x with

$$H't \subset W \text{ for } 0 \leq t \leq s \quad \text{and} \quad H's \subset H.$$

This means that $O^+(H') \subset \overline{W}$. But this contradicts to the fact that $y \in \mathfrak{C}(U, W)$. Hence, we conclude that $\mathfrak{C}(U, W)$ is positively invariant. This completes the proof of this theorem. □

The above lemma yields the following theorem.

THEOREM 3.3. *Let X be a compact space. Then the set of points of characteristic 0^+ in X is the intersection of negatively invariant open sets.*

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