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ON THE OSCILLATION OF CERTAIN FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some new oscillation criteria for the functional differential equations of the form

$$\frac{d}{dt}\left(\frac{1}{a_{n-1}(t)}\frac{d}{dt}\left(\frac{1}{a_{n-2}(t)}\frac{d}{dt}\left(\cdots\left(\frac{1}{a_1(t)}\frac{d}{dt}x(t)\right)\cdots\right)\right)\right)^{\alpha}$$
$$+\delta\left[f_1\left(t, x[g_1(t)], \frac{d}{dt}x[h_1(t)]\right) + f_2\left(t, x[g_2(t)], \frac{d}{dt}x[h_2(t)]\right)\right]=0$$

via comparing it with some other functional differential equations whose oscillatory behavior is known.

1. Introduction

In this paper, we deal with the oscillatory behavior of all solutions of functional differential equations of the form

$$L_n x(t)+\delta\left[f_1\left(t, x[g_1(t)], \frac{d}{dt}x[h_1(t)]\right) + f_2\left(t, x[g_2(t)], \frac{d}{dt}x[h_2(t)]\right)\right] = 0 \tag{1.1;δ}$$

where $n \geq 2$, $\delta = \pm 1$ and

$$\begin{cases} L_0 x(t) = x(t) \\ L_k x(t) = \dfrac{1}{a_k(t)}\dfrac{d}{dt}L_{k-1}x(t), \ k = 1, 2, \cdots, n-1 \\ L_n x(t) = \dfrac{d}{dt}(L_{n-1}x(t))^{\alpha}. \end{cases} \tag{1.2}$$

We assume that:

(i) $a_i(t) \in C\left([t_0, \infty), \mathbb{R}^+ = (0, \infty)\right)$, $t_0 \geq 0$ and

$$\int^{\infty} a_i(s)ds = \infty, \ i = 1, 2, \cdots, n-1; \tag{1.3}$$

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(ii) $g_j(t), h_j(t) \in C([t_0,\infty), \mathbb{R} = (-\infty,\infty))$, $\lim_{t\to\infty} g_j(t) = \infty$ and $\lim_{t\to\infty} h_j(t) = \infty$, $j = 1, 2$;

(iii) $f_j \in C([t_0,\infty) \times \mathbb{R}^2, \mathbb{R})$, $j = 1, 2$;

(iv) α is the quotient of positive odd integers.

We also assume that there exist functions $q_j(t) \in C([t_0,\infty), \mathbb{R}^+)$ and positive constants β_j and γ_j, $j = 1, 2$ such that,

$$f_j(t,x,y) \text{ sgn } x \geq q_j(t)|x|^{\beta_j}|y|^{\gamma_j} \text{ for } xy \neq 0 \text{ and } t \geq t_0. \tag{1.4}$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all non constant functions $x : [t_x,\infty) \to \mathbb{R}$ such that $L_j x(t)$, $0 \leq j \leq n$ exist and are continuous on $[t_x,\infty)$. Our attention is restricted to those solutions $x \in \mathcal{D}(L_n)$ of equation (1.1;δ) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq t_x$. We make the standing hypothesis that equation (1.1;δ) does possess such solutions. A solution of equation (1.1;δ) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called nonoscillatory. Equation (1.1;δ) is called oscillatory if all its solutions are oscillatory.

The problem of obtaining sufficient conditions to ensure that all solutions of n^{th} order functional differential equations which are special cases of equation (1.1;δ) are oscillatory has been studied by a number of researchers. For recent contributions, we refer to [1-13] and the references cited therein.

In this paper we present new criteria for the oscillation of equation (1.1;δ) by comparing it with some first order functional differential equations whose oscillatory behavior is known. Sufficient conditions for the oscillation of equation (1.1;δ) and its special cases when $a_i = 1$, $i = 1, 2, \cdots, n-1$ are also given.

2. Preliminaries

To formulate our results, we shall use the following notation: For $a_i(t) \in C([t_0,\infty), \mathbb{R})$, $i = 1, 2, \cdots$, we define

$$I_0 = 1,$$

$$I_i(t,s;a_i,a_{i-1},\cdots,a_1) = \int_s^t a_i(u) I_{i-1}(u,s;a_{i-1},\cdots,a_1)du,$$

$$i = 1, 2, \cdots.$$

It is easy to verify from the definition of I_i that

$$I_i(t,s;a_1,\cdots,a_i) = (-1)^i I_i(s,t;a_i,\cdots,a_1)$$

and

$$I_i(t,s;a_1,\cdots,a_i)=\int_s^t a_i(u)I_{i-1}(t,u;a_1,\cdots,a_{i-1})du.$$

We will need the following lemmas:

LEMMA 2.1. *Suppose condition (1.3) holds. If $x\in\mathcal{D}(\bar{L}_n)$, where $\bar{L}_n$ is L_n defined by (1.2) with $\alpha=1$ is eventually of one sign, there exist a $t_x\ge t_0\ge 0$ and an integer ℓ, $0\le\ell\le n$ with $n+\ell$ even for $x(t)\bar{L}_nx(t)$ nonnegative eventually, or $n+\ell$ odd for $x(t)\bar{L}_nx(t)$ nonpositive eventually and such that for $t\ge t_x$*

(2.1)
$$\begin{cases}\ell > 0 \text{ implies } x(t)L_kx(t)>0,\ k=0,1,\cdots,\ell\\ \ell\le n-1 \text{ implies } (-1)^{\ell-k}x(t)L_kx(t)>0,\ k=\ell,\ell+1,\cdots,n.\end{cases}$$

This lemma generalizes a well known lemma of Kiguradze and can be proved similarly.

We note that if $x(t)$ is a solution of equation (1.1;δ) which is eventually of one sign, then

$$\frac{d}{dt}\left(L_{n-1}x(t)\right)^{\alpha}=\alpha L_{n-1}^{\alpha-1}x(t)\bar{L}_nx(t)$$

and since α satisfies (iv), we see that L_nx and $\bar{L}_nx$ have the same sign. Moreover, one can easily see that $L_jx(t)$, $0\le j\le n$ are eventually of one sign.

It will be convenient to make use of the following notations in the remainder of this paper. For any $T\ge t_0$ and all $t\ge T$, we let

$$w_\ell[t,T]=\int_T^t I_{\ell-1}(t,s;a_1,\cdots,a_{\ell-1})a_\ell(s)I_{n-\ell-1}(t,s;a_{n-1},\cdots,a_{\ell+1})ds,\quad 1\le\ell\le n-1;$$

$$\overline{w}_\ell[t,T]=a_1(t)\int_T^t I_{\ell-2}(t,s;a_2,\cdots,a_{\ell-1})a_\ell(s)\,I_{n-\ell-1}(t,s;a_{n-1},\cdots,a_{\ell+1})ds,\ 2\le\ell\le n-1;$$

$$w[t,T]=\min_{1\le\ell\le n-1}w_\ell[t,T]\text{ and }\overline{w}[t,T]=\min_{2\le\ell\le n-1}\overline{w}_\ell[t,T].$$

For $t \geq s \geq T$, we let

$$\begin{aligned}
\overline{w}_1[t,s] &= a_1(s)I_{n-2}(t,s;a_{n-1},\cdots,a_2);\\
\overline{w}_\lambda[t,T] &= \min\{\overline{w}_1[t,\lambda t],\ \overline{w}_\ell[\lambda t,T],\ 2\leq \ell\leq n-1\\
&\qquad \text{and for some } \lambda\in(0,1),\ \text{for } t\geq \frac{T}{\lambda}\};\\
w_n[t,s] &= I_{n-1}(t,s;a_1,\cdots,a_{n-1});\\
\overline{w}_n[t,s] &= a_1(t)I_{n-2}(t,s;a_2,\cdots,a_{n-1});\\
w_0[t,s] &= I_{n-1}(t,s;a_{n-1},\cdots,a_1);\\
\overline{w}_0[t,s] &= a_1(s)I_{n-2}(t,s;a_{n-1},\cdots,a_2);\\
I[t,T] &= \int_T^t a_1(s)ds.
\end{aligned}$$

LEMMA 2.2. *Let $x \in \mathcal{D}(L_n)$ be eventually positive, $(-1)^{n-\ell}\delta = -1$ and condition (1.3) hold. Then*

(i_1) *for $1 \leq \ell \leq n-1$ and all $t \geq T \geq t_0$*

$$x(t) \geq w[t,T](\delta L_{n-1}x(t)); \tag{2.2}$$

(i_2) *for $2 \leq \ell \leq n-1$ and all $t \geq T \geq t_0$*

$$x'(t) \geq \overline{w}[t,T](\delta L_{n-1}x(t)); \tag{2.3}$$

(i_3) *for $1 \leq \ell \leq n-1$, let λ be a constant, $0 < \lambda < 1$. Then there exists a $T^* \geq \frac{T}{\lambda}$ such that*

$$x'(\lambda t) \geq \overline{w}_\lambda[t,T](\delta L_{n-1}x(t)) \text{ for } t \geq T^*. \tag{2.4}$$

LEMMA 2.3. *Let $x \in \mathcal{D}(L_n)$ be eventually positive, condition (1.3) hold and $\ell = 0$. Then for $t \geq s \geq T \geq t_0$*

$$x(s) \geq w_0[t,s]((-1)^{n-1}L_{n-1}x(t)) \tag{2.5}$$

and

$$-x'(s) \geq \overline{w}_0[t,s]((-1)^{n-1}L_{n-1}x(t)). \tag{2.6}$$

LEMMA 2.4. *Let $x \in \mathcal{D}(L_n)$ be eventually positive, condition (1.3) hold and $\ell = n$. Then for $t \geq s \geq T \geq t_0$*

$$x(t) \geq w_n[t,s]L_{n-1}x(s) \tag{2.7}$$

and

$$x'(t) \geq \overline{w}_n[t,s]L_{n-1}x(s). \tag{2.8}$$

The proof of these lemmas may be found in [1-9].

We note that if $x \in \mathcal{D}(L_n)$ is a solution of equation (1.1;δ) which is eventually of one sign, and when condition (1.4) holds, then x satisfies the differential inequalities

$$(2.9;\delta) \qquad \left\{\delta L_n x(t) + q_1(t)\,|x[g_1(t)]|^{\beta_1}\left|\frac{d}{dt}x[h_1(t)]\right|^{\gamma_1}\right\} \operatorname{sgn} x[g_1(t)] \le 0$$

and

$$(2.10;\delta) \qquad \left\{\delta L_n x(t) + q_2(t)\,|x[g_2(t)]|^{\beta_2}\left|\frac{d}{dt}x[h_2(t)]\right|^{\gamma_2}\right\} \operatorname{sgn} x[g_2(t)] \le 0.$$

3. Main results

Throughout, we shall assume the following conditions: There exist functions $\xi(t),\ \eta(t) \in C^1([t_0,\infty),\mathbb{R})$ such that

$$(3.1) \qquad h_1(t) \le g_1(t) \le \xi(t) \le t,\ h_1'(t) \ge 0,\ \xi'(t) \ge 0 \text{ for } t \ge t_0$$

and

$$(3.2) \qquad g_2(t) \ge h_2(t) \ge \eta(t) \ge t,\ \eta'(t) \ge 0 \text{ for } t \ge t_0.$$

We let

$$\lambda_1 = \beta_1 + \gamma_1 \le \alpha \text{ and } \lambda_2 = \beta_2 + \gamma_2 \ge \alpha$$

and for all large $T \ge t_0$ with $h_1(t) \ge T$ and $\eta(t) \ge T$

$$Q_1(t) = q_1(t) w^{\beta_1}[h_1(t),T](h_1'(t)\overline{w}[h_1(t),T])^{\gamma_1},$$

$$Q_2(t) = q_1(t) I^{\beta_1}[h_1(t),T](a_1[h_1(t)]h_1'(t))^{\gamma_1}(w_0^*[\xi(t),h_1(t)])^{\lambda_1},$$

$$w_0^*[t,s] = I_{n-2}(t,s;a_{n-1},\cdots,a_2),\ t \ge s \ge T,$$

$$Q_3(t) = q_1(t) w_0^{\beta_1}[\xi(t),g_1(t)](h_1'(t)\overline{w}_0[\xi(t),h_1(t)])^{\gamma_1}$$

and

$$Q_4(t) = q_2(t) w_n^{\beta_2}[g_2(t),\eta(t)](h_2'(t)\overline{w}_n[h_2(t),\eta(t)])^{\gamma_2}.$$

We now present the following results.

THEOREM 3.1. *Let n be even, conditions* (i)-(iv), (1.4) *and* (3.1) *hold. If for all large $T \ge t_0$ with $h_1(t) \ge T$, the first order retarded equations*

$$(3.3) \qquad y'(t) + Q_1(t)|y[h_1(t)]|^{\frac{\lambda_1}{\alpha}} \operatorname{sgn} y[h_1(t)] = 0$$

and

$$(3.4) \qquad z'(t) + Q_2(t)|z[\xi(t)]|^{\frac{\lambda_1}{\alpha}} \operatorname{sgn} z[\xi(t)] = 0$$

are oscillatory, then equation (1.1; 1) (*or inequality* (2.9; 1)) *is oscillatory.*

THEOREM 3.2. *Let n be odd, conditions* (i)-(iv), (1.4) *and* (3.1) *hold. If for all large $T \geq t_0$ with $h_1(t) \geq T$, the first order delay equation* (3.3) *and*

$$v'(t) + Q_3(t)|v[\xi(t)]|^{\frac{\lambda_1}{\alpha}} \, sgn\, v[\xi(t)] = 0 \tag{3.5}$$

are oscillatory, then equation $(1.1;1)$ (*or inequality* $(2.9;1)$) *is oscillatory.*

THEOREM 3.3. *Let n be even, conditions* (i)-(iv), (1.4), (3.1) *and* (3.2) *hold. If for all large $T \geq t_0$ with $h_1(t) \geq T$, and $\eta(t) \geq T$, the first order delay equations* (3.3), (3.5) *and the advanced equation*

$$W'(t) - Q_4(t)|W[\eta(t)]|^{\frac{\lambda_2}{\alpha}} \, sgn\, W[\eta(t)] = 0 \tag{3.6}$$

are oscillatory, then equation $(1.1;-1)$ *is oscillatory.*

THEOREM 3.4. *Let n be odd, conditions* (i)-(iv), (1.4), (3.1) *and* (3.2) *hold. If for all large $T \geq t_0$ with $h_1(t) \geq T$, and $\eta(t) \geq T$, the first order delay equations* (3.3), (3.4) *and the advanced equation* (3.6) *are oscillatory, then equation* $(1.1;-1)$ *is oscillatory.*

PROOFS OF THEOREMS 3.1 - 3.4. Let $x(t)$ be an eventually positive solution of equation $(1.1;\delta)$, say $x(t) > 0$ for $t \geq t_0 \geq 0$. By Lemma 2.1, there exists a $t_1 \geq t_0$ such that (2.1) holds. Next, we consider the following four cases:

(I_1) $1 < \ell \leq n-1$,
(I_2) $\ell = 1$,
(I_3) $\ell = n$,
(I_4) $\ell = 0$.

Case I_1: Assume $1 < \ell \leq n-1$. By Lemma 2.2, there exists a $t_2 \geq t_1$ with $h_1(t) > t_1$ such that for $t \geq t_2$

$$x[g_1(t)] \geq w[h_1(t), t_1](\delta L_{n-1} x[h_1(t)]) \tag{3.7}$$

and

$$\frac{d}{dt} x[h_1(t)] \geq \overline{w}[h_1(t), t_1] h_1'(t)(\delta L_{n-1} x[h_1(t)]). \tag{3.8}$$

Using (3.7) and (3.8) in inequality $(2.9;\, \delta)$ we have

$$\begin{aligned} &\delta \frac{d}{dt}\left(L_{n-1}x(t)\right)^{\alpha} \\ \leq\ & -q(t)x^{\beta_1}[g_1(t)]\left(\frac{d}{dt}x[h_1(t)]\right)^{\gamma_1} \\ \leq\ & -q_1(t)w^{\beta_1}[h_1(t),t_1]\left(\overline{w}[h_1(t),t_1]\right)^{\gamma_1}\left(h'(t)\right)^{\gamma_1}\left(\delta L_{n-1}x[h_1(t)]\right)^{\beta_1+\gamma_1}. \end{aligned}$$

Set $u(t) = (\delta L_{n-1}x(t))^{\alpha}$. Then

$$\frac{d}{dt}u(t) \leq -Q_1(t)u^{\frac{\lambda_1}{\alpha}}[h_1(t)] \text{ for } t \geq t_2. \tag{3.9}$$

Integrating (3.9) from $t \geq t_2$ to v and letting $v \to \infty$, we get

$$u(t) \geq \int_t^{\infty} Q_1(s)u^{\frac{\lambda_1}{\alpha}}[h_1(s)]ds.$$

The function $u(t)$ is clearly strictly decreasing for $t \geq t_2$. Hence by Theorem 1 in [13] there exists a positive solution $y(t)$ of equation (3.3) with $y(t) \to 0$ as $t \to \infty$. But, this contradicts the assumption that equation (3.1) is oscillatory.

Case I_2: Assume $\ell = 1$. This is the case when n is even, $\delta = 1$ and n is odd, $\delta = -1$. There exists a $T_1 \geq t_1$ with $h_1(t) \geq t_1$ such that for $t \geq T_1$, we have

$$x[g_1(t)] \geq x[h_1(t)] \geq I[h_1(t), T_1]L_1x[h_1(t)]. \tag{3.10}$$

Using (3.10) in inequality (2.9;δ) and letting $v(t) = L_1x(t) = \frac{1}{a_1(t)}x'(t)$, $t \geq T_1$, we have

$$\begin{aligned} &\frac{d}{dt}((-1)^n L_{n-2}v(t))^{\alpha} + q_1(t)I^{\beta}[h_1(t), T_1] \\ &(a_1[h_1(t)]h_1'(t))^{\gamma_1}v^{\lambda_1}[h_1(t)] \\ \leq\ & 0 \text{ for } t \geq T_1. \end{aligned} \tag{3.11}$$

It is clear that the function $u(t)$ satisfies

$$(-1)^i L_i v(t) > 0 \text{ for } i = 0, 1, \cdots, n-2 \text{ and } t \geq T_1.$$

Now, by applying Lemma 2.3 with $n-1$ replaced by $n-2$, there exists a $T_2 \geq T_1$ such that for $t \geq T_2$

(3.12)

$$v[h_1(t)] \geq w_0^*[\xi(t), h_1(t)]((-1)^n L_{n-2}v[\xi(t)]) \text{ for } T_2 \leq h_1(t) \leq \xi(t).$$

Using (3.12) in inequality (3.11) we have

$$\begin{aligned} &\frac{d}{dt}z(t) + q_1(t)I^{\beta_1}[h_1(t), T_1](a_1[h_1(t)]h_1'(t))^{\gamma_1} \\ &(w_0^*[\xi(t), h_1(t)])^{\lambda_1}z^{\frac{\lambda_1}{\alpha}}[\xi(t)] \leq 0 \text{ for } t \geq T_2. \end{aligned}$$

The rest of the proof is similar to that of the previous case and hence omitted. This completes the proof of Theorem 3.1.

Case I_3: Assume $\ell = n$. This is the case when $\delta = -1$. By applying Lemma 2.4, there exists a $T^* \geq t_1$ with $\eta(t) \geq t_1$ such that for all $t \geq T^*$

$$x[g_2(t)] \geq w_n[g_2(t), \eta(t)]L_{n-1}x[\eta(t)] \tag{3.13}$$

and
(3.14)
$$\frac{d}{dt}x[h_2(t)] \geq h_2'(t)w_n[h_2(t),\eta(t)]L_nx[\eta(t)] \text{ for } g_2(t) \geq h_2(t) \geq \eta(t) \geq T^*.$$

Using (3.13) and (3.14) in inequality (2.10; -1) we have

$$Y'(t) \geq q_2(t)w_n^{\beta_2}[g_2(t),\eta(t)](h_2'(t)w_n[h_2(t),\eta(t)])^{\gamma_2}Y^{\frac{\lambda_2}{\alpha}}[\eta(t)] \tag{3.15}$$

where $Y(t) = (L_{n-1}x(t))^\alpha > 0$ for $t \geq T^*$. Now by applying a result similar to Corollary 3.2.3 in [10], we arrive at the desired contradiction. This completes the proof of Theorem 3.4.

Case I_4**:** Assume $\ell = 0$. This is the case when $\delta = 1$, n is odd and $\delta = -1$, n is even. Now, by applying Lemma 2.3, there exists a $T_1^* \geq t_1$ with $h_1(t) \geq t_1$ such that for all $t \geq T_1^*$

$$x[g_1(t)] \geq w_0[\xi(t),g_1(t)]((-1)^{n-1}L_{n-1}x[\xi(t)]) \tag{3.16}$$

and

$$\begin{aligned} &-\frac{d}{dt}x[h_1(t)] \\ \geq\ & h_1'(t)\overline{w}_0[\xi(t),h_1(t)]((-1)^{n-1}L_{n-1}x[\xi(t)]) \text{ for } \xi(t) \\ \geq\ & g_1(t) \geq h_1(t) \geq T. \end{aligned} \tag{3.17}$$

Using (3.16) and (3.17) in inequality (2.9;δ) we have

$$\begin{aligned} &\frac{d}{dt}((-1)^{n-1}L_{n-1}x(t))^\alpha + q_1(t)w_0^{\beta_1}[\xi(t),g_1(t)](h_1'(t)\overline{w}_0[\xi(t),h_1(t)])^{\gamma_1} \\ &\times((-1)^{n-1}L_{n-1}x[\xi(t)])^{\lambda_2} \leq 0 \text{ for } t \geq T_1^*. \end{aligned}$$

The rest of the proof is similar to that given in Case I_2 and hence omitted. This completes the proofs of Theorems 3.2 and 3.3.

The following results are immediate.

COROLLARY 3.1. *Let conditions* (i)-(iv), (1.4), (3.1) *and* (3.2) *hold. Equation* (1.1; δ) *is oscillatory if*

(i_1) *for* $\delta = 1$, *n is even and all large* $t \geq T \geq t_0$

$$\begin{cases} \liminf\limits_{t\to\infty} \displaystyle\int_{h_1(t)}^t Q_1(s)ds > \frac{1}{e} \text{ when } \lambda_1 = \alpha; \\ \displaystyle\int^\infty Q_1(s)ds = \infty \text{ when } \lambda_1 < \alpha; \end{cases} \tag{3.18}$$

and

$$(3.19)\qquad \begin{cases} \liminf_{t\to\infty} \int_{\xi(t)}^{t} Q_2(s)ds > \frac{1}{e} \text{ when } \lambda_1 = \alpha; \\ \int^{\infty} Q_2(s)ds = \infty \text{ when } \lambda_1 < \alpha \end{cases}$$

are satisfied;

(i_2) *for* $\delta = 1$, n *is odd and all large* $t \geq T \geq t_0$, *condition* (3.18) *and*

$$(3.20)\qquad \begin{cases} \liminf_{t\to\infty} \int_{\xi(t)}^{t} Q_3(s)ds > \frac{1}{e} \text{ when } \lambda_1 = \alpha; \\ \int^{\infty} Q_3(s)ds = \infty \text{ when } \lambda_1 < \alpha \end{cases}$$

are satisfied;

(i_3) *for* $\delta = -1$, n *is even and all large* $t \geq T \geq t_0$, *conditions* (3.18), (3.20) *and*

$$(3.21)\qquad \begin{cases} \liminf_{t\to\infty} \int_{t}^{\eta(t)} Q_4(s)ds > \frac{1}{e} \textit{ when } \lambda_2 = \alpha; \\ \int^{\infty} Q_4(s)ds = \infty \textit{ when } \lambda_2 > \alpha \end{cases}$$

are satisfied;

(i_4) *for* $\delta = -1$, n *is odd and all large* $t \geq T \geq t_0$, *conditions* (3.18), (3.19) *and* (3.21) *hold.*

Next, we let

$$Q(t) = \min\{Q_i(t),\ i = 1, 2, 3 : t \geq T\}.$$

In this case, conditions (3.18), (3.19) and (3.20) can be replaced with

$$(3.22)\qquad \begin{cases} \liminf_{t\to\infty} \int_{\xi(t)}^{t} Q(s)ds > \frac{1}{e} \text{ when } \lambda_1 = \alpha; \\ \int^{\infty} Q(s)ds = \infty \text{ when } \lambda_1 < \alpha. \end{cases}$$

Our results seem to be new even when specialized to the equation

$$(3.23;\delta)\qquad \frac{d}{dt}\left(x^{n-1}(t)\right)^{\alpha} + \delta\left[f_1\left(t, x[g_1(t)], \frac{d}{dt}x[h_1(t)]\right) + f_2 \left(t, x[g_2(t)], \frac{d}{dt}x[h_2(t)]\right)\right] = 0$$

for which, conditions (ii)-(iv), (1.4), (3.1) and (3.2) are satisfied. So, we shall state them as a corollary by observing that in this case for $T \geq t_0$ and all $t \geq T$, there exists a constant θ, $0 < \theta < 1$ such that

$$w[t,T] = \frac{\theta}{(n-1)!}t^{n-1} \text{ and } \overline{w}[t,T] = \frac{\theta}{(n-2)!}t^{n-2}$$

$$w_0[t,s] = w_n[t,s] = \frac{(t-s)^{n-1}}{(n-1)!} \text{ for } t \geq s \geq T$$

and

$$\overline{w}_0[t,s] = w_0^*[t,s] = \overline{w}_n[t,s] = \frac{(t-s)^{n-2}}{(n-2)!} \text{ for } t \geq s \geq T.$$

Now, we have for all large $t \geq T$ and some constants θ, θ_1, $0 < \theta, \ \theta_1 < 1$

$$Q_1(t) = \left(\frac{\theta}{(n-2)!}\right)^{\lambda_1} \left(\frac{h_1(t)}{n-1}\right)^{\beta_1} q_1(t)(h_1'(t))^{\gamma_1} h_1^{(n-2)\lambda_1}(t);$$

$$Q_2(t) = \theta_1 q_1(t)(h_1^{\beta}(t))(h_1'(t))^{\gamma_1} \left(\frac{[\xi(t)-\eta(t)]^{n-2}}{(n-2)!}\right)^{\lambda_1};$$

$$Q_3(t) = q_1(t)\left(\frac{[\xi(t)-g_1(t)]^{n-1}}{(n-1)!}\right)^{\beta_1} \left(\frac{h_1'(t)[\xi(t)-h_1(t)]^{n-2}}{(n-2)!}\right)^{\gamma_1}$$

and

$$Q_4(t) = q_2(t)\left(\frac{[g_2(t)-\eta(t)]^{n-1}}{(n-1)!}\right)^{\beta_2} \left(\frac{h_2'(t)[h_2(t)-\eta(t)]^{n-2}}{(n-2)!}\right)^{\gamma_2}.$$

Now, we have the following immediate result.

COROLLARY 3.2. *Let conditions* (ii)-(iv), (1.4), (3.1) *and* (3.2) *hold. Equation* (3.23; δ) *is oscillatory if*

(o_1) *for* $\delta = 1$ *and n is even*

$$(3.24) \quad \begin{cases} \liminf_{t\to\infty} \int_{h_1(t)}^{t} q_1(s)h_1^{\beta_1}(s)(h_1'(s))^{\gamma_1} h_1^{(n-2)\lambda_1}(s)ds \\ > \frac{(n-1)^{\beta_1}((n-2)!)^{\lambda_1}}{e}, \text{ when } \lambda_1 = \alpha; \\ \int^{\infty} q_1(s)h_1^{\beta_1}(s)(h_1'(s))^{\gamma_1} h_1^{(n-2)\lambda_1}(s)ds = \infty, \text{ when } \lambda_1 < \alpha \end{cases}$$

and

(3.25)
$$\begin{cases} \liminf_{t\to\infty} \int_{\xi(t)}^{t} q_1(s)h_1^{\beta_1}(s)(h_1'(s))^{\gamma_1}[\xi(s)-h_1(s)]^{(n-2)\lambda_1}ds \\ > \frac{((n-2)!)^{\lambda_1}}{e}, \text{ when } \lambda_1=\alpha; \\ \int^{\infty} q_1(s)h_1^{\beta_1}(s)(h_1'(s))^{\gamma_1}[\xi(s)-h_1(s)]^{(n-2)\lambda_1}ds=\infty, \text{ when } \lambda_1<\alpha \end{cases}$$

are satisfied;

(o_2) *for* $\delta = 1$ *and n is odd, condition* (3.24) *and*

(3.26)
$$\begin{cases} \liminf_{t\to\infty} \int_{\xi(t)}^{t} q_1(s)[\xi(s)-g_1(s)]^{(n-1)\beta_1}(h_1'(s)[\xi(s)-h_1(s)]^{n-2})^{\gamma_1}ds \\ > \frac{[(n-1)!]^{\beta_1}[(n-2)!]^{\gamma_1}}{e}, \text{ when } \lambda_1=\alpha; \\ \int^{\infty} q_1(s)[\xi(s)-g_1(s)]^{(n-1)\beta_1}(h_1'(s)[\xi(s)-h_1(s)]^{n-2})^{\gamma_1}ds=\infty, \\ \text{when } \lambda_1<\alpha \end{cases}$$

hold;

(o_3) *for* $\delta = -1$ *and n is even, conditions* (3.24), (3.26) *and*

(3.27)
$$\begin{cases} \liminf_{t\to\infty} \int_{t}^{\eta(t)} q_2(s)[g_2(s)-\eta(s)]^{(n-1)\beta_2}(h_2'(s)[h_2(s)-\eta(s)]^{n-2})^{\gamma_2}ds \\ > \frac{[(n-1)!]^{\beta_2}[(n-2)!]^{\gamma_2}}{e}, \text{ when } \lambda_2=\alpha; \\ \int^{\infty} q_2(s)[g_2(s)-\eta(s)]^{(n-1)\beta_2}(h_2'(s)[h_2(s)-\eta(s)]^{n-2})^{\gamma_2}ds=\infty, \\ \text{when } \lambda_2>\alpha \end{cases}$$

are satisfied;

(o_4) *for* $\delta = -1$ *and n is odd, conditions* (3.24), (3.25) *and* (3.27) *hold.*

As example, we consider a special case of inequality (2.9;1) namely the equation

$$\frac{d}{dt}\left(x^{(n-1)}(t)\right)^{\alpha} + q_1(t)|x[\sqrt{t}]|^{\beta_1}\left|\frac{d}{dt}x[\sqrt{t}]\right|^{\gamma_1} \operatorname{sgn} x[\sqrt{t}] = 0. \tag{3.28}$$

Here, we take $g_1(t) = h_1(t) = \sqrt{t}$ and let $\xi(t) = \frac{t+\sqrt{t}}{2}$, $t \geq 0$.

Now, equation (3.28) is oscillatory if

(e_1) for n even, conditions (3.24) and (3.25) are satisfied;

(e_2) for n odd, conditions (3.24) and (3.26) are satisfied.

Also, for the special case of equation (3.23; δ), namely the equation

$$\begin{aligned}(3.29)\qquad &\frac{d}{dt}(x^{(n-1)}(t))^{\alpha}\\ =\ & q_1(t)|x[\sqrt{t}]|^{\beta_1}\left|\frac{d}{dt}x[\sqrt{t}]\right|^{\gamma_1}\ \operatorname{sgn} x[\sqrt{t}]\\ &+q_2(t)|x[t\sqrt{t}]|^{\beta_2}\left|\frac{d}{dt}x[t\sqrt{t}]\right|^{\gamma_2}\ \operatorname{sgn} x[t\sqrt{t}].\end{aligned}$$

Here, we take $g_1(t) = h_1(t) = \sqrt{t}$ and $g_2(t) = h_2(t) = t\sqrt{t}$. We let $\xi(t) = \frac{t+\sqrt{t}}{2}$ and $\eta(t) = \frac{t}{2}(1+\sqrt{t})$, $t \geq 0$.

Now, equation (3.29) is oscillatory if

(s_1) for n even, conditions (3.24), (3.26) and (3.27) hold;

(s_2) for n odd, conditions (3.24), (3.25) and (3.27) hold.

We may note that Theorems 3.1 and 3.2 are applicable to equations of type (1.1;1) with $f_2 = 0$ and both $g_1(t)$, $h_1(t) \leq t$ while Theorems 3.3 and 3.4 are applicable to the mixed equation (1.1;-1) with $g_1(t)$, $h_1(t) < t$ and $g_1(t)$, $h_1(t) > t$.

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