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TROTTER-KATO TYPE APPROXIMATIONS OF CONVOLUTED SOLUTION OPERATOR FAMILIES

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ABSTRACT. Trotter-Kato type approximations of the convoluted solution operator families for the Volterra integral equations (VE_n): $v(t) = A_n \int_0^t v(t-s)d\mu_n(s) + f_n(t),\ t \geq 0$ and the convergence of the solutions to the equations (VE_n) are studied.

1. Introduction

Approximations of C_0- and integrated- semigroups for the abstract Cauchy problems

$$(ACP_n) \qquad u'(t) = A_n u(t), \quad t \geq 0 \quad ; \quad u(0) = x_n$$

and those of the cosine families for the second order Cauchy problems

$$(CP_n) \quad u''(t) - B_n u'(t) - A_n u(t) = 0, \ t \geq 0 \quad ; \ u(0) = x_n, \ u'(0) = y_n$$

for $n \in \mathbb{N}$ and their applications have been studied in the papers [1], [3], [6], [7], and etc.. Integrated and convoluted solution operator families which are general notion of the integrated and convoluted semigroups are suitable for studying the generalized well-posedness of the Volterra equation (VE) that follows (see [5]). The approximations of the integrated or convoluted solution operator families for the Volterra integral equations

$$(VE_n) \qquad v(t) = A_n \int_0^t v(t-s)d\mu_n(s) + f_n(t), \qquad t \geq 0$$

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for $n \in \mathbb{N}$ to an integrated or a convoluted solution operator family for the equation

$$(VE) \qquad v(t) = A \int_0^t v(t-s)d\mu(s) + f(t), \qquad t \geq 0,$$

also have been studied in [4] and [5], where it is assumed that A and A_n, $n \in \mathbb{N}$ are closed linear operators on a Banach space X, that μ and μ_n, $n \in \mathbb{N}$ are scalar valued functions of local bounded variation on $[0, \infty)$ which are normalized, i.e., vanish at 0, and that f and f_n, $n \in \mathbb{N}$ are X valued Laplace transformable functions defined on $[0, \infty)$.

The objective of this paper is to improve a Trotter-Kato type approximation result of convoluted solution operator families (Theorem 4.2 in [5]) by showing that the result holds under weaker conditions on the scalar functions μ and μ_n and additionally, to formulate the approximation results (Theorems 4.1 and 4.2 in [5]) to the case that the spaces where the operators A_n are defined vary depending on $n \in \mathbb{N}$. In practical examples however, only the norms might vary on a fixed set depending on n. Since the former result with weaker conditions is immediately deduced from the latter ones with operators on varying spaces, we will prove the latter first. A convergence of the solutions of the equations (VE_n) to a solution of the equation (VE) is deduced from a convergence theorem of the functions in $Lip_\omega([0, \infty); X)$ in terms of their Laplace-Stieltjes transforms and a solution characterization for (VE). Another convergence of the solutions of (VE_n) to a solution of (VE) is deduced from an integrated solution operator family result in [4] and the approximation result of convoluted solution operator families. For examples, those in [2] can be referred to.

2. Approximations of convoluted solution operator families and the convergence of the solutions of (VE_n)

We set assumptions on the operator A and the function μ for the equation (VE) in this section :

(A) Let A be a closed linear operator with domain $D(A)$ and range in a Banach space X and for some constant $\epsilon \geq 0$, $\mu \in BV_\epsilon([0, \infty); \mathbb{C})$, i.e., μ is a $\mathbb{C}$-valued, normalized function of local bounded variation on $[0, \infty)$ whose variation on $[0, t]$ does not exceed $Me^{\epsilon t}$ for some constant $M \geq 0$ and for every $t \geq 0$.

Theorem 1 through Theorem 5 are preliminaries for the new results. See [1], [3], [4] or [5] for Theorems 1 and 2, Definition 3, Theorem 5, notation, and details. By $Lip_\omega([0,\infty);X)$ for $\omega \in \mathbb{R}$ we denote the space consisting of those functions $F : [0,\infty) \to X$ with $F(0) = 0$ and for which $\|F\|_{Lip_\omega}$ defined as

$$\inf\{M \mid \|F(t+h) - F(t)\| \leq M \int_t^{t+h} e^{\omega r} dr \ \text{ for } t,\ h \geq 0\}$$

is finite. It is clear that if $F \in Lip_\omega([0,\infty);X)$ for some $\omega \geq 0$, the exponential bound $\omega(F)$ of F is less than or equal to ω. If $f \in L^1_{loc}([0,\infty);X)$ with $\omega(f) < \infty$, $f^{[1]} \in Lip_\omega([0,\infty);X)$ for any number $\omega > \omega(f)$. By $C^\infty_W((\omega,\infty);X)$ we denote the Widder space consisting of all those functions $r \in C^\infty((\omega,\infty);X)$ for which

$$\|r\|_{W,\omega} := \sup_{k\in\mathbb{N}_0,\ \lambda>\omega} \|(\lambda-\omega)^{k+1}\frac{1}{k!}r^{(k)}(\lambda)\| < \infty.$$

It is well-known that the Laplace-Stieltjes transform $F \mapsto \widehat{dF}(\lambda) := \int_0^\infty e^{-\lambda t} dF(t)$, $\lambda > \omega$ is an isometric isomorphism from $Lip_\omega([0,\infty);X)$ onto $C^\infty_W((\omega,\infty);X)$ (see [3], [4], or [5] for example).

THEOREM 1. *Let $\{F_n\}_n$ be a sequence of functions in $Lip_\omega([0,\infty);X)$ for which there exists a constant $M \geq 0$ such that $\|F_n\|_{Lip_\omega} \leq M$ for all $n \in \mathbb{N}$. Then the following are equivalent.*

(i) *There exist constants $a > \omega$ and $b > 0$ such that $\lim_{n\to\infty} \widehat{dF_n}(\lambda_k)$ exists for all $\lambda_k := a + kb$, $k \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.*

(ii) *There exists an $F \in Lip_\omega([0,\infty);X)$ such that $\|\widehat{dF}\|_{W,\omega} \leq M$ and $\{\widehat{dF_n}(\cdot)\}_n$ converges to $\widehat{dF}(\cdot)$ uniformly on compact subsets of (ω,∞).*

(iii) *$\lim_{n\to\infty} F_n(t)$ exists for every $t \geq 0$.*

(iv) *There exists an $F \in Lip_\omega([0,\infty);X)$ with $\|F\|_{Lip_\omega} \leq M$ such that $\{F_n(\cdot)\}_n$ converges to $F(\cdot)$ uniformly on compact subsets of $[0,\infty)$.*

The following is a characterization of solutions of (VE) in [4]. By $\widehat{f}(\lambda)$ we denote the Laplace transform $\int_0^\infty e^{-\lambda t} f(t)dt$ of f.

THEOREM 2. *Suppose that the assumptions in* **(A)** *hold. Let $f \in L^1_{loc}([0,\infty);X)$ be Laplace transformable. Let $v \in C([0,\infty);X)$ with $\omega(v) < \infty$ and let ω be a number such that $\omega \geq \max\{\epsilon,\ \text{abs}(f),\ \omega(v)\}$. Then the following are equivalent.*

(i) v *solves* (VE).
(ii) $\widehat{v}(\lambda) \in D(A)$ *and* $(I - \widehat{d\mu}(\lambda)A)\widehat{v}(\lambda) = \widehat{f}(\lambda)$ *if* $\lambda \in \mathbb{C}_\omega$.
(iii) $\widehat{v}(k) \in D(A)$ *and* $(I - \widehat{d\mu}(k)A)\widehat{v}(k) = \widehat{f}(k)$ *if* $\omega < k \in \mathbb{N}$.

The following definition of convoluted solution operator families is taken from [5].

DEFINITION 3. Suppose that the assumptions for A and μ in **(A)** hold. Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be Laplace transformable. Let $M > 0$ and $\omega \geq \max\{\epsilon, \text{ abs}(k)\}$ be some constants. Suppose that $(I - \widehat{d\mu}(\lambda)A)^{-1} \in L(X)$ for all $\lambda > \omega$. A strongly continuous mapping $S : [0,\infty) \to L(X)$ is said to be a k-convoluted solution operator family (k-c.s.o.f. for short) of exponential type $(M;\omega)$ with generator (A,μ) if the following hold.

(i) $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.
(ii) $\widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \widehat{S}(\lambda)x = \int_0^\infty e^{-\lambda t}S(t)x\,dt$ for every $\lambda > \omega$ and every $x \in X$.

REMARK 4. (i) If S is a k-c.s.o.f. of exponential type $(M;\omega)$ with generator (A,μ), then

$$S(t)x = \int_0^t S(t-s)Axd\mu(s) + k(t)x$$

for every $t \geq 0$ and $x \in D(A)$. (ii) If $k(t) = \frac{t^m}{m!}$, $t \geq 0$ for some $m \in \mathbb{N}_0$, a k-convoluted solution operator family with generator (A,μ) is an m-times intergated solution operator family with generator (m-i.s.o.f. for short) (A,μ).

One can refer to Corollary 3.1.12 in [4] for the following.

THEOREM 5. *Let S be an m-i.s.o.f. with generator (A,μ) for some $m \in \mathbb{N}_0$. Suppose that $f \in C^{(m+1)}([0,\infty);X)$, i.e., $f = g^{[m+1]}$ for some $g \in C([0,\infty);X)$. Then the function v defined as $v(t) := \int_0^t S(t-s)g(s)ds$, $t \geq 0$ is a solution of (VE).*

Theorems 1 and 2 imply a convergence of the solutions of (VE_n) to a solution of (VE).

THEOREM 6. *Suppose that the assumptions in* **(A)** *hold. Let A_n, $n \in \mathbb{N}$ be closed linear operators on the Banach space X and $\mu_n \in BV_\epsilon([0,\infty);\mathbb{C})$ for every $n \in \mathbb{N}$. Let f, $f_n \in L^1_{loc}([0,\infty);X)$ be Lapalce transformable functions for which there exists a constant $a \geq 0$ such that* $\text{abs}(f)$, $\text{abs}(f_n) \leq a$ *for all $n \in \mathbb{N}$. Let $v_n \in Lip_\omega([0,\infty);X)$ be* *a solution of* (VE_n) *for every* $n \in \mathbb{N}$ *for which there exist constants* $M > 0$ *and* $\omega \geq \max\{\epsilon, a\}$ *such that* $\|v_n\|_{Lip_\omega} \leq M$ *for all* $n \in \mathbb{N}$. *Suppose that* $(I - \widehat{d\mu}(\lambda)A)^{-1}$ *and* $(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}$ *exist in* $L(X)$ *for all* $n \in \mathbb{N}$ *and that* $\lim_{n\to\infty}(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}\widehat{f_n}(\lambda) = (I - \widehat{d\mu}(\lambda)A)^{-1}\widehat{f}(\lambda)$ *for every* $\lambda > \omega$. *Then* $v_n(t)$ *converges to a solution of* (VE) *uniformly on compact subsets of* $[0, \infty)$.

PROOF. Let $\lambda > \omega$. Since v_n is an exponentially bounded solution of (VE_n) with $\omega(v_n) \leq a \leq \omega$, by Theorem 2, $\lambda(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}\widehat{f_n}(\lambda) = \lambda\widehat{v_n}(\lambda) = \widehat{dv_n}(\lambda)$ for every $n \in \mathbb{N}$. Since $\|v_n\|_{Lip_\omega} \leq M$ for all $n \in \mathbb{N}$ and since by the hypothesis, $\lim_{n\to\infty}\widehat{dv_n}(\lambda) = \lambda(I - \widehat{d\mu}(\lambda)A)^{-1}\widehat{f}(\lambda)$ for every $\lambda > \omega$, it follows from Theorem 1 that the solution $v_n(t)$ of (VE_n) converges to a function $v(t)$ in $Lip_\omega([0,\infty); X)$ uniformly on compact subsets of $[0, \infty)$. By Theorem 2 and the uniqueness of a limit, $\lambda\widehat{v}(\lambda) = \widehat{dv}(\lambda) = \lambda(I - \widehat{d\mu}(\lambda)A)^{-1}\widehat{f}(\lambda)$ for every $\lambda > \omega$. Thus, $(I - \widehat{d\mu}(\lambda)A)\widehat{v}(\lambda) = \widehat{f}(\lambda)$ for every $\lambda > \omega$. By Theorem 2, $v(t)$ is a solution of the equation (VE). □

Let X be a Banach space and let $M > 0$ and $\omega \geq 0$ be some constants. A sequence $\{S_n\}_n$ of operator valued functions $S_n : [0, \infty) \to L(X)$ is said to be $(M; \omega)$-stable (or simply stable) if $\|S_n(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and $n \in \mathbb{N}$. The Trotter-Kato type approximations of convoluted solution operator families, Theorems 4.1 and 4.2 in [5] can be formulated to the case that the spaces on which the operators A_n are defined vary depending on n. For the results we set assumptions for (VE) and $(VE)_n$:

(B) Let $(X, \|\cdot\|)$ and $(X_n, \|\cdot\|_n)$, $n \in \mathbb{N}$ be Banach spaces. Let A and A_n, $n \in \mathbb{N}$ be closed linear operators on X and X_n, $n \in \mathbb{N}$ respectively. For some $\epsilon \geq 0$, let μ, $\mu_n \in BV_\epsilon([0, \infty); \mathbb{C})$ for all $n \in \mathbb{N}$.

In addition we assume that $f \in L^1_{loc}([0, \infty); X)$ and $f_n \in L^1_{loc}([0, \infty); X_n)$, $n \in \mathbb{N}$ are Laplace transformable. We prove the new results analogously to Theorems 4.1 and 4.2 in [5] but under weaker conditions on the scalar functions in Theorem 9 that follows.

THEOREM 7. *Suppose that the assumptions in* **(B)** *hold. Let* k *which is in* $L^1_{loc}([0, \infty); \mathbb{C})$ *be Laplace transformable. Let* $M > 0$, $L > 0$, *and* $\omega \geq \epsilon$ *be some constants. Let* $\{S_n\}_n$ *be an* $(M; \omega)$*-stable sequence of* k*-convoluted solution operator families* S_n *with generators* (A_n, μ_n) *for* $n \in \mathbb{N}$. *Let* $P_n \in L(X; X_n)$ *and* $Q_n \in L(X_n; X)$ *with* $\|P_n\|$, $\|Q_n\| \leq L$ *for all* $n \in \mathbb{N}$. *Suppose that* $(I - \widehat{d\mu}(\lambda)A)^{-1}$ *exists as an operator in* $L(X)$ *for* *every* $\lambda > \omega$ *and that* $\lim_{n\to\infty} Q_n(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}P_n x = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ *for every* $\lambda > \omega$ *and* $x \in X$*. Then* (A, μ) *generates a* $k^{[1]}$-c.s.o.f. $T \in Lip_\omega([0,\infty); L(X))$ *with* $\|T\|_{Lip_\omega} \le L^2 M$*. Moreover, for every* $x \in X$*, the sequence* $\{Q_n S_n^{[1]}(t)P_n x\}_n$ *converges to* $T(t)x$ *uniformly on compact subsets of* $[0,\infty)$*. If in addition,* A *is densely defined and* μ *is absolutely continuons on* $[0,\infty)$*, then there exists a* k-c.s.o.f. S *of exponential type* $(M;\omega)$ *with generator* (A,μ)*. In fact,* $S(t)x = \frac{dT(t)x}{dt}$ *for every* $t \ge 0$ *and* $x \in X$*.*

Proof. Define $T_n(t)x := Q_n S_n^{[1]}(t)P_n x := Q_n \int_0^t S_n(s)P_n x ds$ for every $n \in \mathbb{N}$, $t \ge 0$, and $x \in X$. Then clearly, $T_n \in Lip_\omega([0,\infty); L(X))$ with $\|T_n\|_{Lip_\omega} \le L^2 M$ for all $n \in \mathbb{N}$ and so $\|T_n(\cdot)x\|_{Lip_\omega} \le L^2 M\|x\|$ for all $n \in \mathbb{N}$ and all $x \in X$. Then it follows from $\lim_{n\to\infty} Q_n(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}P_n x = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ that $\widehat{dT_n}(\lambda)x = Q_n\widehat{S_n}(\lambda)P_n x = \widehat{k}(\lambda)Q_n(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}P_n x$ converges to $\widehat{k}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$. Thus, by Theorem 1, it holds that for every $x \in X$, there exists $T_x \in Lip_\omega([0,\infty); X)$ with $\|T_x\|_{Lip_\omega} \le L^2 M\|x\|$ such that $\{T_n(\cdot)x\}_n$ converges to $T_x(\cdot)$ uniformly on compact subsets of $[0,\infty)$. Define $T(t)x := T_x(t)$ for every $t \ge 0$ and $x \in X$. Then by the uniqueness of a limit, $T(t) : X \to X$ is linear for every $t \ge 0$. Moreover, $T \in Lip_\omega([0,\infty); L(X))$ with $\|T\|_{Lip_\omega} \le L^2 M$. Thus, it follows from Theorem 1 that for every $x \in X$, $\{\widehat{dT_n}(\lambda)x\}_n$ converges to $\widehat{dT}(\lambda)x$ uniformly on compact subsets of (ω,∞). By the uniqueness of a limit, $\widehat{k^{[1]}}(\lambda)(I - \widehat{d\mu}(\lambda)A)^{-1}x = \frac{\widehat{k}(\lambda)}{\lambda}(I - \widehat{d\mu}(\lambda)A)^{-1}x = \frac{\widehat{dT}(\lambda)}{\lambda}x = \widehat{T}(\lambda)x$ for every $\lambda > \omega$ and $x \in X$. Thus, T is a $k^{[1]}$-c.s.o.f. with generator (A,μ). Assuming that A is densely defined and μ is absolutely continuous on$[0,\infty)$, it follows from the second half of the proof of Theorem 3.4 in [5] that $S(t)x := \frac{dT(t)}{dt}x$ exists for all $t \ge 0$ and $x \in X$, that $S(t) \in L(X)$ for every $t \ge 0$, and that $S(t)x$ is continuous on $[0,\infty)$ for every $x \in X$, and finally that S is a k-c.s.o.f. with generator (A,μ). □

If $P_n Q_n$ is the identity operator on X_n and $\lim_{n\to\infty} Q_n P_n x = x$ for all $x \in X$ in Theorem 7, it is deduced under additional assumptions on μ and μ_n that for every $x \in X$, the sequence $\{Q_n S_n(t)P_n x\}_n$ converges to $S(t)x$ uniformly on compact subsets of $[0,\infty)$. We use the following elementary fact for the result.

Lemma 8. *Let* $\{T_n\}_n$ *be an* $(M;\omega)$*-stable sequence of* k*-convoluted solution operator families* $T_n : [0,\infty) \to L(X)$ *for every* $n \in \mathbb{N}$ *and for* *some* $\epsilon \geq 0$, *let* $\mu_n \in BV_\epsilon([0,\infty);\mathbb{C})$ *for all* $n \in \mathbb{N}$. *If* $\lim_{n\to\infty} \mu_n(t) = 0$, *then* $\lim_{n\to\infty} \int_0^t T_n(s)x d\mu_n(s) = 0$ *for every* $t \geq 0$ *and every* $x \in X$.

PROOF. Case 1. Suppose that μ_n is an increasing function in the space $BV_\epsilon([0,\infty);\mathbb{R})$ for every $n \in \mathbb{N}$. Let $t \geq 0$ and $x \in X$. Since $\| \int_0^t T_n(s)x d\mu_n(s)\| \leq \int_0^t \|T_n(s)x\| d\mu_n(s) \leq Me^{\omega t}\|x\|\mu_n(t)$ and since the last term converges to 0 as $n \to \infty$, it holds that $\lim_{n\to\infty} \int_0^t T_n(s)x d\mu_n(s) = 0$.

Case 2. Suppose that $\mu_n \in BV_\epsilon([0,\infty);\mathbb{R})$ for every $n \in \mathbb{N}$. Then $\mu_n = \alpha_n - \beta_n$ for some increasing functions α_n and β_n in $BV_\epsilon([0,\infty);\mathbb{R})$ and so $\int_0^t T_n(s)x d\mu_n(s) = \int_0^t T_n(s)x d\alpha_n(s) - \int_0^t T_n(s)x d\beta_n(s)$ for every $n \in \mathbb{N}$. Thus, it follows from Case 1 that $\lim_{n\to\infty} \int_0^t T_n(s)x d\mu_n(s) = 0$ for every $t \geq 0$ and $x \in X$.

Case 3. Suppose that $\mu_n \in BV_\epsilon([0,\infty);\mathbb{C})$ for every $n \in \mathbb{N}$. Then $\mu_n = g_n + ih_n$ for some g_n, $h_n \in BV_\epsilon([0,\infty);\mathbb{R})$ for every $n \in \mathbb{N}$ implies that $\lim_{n\to\infty} \int_0^t T_n(s)x d\mu_n(s) = 0$ for every $t \geq 0$ and $x \in X$. □

THEOREM 9. *Suppose that the assumptions in* **(B)** *hold. Additionally suppose that* A *and* A_n, $n \in \mathbb{N}$ *are densely defined and that* μ *and* μ_n, $n \in \mathbb{N}$ *are absolutely continuous. Suppose that there exist constants* R, $a > 0$ *such that* $\mu_n' \in Lip_a([0,\infty);\mathbb{C})$ *with* $|\mu_n'|_{Lip_a} \leq R$ *for all* $n \in \mathbb{N}$ *and that the sequence* $\{\mu_n(t)\}_n$ *converges to* $\mu(t)$ *uniformly on compact subsets of* $[0,\infty)$. *Let* $k \in L^1_{loc}([0,\infty);\mathbb{C})$ *be a Laplace transformable function which is bounded on compact subsets of* $[0,\infty)$. *Let* $M > 0$ *and* $\omega \geq \max\{\epsilon, a, \mathrm{abs}(k)\}$ *be some constants. Let* $\{S_n\}_n$ *be an* $(M;\omega)$*-stable sequence of* k*-convoluted solution operator families* S_n *with generators* (A_n,μ_n) *for* $n \in \mathbb{N}$. *Suppose that* $\widehat{d\mu} \not\equiv 0$ *on* (ω,∞) *and that* $(I - \widehat{d\mu}(\lambda)A)^{-1}$ *exists in* $L(X)$ *for every* $\lambda > \omega$. *Let* $P_n \in L(X;X_n)$ *and* $Q_n \in L(X_n;X)$ *such that* $P_nQ_n = I_n$, *the identity operator on* X_n *and* $\lim_{n\to\infty} Q_nP_nx = x$ *for all* $x \in X$. *Suppose that there exists a constant* $L \geq 0$ *such that* $\|P_n\|$, $\|Q_n\| \leq L$ *for all* $n \in \mathbb{N}$ *and that* $\lim_{n\to\infty} Q_n(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}P_nx = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ *for every* $\lambda > \omega$ *and* $x \in X$. *Then there exists a* k-c.s.o.f. $S = \{S(t)\}_{t\geq 0}$ *of exponential type* $(L^2M;\omega)$ *with generator* (A,μ) *such that for every* $x \in X$, *the sequence* $\{Q_nS_n(t)P_nx\}_n$ *converges to* $S(t)x$ *uniformly on compact subsets of* $[0,\infty)$.

PROOF. Since the hypotheses include the assumptions for Theorem 7, (A,μ) generates a k-c.s.o.f. $S = \{S(t)\}_{t\geq 0}$ of exponential type

$(L^2M;\omega)$. For the uniform convergence of $\{S_n(t)\}_n$ to $S(t)$ on compact subsets of $[0,\infty)$, we first show that for every $y \in D(A)$, the sequence $\{Q_nS_n(t)P_ny\}_n$ converges to $S(t)y$ uniformly on compact subsets of $[0,\infty)$. Let $y \in D(A)$ and $y_n \in D(A_n)$ for $n \in \mathbb{N}$. Since S and S_n, $n \in \mathbb{N}$ are k-convoluted solution operator families with generators (A,μ) and (A_n,μ_n), $n \in \mathbb{N}$, respectively and Q_n are bounded linear operators, it follows from Remark 4 following Definition 3 that

$$S(t)y = \int_0^t S(t-s)Ayd\mu(s) + k(t)y \tag{1}$$

and

$$Q_nS_n(t)y_n = \int_0^t Q_nS_n(t-s)A_ny_nd\mu_n(s) + k(t)Q_ny_n \tag{2}$$

hold for every $t \geq 0$. Let $h(\lambda) := (I - \widehat{d\mu}(\lambda)A)^{-1}$ and let $h_n(\lambda) := (I - \widehat{d\mu_n}(\lambda)A_n)^{-1}$ for $\lambda > \omega$ and $n \in \mathbb{N}$. Then from the hypothesis, $\lim_{n\to\infty} Q_nh_n(\lambda)P_nx = h(\lambda)x$ for every $\lambda > \omega$ and $x \in X$. Let $\lambda_0 > \omega$ such that $\widehat{d\mu}(\lambda_0) \neq 0$ and let $z := (I - \widehat{d\mu}(\lambda_0)A)y$. Then $y = h(\lambda_0)z$.

$$\begin{aligned} &\|Q_nS_n(t)P_ny - S(t)y\| \\ &\leq \|Q_nS_n(t)P_n\Big(h(\lambda_0)z \;-\; Q_nh_n(\lambda_0)P_nz\Big)\| \\ &\quad + \;\|Q_nS_n(t)h_n(\lambda_0)P_nz - S(t)h(\lambda_0)z\|. \end{aligned} \tag{3}$$

Since $Q_nS_n(t)P_n$ are uniformly bounded on compact subsets of $[0,\infty)$ and since $\lim_{n\to\infty} Q_nh_n(\lambda_0)P_nz = h(\lambda_0)$, it suffices to estimate the convergence of the second term in (3). It is deduced from the condition $\mu_n' \in Lip_a([0,\infty);\mathbb{C})$ with $|\mu_n'|_{Lip_a} \leq R$ for all $n \in \mathbb{N}$ that $\mu_n \in Lip_a([0,\infty);\mathbb{C})$ and $\|\mu_n\|_{Lip_a} \leq R$ for all $n \in \mathbb{N}$ (see [4] or [5] for example). Since $\lim_{n\to\infty} \mu_n(t) = \mu(t)$ for every $t \geq 0$ with $\|\mu_n\|_{Lip_a} \leq R$ for all $n \in \mathbb{N}$, by Theorem 1, $\lim_{n\to\infty} \widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0)$. Since $\lim_{n\to\infty} \widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0) \neq 0$, to estimate the convergence of the second term

$$\|Q_nS_n(t)h_n(\lambda_0)P_nz - S(t)h(\lambda_0)z\|$$

in (3) is equivalent to estimate that of the sequence

$$\{\|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\big(Q_nS_n(t)h_n(\lambda_0)P_nz - S(t)h(\lambda_0)z\big)\|\}_n.$$

By (1) and (2),

$$
\begin{aligned}
&\|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\big(Q_nS_n(t)h_n(\lambda_0)P_nz - S(t)h(\lambda_0)z\big)\| \\
&\le \|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\Big(\int_0^t Q_nS_n(t-s)A_nh_n(\lambda_0)P_nz\ d\mu_n(s) \\
&\qquad\qquad - \int_0^t S(t-s)Ah(\lambda_0)z\ d\mu(s)\Big)\| \\
&\quad + \|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)k(t)\Big(Q_nh_n(\lambda_0)P_nz - h(\lambda_0)z\Big)\|.
\end{aligned}
\tag{4}
$$

Since the second term converges to 0 uniformly on compact subsets of $[0,\infty)$, it suffices to estimate the convergence of the first term in (4).

$$
\begin{aligned}
&\|\widehat{d\mu}(\lambda_0)\widehat{d\mu_n}(\lambda_0)\Big(\int_0^t Q_nS_n(t-s)A_nh_n(\lambda_0)P_nzd\mu_n(s) \\
&\qquad\qquad - \int_0^t S(t-s)Ah(\lambda_0)zd\mu(s)\Big)\| \\
&= \|\widehat{d\mu}(\lambda_0)\int_0^t Q_nS_n(t-s)(h_n(\lambda_0)-I)P_nzd\mu_n(s) \\
&\qquad\qquad - \widehat{d\mu_n}(\lambda_0)\int_0^t S(t-s)(h(\lambda_0)-I)zd\mu(s)\| \\
&\le \|\Big(\widehat{d\mu}(\lambda_0)-\widehat{d\mu_n}(\lambda_0)\Big)\int_0^t Q_nS_n(t-s)(h_n(\lambda_0)-I)P_nzd\mu_n(s)\| \\
&\quad + |\widehat{d\mu_n}(\lambda_0)|\ \Big\|\int_0^t Q_nS_n(t-s)(h_n(\lambda_0)-I)P_nzd\mu_n(s) \\
&\qquad\qquad - \int_0^t S(t-s)(h(\lambda_0)-I)zd\mu(s)\Big\|.
\end{aligned}
\tag{5}
$$

Since $\|\int_0^t Q_nS_n(t-\cdot)(h_n(\lambda_0)-I)P_nz\ d\mu_n(s)\|$ are uniformly bounded on compact subsets of $[0,\infty)$ and since $\lim_{n\to\infty}\widehat{d\mu_n}(\lambda_0) = \widehat{d\mu}(\lambda_0)$ in the first term of (5), it suffices to estimate the convergence of the term

$$
\|\int_0^t Q_nS_n(t-s)(h_n(\lambda_0)-I)P_nzd\mu_n(s) - \int_0^t S(t-s)(h(\lambda_0)-I)zd\mu(s)\|
$$

in the second term of (5).

$$
\begin{aligned}
&\Big\| \int_0^t Q_n S_n(t-s)(h_n(\lambda_0)-I)P_n z d\mu_n(s) \\
&\qquad\qquad - \int_0^t S(t-s)(h(\lambda_0)-I)z d\mu(s)\Big\| \\
&= \Big\| \int_0^t Q_n S_n(s)(h_n(\lambda_0)-I)P_n z d\mu_n(t-s) \\
&\qquad\qquad - \int_0^t S(s)(h(\lambda_0)-I)z d\mu(t-s)\Big\| \\
&= \Big\| \int_0^t Q_n S_n(s)P_n\Big(Q_n h_n(\lambda_0)P_n z - Q_n P_n z\Big) d\mu_n(t-s) \\
&\qquad\qquad - \int_0^t S(s)(h(\lambda_0)-I)z d\mu(t-s)\Big\| \\
&\le \Big\| \int_0^t Q_n S_n(s)P_n\Big(Q_n h_n(\lambda_0)P_n z - h(\lambda_0)z\Big) d\mu_n(t-s)\Big\| \\
&\quad + \Big\| \int_0^t \Big(Q_n S_n(s)P_n - S(s)\Big)(h(\lambda_0) - Q_n P_n)z\, d\mu_n(t-s)\Big\| \\
&\quad + \Big\| \int_0^t S(s)\Big(I - Q_n P_n\Big) z d\mu_n(t-s)\Big\| \\
&\quad + \Big\| \int_0^t S(s)(h(\lambda_0)-I)z d\Big(\mu_n(t-s) - \mu(t-s)\Big)\Big\| \\
&= \Big\| \int_0^t Q_n S_n(s)P_n(Q_n h_n(\lambda_0)P_n z - h(\lambda_0)z)\mu_n'(t-s) ds\Big\| \\
&\quad + \Big\| \int_0^t \Big(Q_n S_n(s)P_n - S(s)\Big)(h(\lambda_0) - Q_n P_n)z \mu_n'(t-s) ds\Big\| \\
&\quad + \Big\| \int_0^t S(s)(z - Q_n P_n z)\mu_n'(t-s)\, ds\Big\| \\
&\quad + \Big\| \int_0^t S(t-s)(h(\lambda_0)-I)z d\Big(\mu_n(s) - \mu(s)\Big)\Big\|.
\end{aligned}
\tag{6}
$$

Since $\|Q_n S_n(s) P_n\|\,|\mu_n'(t-s)|$ are uniformly bounded on compact subsets of $[0,\infty)$ and since $\lim_{n\to\infty} Q_n h_n(\lambda_0) P_n z = h(\lambda_0)z$, the first term in (6) converges to 0 uniformly on compact subsets of $[0,\infty)$. Since $\|S(s)\|\,|\mu_n'(t-s)|$ are uniformly bounded on compact subsets of $[0,\infty)$

and $\lim_{n\to\infty} Q_nP_nz = z$, the third term in (6) converges to 0 uniformly on compact subsets of $[0,\infty)$. Lemma 8 implies that the fourth term in (6) converges uniformly on compact subsets of $[0,\infty)$. Thus, it suffices to estimate the second term in (6). By the integration by parts,

$$
\begin{aligned}
&\| \int_0^t \Big(Q_nS_n(s)P_n - S(s)\Big)(h(\lambda_0) - Q_nP_n)z\mu_n'(t-s)ds\| \\
&\le \| \int_0^t \Big(Q_nS_n^{[1]}(s)P_n - S^{[1]}(s)\Big)(h(\lambda_0)z - Q_nP_nz)\mu_n''(t-s)ds\| \\
(7)\quad &\le \sup_{n\in\mathbb{N}} \|h(\lambda_0)z - Q_nP_nz\| \cdot \sup_{n\in\mathbb{N}} \operatorname{ess} \sup_{s\in[0,t]} |\mu_n''(s)| \cdot \\
&\qquad\qquad \int_0^t \|Q_nS_n^{[1]}(s)P_n - S^{[1]}(s)\|ds.
\end{aligned}
$$

Since $\sup_{n\in\mathbb{N}} |\mu_n'|_{Lip_a} \le R$, $\sup_{n\in\mathbb{N}} \operatorname{ess}\sup_{s\in[0,t]} |\mu_n''(s)| < \infty$ for every $t \ge 0$. Since $\lim_{n\to\infty} Q_nP_nx = x$ for all $x \in X$, $\sup_{n\in\mathbb{N}} \|h(\lambda_0)z - Q_nP_nz\|$ is finite. Since for every $x \in X$, $\{Q_nS_n^{[1]}(s)P_nx\}_n$ converges to $S^{[1]}(s)x$ uniformly on compact subsets of $[0,\infty)$, (7) converges to 0 uniformly on compact subsets of $[0,\infty)$. Thus, $Q_nS_n(t)P_ny$ converges to $S(t)y$ uniformly on compact subsets of $[0,\infty)$ for every $y \in D(A)$. Since $\overline{D(A)} = X$ and since $Q_nS_n(t)P_n$ are uniformly bounded on compact subsets of $[0,\infty)$, $Q_nS_n(t)P_nx$ converges to $S(t)x$ uniformly on compact subsets of $[0,\infty)$ for every $x \in X$. □

If $X_n = X$ and $Q_n = P_n = I$, the identity operator on X for all $n \in \mathbb{N}$ Theorem 7 becomes Theorem 4.1 in [5] and Theorem 9 does almost Theorem 4.2 in [5] but with weaker conditions on the scalar functions μ_n and μ as follows.

COROLLARY 10. (a) *Let A and A_n, $n \in \mathbb{N}$ be densely defined closed linear operators on a Banach space X and let μ and μ_n, $n \in \mathbb{N}$ be absolutely continuous functions in $BV_\epsilon([0,\infty);\mathbb{C})$ for some $\epsilon \ge 0$ for all $n \in \mathbb{N}$. Additionally suppose that $D(A) \cap \bigcap_{n\in\mathbb{N}} D(A_n)$ contains a dense subset D of X, that $\mu_n(t)$ converges to $\mu(t)$ uniformly on compact susets of $[0,\infty)$, and that there exist constants L, $a > 0$ such that $\mu_n' \in Lip_a([0,\infty);\mathbb{C})$ and $|\mu_n'|_{Lip_a} \le L$ for all $n \in \mathbb{N}$.*

(b) *Let $k \in L^1_{loc}([0,\infty);\mathbb{C})$ be a Laplace transformable function which is bounded on compact subsets of $[0,\infty)$. Let $\omega \ge \max\{\epsilon,\ a,\ \mathrm{abs}(k)\}$*

and $M > 0$ be some constants. Let $\{S_n\}_n$ be an $(M;\omega)$-stable sequence of k-convoluted solution operator families S_n with generators (A_n, μ_n) for $n \in \mathbb{N}$.

(c) *Suppose that $\widehat{d\mu} \not\equiv 0$ on (ω, ∞) and that $(I - \widehat{d\mu}(\lambda)A)^{-1}$ exists in $L(X)$ for every $\lambda > \omega$ and $\lim_{n\to\infty}(I - \widehat{d\mu_n}(\lambda)A_n)^{-1}x = (I - \widehat{d\mu}(\lambda)A)^{-1}x$ for every $\lambda > \omega$ and $x \in X$.*

Then there exists a k-c.s.o.f. S of exponential type $(M;\omega)$ with generator (A, μ) for which for every $x \in X$, the sequence $\{S_n(\cdot)x\}_n$ converges to $S(\cdot)x$ uniformly on compact subsets of $[0, \infty)$.

Suppose that for some $x \in X$, $f(t) = f_n(t) = \frac{t^{m+1}}{(m+1)!}x$ for all $t \geq 0$ and $n \in \mathbb{N}$ in (VE_n) and (VE) so that the equations (VE_n) and (VE) become

$$(VE_n{}') \qquad v(t) = A_n \int_0^t v(t-s)d\mu_n(s) + \frac{t^{m+1}}{(m+1)!}x, \qquad t \geq 0$$

$$(VE{}') \qquad v(t) = A \int_0^t v(t-s)d\mu(s) + \frac{t^{m+1}}{(m+1)!}x, \qquad t \geq 0,$$

respectively. If $k(t) = \frac{t^m}{m!}$ for some $m \in \mathbb{N}_0$ in Corollary 10 so that (A_n, μ_n), $n \in \mathbb{N}$ generate m-times integrated solution operator families, a solution of the equation (VE') is obtained as a limit of the solutions of the equations $(VE_n{}')$ as follows.

Assume (a′) through (c′) :

(a′) the assumption (a) in Corollary 10.

(b′) Let $m \in \mathbb{N}_0$. For some constants $M > 0$ and $\omega \geq \max\{\epsilon,\ a\}$, suppose that $\{S_n\}_n$ is an $(M;\omega)$-stable sequence of m-times intergated solution operator families S_n with generators (A_n, μ_n) for $n \in \mathbb{N}$.

(c′) the assumption (c) in Corollary 10.

Then by Corollary 10, (A, μ) generates an m-i.s.o.f. S of exponential type $(M;\omega)$ for which for every $x \in X$, $\{S_n(\cdot)x\}_n$ converges to $S(\cdot)x$ uniformly on compact subsets of $[0, \infty)$. Thus, $v_n(t) := \int_0^t S_n(s)x\, ds$ converges to $\int_0^t S(s)xds$ uniformly on compact subsets of $[0, \infty)$. Note that by Theorem 5, $v(t) := \int_0^t S(s)xds$ is the unique solution of (VE').

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