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ALMOST STABILITY OF ISHIKAWA ITERATIVE SCHEMES WITH ERRORS FOR ϕ-STRONGLY QUASI-ACCRETIVE AND ϕ-HEMICONTRACTIVE OPERATORS

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ABSTRACT. In this paper, we establish almost stability of Ishikawa iterative schemes with errors for the classes of Lipschitz ϕ-strongly quasi-accretive operators and Lipschitz ϕ-hemicontractive operators in arbitrary Banach spaces. The results of this paper extend a few well-known recent results.

1. Introduction

Let X be an arbitrary Banach space, X^* its dual space and $\langle x, f\rangle$ the generalized duality pairing between $x \in X$ and $f \in X^*$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : Re\langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \quad x \in X.$$

DEFINITION 1.1. ([1], [4], [16]) Let T be an operator with domain $D(T)$ and range $R(T)$ in X. Let $F(T) = \{x \in D(T) : Tx = x\}, N(T) = \{x \in D(T) : Tx = 0\}$ and I denote the identity operator on X.

(i) T is said to be *strongly accretive* if there exists a constant $k \in (0, 1)$ such that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - Ty, j(x - y)\rangle \geq k\|x - y\|^2;$$

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(ii) T is said to be *ϕ-strongly accretive* if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|;$$

(iii) T is said to be *strongly quasi-accretive* if $N(T) \neq \emptyset$ and if there exists a constant $k \in (0,1)$ such that for all $x \in D(T), y \in N(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2;$$

(iv) T is said to be *ϕ-strongly quasi-accretive* if $N(T) \neq \emptyset$ and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x \in D(T), y \in N(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re\langle Tx - Ty, j(x-y)\rangle \geq \phi(\|x-y\|)\|x-y\|;$$

(v) T is said to be *strongly pseudocontractive* (*ϕ-strongly pseudocontractive, strictly hemicontractive, ϕ-hemicontractive*, resp.) if $I - T$ is strongly accretive (ϕ-strongly accretive, strongly quasi-accretive, ϕ-strongly quasi-accretive, resp.).

Let K be a nonempty convex subset of an arbitrary Banach space X and $T : K \to K$ be an operator. Assume that $x_0 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iterative scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset K$. Suppose, furthermore, that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in K and put $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$.

DEFINITION 1.2. ([8-10], [17]) (i) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *T-stable* on K if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$;

(ii) The iteration scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be *almost T-stable* on K if $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

It is easy to verify that an iterative scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable on K is almost T-stable on K. Osilike [17] proved that the converse is not true.

Let us recall the following three iterative processes due to Mann [15], Ishikawa [11] and Xu [22], respectively.

DEFINITION 1.3. Let K be a nonempty convex subset of an arbitrary Banach space X and $T : K \to K$ be an operator.

(i) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_n T y_n, \\ y_n = (1 - b_n)x_n + b_n T x_n, & n \geq 0, \end{cases}$$

is called the *Ishikawa iterative sequence*, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ satisfying appropriate conditions.

(ii) If $b_n = 0$ for all $n \geq 0$ in (i), then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, x_{n+1} = (1 - a_n)x_n + a_n T x_n, \qquad n \geq 0,$$

is called the *Mann iterative sequence*.

(iii) For any given $x_0 \in K$ the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \\ y_n = a'_n x_n + b'_n T x_n + c'_n v_n, & n \geq 0, \end{cases}$$

where $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for all $n \geq 0$, is called the *Ishikawa iterative sequence with errors*.

(iv) If $b'_n = c'_n = 0$ for all $n \geq 0$ in (iii), then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, n \geq 0,$$

is called the *Mann iterative sequence with errors*.

Many stability results for certain classes of nonlinear mappings have been established by several authors (see, [8-10], [17]). Rhoades [19] proved that the Mann and Ishikawa iterative methods may exhibit different behaviors for different classes of nonlinear mappings. Harder-Hicks [10] revealed the importance of investigating the stability of various iterative procedures for various classes of nonlinear mappings. Harder [8] established applications of stability results to first order differential equations. In [17], Osilike proved that certain Ishikawa iterative sequences are almost stable for Lipschitz ϕ-strongly pseudocontractive operators and Lipschitz ϕ-strongly accretive operators in real Banach spaces.

These classes of nonlinear operators in Definition 1.1 have been studied by various researchers (see, [1-7], [12-14], [16-22]). Osilike [16] proved that the class of strongly pseudocontractive operators is a proper subclass of the class of ϕ-strongly pseudocontractive operators, and pointed out that the class of ϕ-strongly pseudocontractive operators with a fixed point is a proper subclass of the class of ϕ-hemicontractive operators. Chidume-Osilike [4] proved that each strongly pseudocontractive operator with a fixed point is strictly hemicontractive, but the converse does not hold in general. It is known that any strictly hemicontractive operator is ϕ-hemicontractive.

On the other hand, Chidume [1] obtained the Mann iterative method can be used to approximate fixed points of Lipschitz strongly pseudocontractive operators in L_p(or l_p) spaces for $p \in [2, \infty)$. Afterwards, authors extended the result in many directions. Schu [20] generalized the result in [1] to real Banach spaces with property $(U, \lambda, m+1, m)$. In [16] and [18], Osilike extended the result in [1] to both Lipschitz ϕ-strongly pseudocontractive operators, Lipschitz ϕ-strongly accretive operators or Lipschitz ϕ-hemicontractive operators and real q-uniformly smooth Banach spaces or real Banach spaces.

In this paper, we establish the almost stability of Ishikawa iterative schemes with errors for the classes of Lipschitz ϕ-strongly quasi-accretive operators and Lipschitz ϕ-hemicontractive operators in arbitrary Banach spaces. We prove that the class of strictly hemicontractive operators is a proper subclass of the class of ϕ-hemicontractive operators. The results of this paper extend the corresponding results in [1], [16-18] and [20].

LEMMA 2.1. [13] *Let $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ be nonnegative sequences satisfying*

$$\alpha_{n+1} \le (1+\beta_n)\alpha_n + \omega_n, \quad n \ge 0,$$
$$\sum_{n=0}^{\infty} \beta_n < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \omega_n < \infty.$$

Then $\{\alpha_n\}_{n=0}^{\infty}$ is bounded.

LEMMA 2.2. [21] *Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be nonnegative sequences satisfying*

$$\alpha_{n+1} \le \alpha_n + \beta_n, \qquad n \ge 0,$$

and $\sum_{n=0}^{\infty} \beta_n < \infty$. Then $\lim_{n\to\infty} \alpha_n$ exists.

LEMMA 2.3. *Suppose that $\phi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $\phi(0) = 0$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ are nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} \gamma_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n < \infty, \quad \sum_{n=0}^{\infty} \omega_n < \infty, \tag{2.1}$$

and

$$\alpha_{n+1} + \min\{\alpha_n, \alpha_{n+1}\} \frac{\phi(\alpha_{n+1})}{1 + \alpha_{n+1} + \phi(\alpha_{n+1})} \gamma_n \le \alpha_n + \beta_n \alpha_n + \omega_n \tag{2.2}$$

for all $n \ge 0$. Then $\lim_{n\to\infty} \alpha_n = 0$.

PROOF. It follows from (2.2) that

$$\alpha_{n+1} \le (1 + \beta_n)\alpha_n + \omega_n, \qquad n \ge 0. \tag{2.3}$$

Lemma 2.1, (2.1) and (2.3) yield that there exists $M > 0$ such that $\alpha_n \le M$ for all $n \ge 0$. Using (2.3), we have

$$\alpha_{n+1} \le \alpha_n + (M\beta_n + \omega_n), \quad n \ge 0.$$

From the above inequality, (2.1) and Lemma 2.2, we infer that $\lim_{n\to\infty}\alpha_n = r \ge 0$. Suppose that $r > 0$. Then there exists a positive integer N such that

$$\frac{1}{2} r \le \alpha_n \le \frac{3}{2} r, \quad n \ge N. \tag{2.4}$$

In view of (2.2) and (2.4), we obtain that for any $n \ge N$,

$$\begin{aligned} \frac{\frac{r}{2}\phi(\frac{r}{2})}{1 + \frac{3}{2}r + \phi(\frac{3}{2}r)} \gamma_n &\le \min\{\alpha_n, \alpha_{n+1}\} \frac{\phi(\alpha_{n+1})}{1 + \alpha_{n+1} + \phi(\alpha_{n+1})} \gamma_n \\ &\le \alpha_n - \alpha_{n+1} + (M\beta_n + \omega_n), \end{aligned}$$

which implies that

$$\frac{\frac{r}{2}\phi(\frac{r}{2})}{1 + \frac{3}{2}r + \phi(\frac{3}{2}r)} \sum_{n=N}^{\infty} \gamma_n \le \alpha_N + M \sum_{n=N}^{\infty} \beta_n + \sum_{n=N}^{\infty} \omega_n < \infty,$$

which is impossible. Hence $\lim_{n\to\infty} \alpha_n = 0$. This completes the proof.□

REMARK 2.1. The Lemma in [7] and the Lemma in [18] are special cases of Lemma 2.1.

LEMMA 2.4. [14] *Let X be a Banach space and $x, y \in X$. Then $\|x\| \le \|x + ty\|$ for each $t > 0$ if and only if there exists $j(x) \in J(x)$ such that $Re\langle y, j(x)\rangle \ge 0$.*

3. Convergence and almost stability

In the sequel, d_n and d'_n denote $b_n + c_n$ and $b'_n + c'_n$, respectively. Let $L_* = 1 + L$, $L \geq 1$ denote the Lipschitzian constant of T, and $A(x,y) = \frac{\phi(\|x-y\|)}{1+\|x-y\|+\phi(\|x-y\|)}$ for each $x, y \in X$.

THEOREM 3.1. *Let X be a Banach space and let $T : X \to X$ be a Lipschitz ϕ-strongly quasi-accretive operator. Suppose that $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are arbitrary sequences in $[0,1]$ satisfying*

$$a_n + d_n = a'_n + d'_n = 1, \quad n \geq 0; \tag{3.1}$$

$$\sum_{n=0}^{\infty} b_n = \infty; \tag{3.2}$$

$$\sum_{n=0}^{\infty} c_n < \infty, \quad \sum_{n=0}^{\infty} b_n^2 < \infty, \quad \sum_{n=0}^{\infty} b_n d'_n < \infty. \tag{3.3}$$

Define $S : X \to X$ by $Sx = x - Tx$ for all $x \in X$. Assume that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in X. Suppose that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from arbitrary $x_0 \in X$ by

$$\begin{cases} z_n = a'_n x_n + b'_n S x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n S z_n + c_n u_n, \quad n \geq 0. \end{cases} \tag{3.4}$$

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in X and define $\{\epsilon_n\}_{n=0}^{\infty}$ by

$$w_n = a'_n y_n + b'_n S y_n + c'_n v_n, \quad \epsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0, \tag{3.5}$$

where $p_n = a_n y_n + b_n S w_n + c_n u_n$. Then

(i) The sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique zero q of T;

(ii)
$$\begin{aligned} &\|y_{n+1} - q\| \\ &\leq [1 - A(p_n, q) d_n] \|y_n - q\| + [(2 + L_* + L_*^2) d_n^2 \\ &\quad + L_* d_n d'_n + L_*^2 d_n b'_n + L_*^3 d_n b_n + L_*^2 (1 + d_n) c_n] \|y_n - q\| \\ &\quad + (2 + L_*) c_n \|u_n - q\| + L_* c'_n [d_n^2 + (1 + d_n) c_n \\ &\quad + (1 + L_*) d_n] \|v_n - q\| + \epsilon_n, \quad n \geq 0; \end{aligned}$$

(iii) $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$, so that $\{x_n\}_{n=0}^{\infty}$ is almost S-stable on X;

(iv) $\lim_{n\to\infty} y_n = q$ implies that $\lim_{n\to\infty} \epsilon_n = 0$.

PROOF. Since T is ϕ-strongly quasi-accretive, it follows that T has a unique zero. Let q be the zero of T. Then $Sq = q$. From (3.1) and (3.4), we get that

$$\begin{aligned} \|z_n - q\| &\le (1 - d'_n)\|x_n - q\| + b'_n\|Sx_n - q\| + c'_n\|x_n - q\| \\ &\le (1 - d'_n + b'_n L_*)\|x_n - q\| + c'_n\|v_n - q\| \qquad (3.6) \\ &\le L_*\|x_n - q\| + c'_n\|v_n - q\| \end{aligned}$$

and

$$\begin{aligned} \|w_n - q\| &\le (1 - d'_n)\|y_n - q\| + b'_n\|Sy_n - q\| + c'_n\|v_n - q\| \qquad (3.7) \\ &\le L_*\|y_n - q\| + c'_n\|v_n - q\| \end{aligned}$$

for all $n \ge 0$. Using (3.1) and (3.4) again, we obtain that

$$\begin{aligned} x_n = {} & (1 + d_n A(x_{n+1}, q))x_{n+1} + d_n(I - S - A(x_{n+1}, q))x_{n+1} \\ & + d_n^2(x_n - Sz_n) - (1 + d_n)c_n(u_n - Sz_n) \qquad (3.8) \\ & + d_n(Sx_{n+1} - Sz_n) \end{aligned}$$

and

$$q = (1 + d_n A(x_{n+1}, q))q + d_n(I - S - A(x_{n+1}, q))q \qquad (3.9)$$

for all $n \ge 0$. It follows from (3.8), (3.9) and Lemma 2.4 that for all $n \ge 0$,

$$\begin{aligned} & \|x_n - q\| \\ & \ge [1 + d_n A(x_{n+1}, q)]\|x_{n+1} - q + \frac{d_n}{1 + d_n A(x_{n+1}, q)}[(I - S \\ & \quad - A(x_{n+1}, q))x_{n+1} - (I - S - A(x_{n+1}, q))q]\| \\ & \quad - d_n^2\|x_n - Sz_n\| - (1 + d_n)c_n\|u_n - Sz_n\| - d_n\|Sx_{n+1} - Sz_n\| \\ & \ge [1 + d_n A(x_{n+1}, q)]\|x_{n+1} - q\| - d_n^2\|x_n - Sz_n\| \\ & \quad - (1 + d_n)c_n\|u_n - Sz_n\| - d_n L_*\|x_{n+1} - z_n\| \end{aligned}$$

which implies that for all $n \ge 0$,

$$\begin{aligned} & [1 + d_n A(x_{n+1}, q)]\|x_{n+1} - q\| \\ & \le \|x_n - q\| + d_n^2\|x_n - Sz_n\| + (1 + d_n)c_n\|u_n - Sz_n\| \\ & \quad + L_* d_n\|x_{n+1} - z_n\| \\ & \le (1 + d_n^2)\|x_n - q\| + [d_n^2 + (1 + d_n)c_n]\|Sz_n - q\| \qquad (3.10) \\ & \quad + (1 + d_n)c_n\|u_n - q\| + L_* d_n(b_n\|Sz_n - x_n\| + c_n\|u_n - x_n\| \\ & \quad + b'_n\|Sx_n - x_n\| + c'_n\|v_n - x_n\|) \\ & \le \|x_n - q\| + \beta_n\|x_n - q\| + \omega_n \end{aligned}$$

by (3.4) and (3.6), where

$$\beta_n = (1 + L_* + L_*^2)d_n^2 + L_* d_n d_n' + L_*^2 d_n b_n' + L_*^3 d_n b_n + L_*^2(1 + d_n)c_n,$$

and

$$\begin{aligned}\omega_n = &[1 + (1 + L_*)d_n]c_n\|u_n - q\| \\ &+ L_* c_n'[d_n^2 + (1 + d_n)c_n + L_* d_n b_n + d_n]\|v_n - q\|\end{aligned}$$

for all $n \geq 0$. (3.3) ensures that $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\sum_{n=0}^{\infty} \omega_n < \infty$. In view of Lemma 2.3, (3.2) and (3.3), we conclude immediately that $\lim_{n\to\infty} x_n = q$.

From (3.1) and (3.5), we get that for all $n \geq 0$,

$$\begin{aligned}y_n = &(1 + d_n A(p_n, q))p_n + d_n(I - S - A(p_n, q))p_n \\ &+ d_n^2(y_n - Sw_n) - (1 + d_n)c_n(u_n - Sw_n) \\ &+ d_n(Sp_n - Sw_n)\end{aligned} \tag{3.11}$$

and

$$q = (1 + d_n A(p_n, q))q + d_n(I - S - A(p_n, q))q. \tag{3.12}$$

It follows from Lemma 2.4, (3.11) and (3.12) that for all $n \geq 0$,

$$\begin{aligned}&\|y_n - q\| \\ &\geq [1 + d_n A(p_n, q)]\|p_n - q + \frac{d_n}{1 + A(p_n, q)d_n}[(I - S - A(p_n, q))x_{n+1} \\ &\quad - (I - S - A(p_n, q))q]\| - d_n^2\|y_n - Sw_n\| \\ &\quad - (1 + d_n)c_n\|u_n - Sw_n\| - d_n\|Sp_n - Sw_n\| \\ &\geq [1 + d_n A(p_n, q)]\|p_n - q\| - d_n^2\|y_n - Sw_n\| \\ &\quad - (1 + d_n)c_n\|u_n - Sw_n\| - L_* d_n\|p_n - w_n\|,\end{aligned}$$

which means that

$$\begin{aligned}&[1 + d_n A(p_n, q)]\|p_n - q\| \\ &\leq \|y_n - q\| + d_n^2\|y_n - Sw_n\| + (1 + d_n)c_n\|u_n - Sw_n\| \\ &\quad + L_* d_n\|p_n - w_n\| \\ &\leq (1 + d_n^2)\|y_n - q\| + [d_n^2 + (1 + d_n)c_n]\|Sw_n - q\| \\ &\quad + (1 + d_n)c_n\|u_n - q\| + L_* d_n(b_n\|Sw_n - y_n\| + c_n\|u_n - y_n\| \\ &\quad + b_n'\|Sy_n - y_n\| + c_n'\|v_n - y_n\|) \\ &\leq [1 + (1 + L_* + L_*^2)d_n^2 + L_* d_n d_n' + L_*^2 d_n b_n' + L_*^3 d_n b_n\end{aligned}$$

$$+ L_*^2(1+d_n)c_n]\|y_n - q\| + [1+(1+L_*)d_n]c_n\|u_n - q\|$$
$$+ L_*c_n'[d_n^2 + (1+d_n)c_n + L_*d_nb_n + d_n)\|v_n - q\|.$$

Hence, we have

(3.13)

$$\begin{aligned}
&\|p_n - q\| \\
&\le \frac{1}{1+A(p_n,q)d_n}[1+(1+L_*+L_*^2)d_n^2 + L_*d_nd_n' + L_*^2d_nb_n' \\
&\quad + L_*^3d_nb_n + L_*^2(1+d_n)c_n]\|y_n - q\| + [1+(1+L_*)d_n]c_n\|u_n - q\| \\
&\quad + L_*c_n'[d_n^2 + (1+d_n)c_n + L_*d_nb_n + d_n]\|v_n - q\| \\
&\le [1-A(p_n,q)d_n]\|y_n - q\| + [(2+L_*+L_*^2)d_n^2 + L_*d_nd_n' + L_*^2d_nb_n' \\
&\quad + L_*^3d_nb_n + L_*^2(1+d_n)c_n]\|y_n - q\| + (2+L_*)c_n\|u_n - q\| \\
&\quad + L_*c_n'[d_n^2 + (1+d_n)c_n + (1+L_*)d_n]\|v_n - q\|
\end{aligned}$$

for any $n \ge 0$. Thus (3.13) yields that

(3.14)

$$\begin{aligned}
\| y_{n+1} - q\| \le\ & [1-A(p_n,q)d_n]\|y_n - q\| \\
&+ [(2+L_*+L_*^2)d_n^2 + L_*d_nd_n' + L_*^2d_nb_n' + L_*^3d_nb_n \\
&+ L_*^2(1+d_n)c_n]\|y_n - q\| + (2+L_*)c_n\|u_n - q\| \\
&+ L_*c_n'[d_n^2 + (1+d_n)c_n + (1+L_*)d_n]\|v_n - q\| + \epsilon_n
\end{aligned}$$

for all $n \ge 0$.

Suppose that $\sum_{n=0}^{\infty} \epsilon_n < \infty$. Let

$$\begin{aligned}
r_n &= (2+L_*+L_*^2)d_n^2 + L_*d_nd_n' + L_*^2d_nb_n' + L_*^3d_nb_n \\
&\quad + L_*^2(1+d_n)c_n, \\
s_n &= (2+L_*)c_n\|u_n - q\| \\
&\quad + L_*c_n'[d_n^2 + (1+d_n)c_n + (1+L_*)d_n]\|v_n - q\| + \epsilon_n
\end{aligned}$$

for all $n \ge 0$. Then $\sum_{n=0}^{\infty} r_n < \infty$ and $\sum_{n=0}^{\infty} s_n < \infty$ and

$$\|y_{n+1} - q\| \le (1+r_n)\|y_n - q\| + s_n, \quad n \ge 0.$$

It follows from Lemma 2.1 that $\{\|y_n - q\|\}_{n=0}^{\infty}$ is bounded. Hence there exists a constant $M > 0$ such that

$$\|y_n - q\| \le M, \quad n \ge 0$$

and

$$\|y_{n+1} - q\| \le \|y_n - q\| + (Mr_n + s_n), \quad n \ge 0.$$

Lemma 2.2 and the above inequality ensure that

$$\lim_{n\to\infty} \|y_n - q\| = r \ge 0. \tag{3.15}$$

We assert that $r = 0$. If not, then $r > 0$. Note that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are bounded. Then

$$\begin{aligned} B :=&(1 + L_*^2)M + (1 + L_*) \max\{\sup\{\|u_n - q\| : n \ge 0\}, \\ &\sup\{\|v_n - q\| : n \ge 0\}\} < \infty. \end{aligned}$$

Observe that

$$\begin{aligned} \|p_n - q\| &\le a_n\|y_n - q\| + b_n\|Sw_n - q\| + c_n\|u_n - q\| \\ &\le (a_n + b_nL_*^2)\|y_n - q\| + c_n\|u_n - q\| + L_*b_nc_n'\|v_n - q\| \\ &\le B \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \|p_n - q\| &\ge a_n\|y_n - q\| - b_n\|Sw_n - q\| - c_n\|u_n - q\| \\ &\ge (a_n - b_nL_*^2)\|y_n - q\| - c_n\|u_n - q\| - L_*b_nc_n'\|v_n - q\| \end{aligned} \tag{3.17}$$

for all $n \ge 0$. From (3.3), (3.16) and (3.17), we obtain that

$$\lim_{n\to\infty} \|p_n - q\| = r. \tag{3.18}$$

Using (3.15) and (3.18), we know that there exists a positive integer N such that

$$\max\{\|p_n - q\|, \|y_n - q\|\} \ge \frac{1}{2}r, \quad n \ge N. \tag{3.19}$$

From (3.14) and (3.19), we conclude that

$$\begin{aligned} \frac{\frac{r}{2}\phi\left(\frac{r}{2}\right)}{1 + M + \phi(M)}d_n &\le \frac{\phi(\|p_n - q\|)}{1 + \|p_n - q\| + \phi(\|p_n - q\|)}d_n\|y_n - q\| \\ &\le \|y_n - q\| - \|y_{n+1} - q\| + r_nM + s_n, \end{aligned}$$

for all $n \geq N$. This means that

$$\frac{\frac{r}{2}\phi\left(\frac{r}{2}\right)}{1+M+\phi(M)} \sum_{n=N}^{\infty} d_n \leq \|y_N - q\| + M \sum_{n=N}^{\infty} r_n + \sum_{n=N}^{\infty} s_n < \infty.$$

This is a contradiction to $\sum_{n=0}^{\infty} d_n = \infty$. Therefore $r = 0$. That is, $\lim_{n\to\infty} y_n = q$.

Suppose that $\lim_{n\to\infty} y_n = q$. Then

$$\begin{aligned} \epsilon_n &\leq \|y_{n+1} - q\| + (a_n + b_n L_*^2)\|y_n - q\| \\ &\quad + c_n\|u_n - q\| + L_* b_n c'_n \|v_n - q\| \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Hence $\lim_{n\to\infty} \epsilon_n = 0$. This completes the proof. □

Similarly we have the following results.

THEOREM 3.2. *Let X be a Banach space and let $T : X \to X$ be a Lipschitz ϕ-strongly accretive operator. Suppose that the equation $Tx = f$ has a solution for each $f \in X$. Define $S : X \to X$ by $Sx = f + x - Tx$ for each $x \in X$. Assume that $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$, $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are as in Theorem 3.1. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution q of the equation $Tx = f$ and* (ii)∼(iv) *in Theorem 3.1 hold.*

THEOREM 3.3. *Let K be a nonempty convex subset of a Banach space X and let $T : K \to K$ be a Lipschitz ϕ-hemicontractive operator. Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are arbitrary bounded sequences in K. Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ be as in Theorem 3.1. Assume that $\{x_n\}_{n=0}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by*

$$\begin{cases} z_n = a'_n x_n + b'_n T x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T z_n + c_n u_n, \qquad n \geq 0. \end{cases}$$

Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in K and define $\{\epsilon_n\}_{n=0}^{\infty}$ by

$$w_n = a'_n y_n + b'_n T y_n + c'_n v_n, \quad \epsilon_n = \|y_{n+1} - p_n\|, \quad n \geq 0,$$

where $p_n = a_n y_n + b_n T w_n + c_n u_n$. Then

(i) *The sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point* q *of* T;

(ii)
$$\begin{aligned} &\|y_{n+1} - q\| \\ &\leq [1 - A(p_n, q)d_n]\|y_n - q\| + [(2 + L + L^2)d_n^2 \\ &\quad + Ld_n d_n' + L^2 d_n b_n' + L^3 d_n b_n + L^2(1 + d_n)c_n]\|y_n - q\| \\ &\quad + (2 + L)c_n\|u_n - q\| + Lc_n'[d_n^2 + (1 + d_n)c_n \\ &\quad + (1 + L)d_n]\|v_n - q\| + \epsilon_n, \quad n \geq 0; \end{aligned}$$

(iii) $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that* $\{x_n\}_{n=0}^{\infty}$ *is almost* T

-stable on K;

(iv) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \epsilon_n = 0$.

REMARK 3.1. Theorem 3.2 and Theorem 3.3 are generalizations and improvements of Theorem 2 and Theorem 1 in [17], respectively. On the other hand, the convergence results in Theorem 3.2 and Theorem 3.3 extend Theorem in [1], Theorem 1 and Theorem 2 in [16], Theorem 1 and Theorem 2 in [18] and Theorem 2 in [20] in the following sense.

(a) The Mann iterative scheme in [1] and the Ishikawa iterative scheme in [16, 18, 20] are replace by the more general Ishikawa iterative scheme with errors;

(b) The L_p (or l_p) space in [1], the real q-uniformly smooth Banach space in [16] and the real Banach space with property $(U, \lambda, m, m+1)$ are replaced by arbitrary Banach space;

(c) The boundedness assumptions of the subset K in [1] and the sequences $\{Tx_n\}_{n=0}^{\infty}$ and $\{Ty_n\}_{n=0}^{\infty}$ in [20] are removed;

(d) The condition $\beta_n \leq \alpha_n^{q-1}$ in [16] is superfluous.

REMARK 3.2. The following example reveals that Theorem 3.3 extends substantially Theorem in [1], Theorem 2 in [16], Theorem 1 in [17], Theorem 2 in [18] and Theorem 2 in [20], and that the class of strictly hemicontractive operators is a proper subclass of ϕ-hemicontractive operators.

EXAMPLE 3.1. Let $X = (-\infty, \infty)$ with the usual norm and $K = [0, \infty)$. Define an operator $T : K \to K$ by $Tx = \frac{x}{1+2x}$ for all $x \in K$. Let $\phi : [0, \infty) \to [0, \infty)$ be a function satisfying $\phi(t) = \frac{t^2}{1+2t}$ for all $t \geq 0$. Clearly, $F(T) = \{0\}$ and

$$\|Tx - Ty\| = \frac{\|x - y\|}{(1 + 2x)(1 + 2y)} \leq \|x - y\|, \quad x, y \in K.$$

Hence T is a Lipschitzian mapping with $L = 1$. Note that for all $x, y \in K$,

$$\langle Tx - Ty, j(x-y)\rangle = \langle \frac{x-y}{(1+2x)(1+2y)}, j(x-y)\rangle$$
$$= \frac{\|x-y\|^2}{(1+2x)(1+2y)}.$$

For a given $k \in (0,1)$, there exists $x = \frac{1-k}{4k} \in K$ such that

$$\langle Tx, j(x)\rangle = \frac{\|x\|^2}{1+2x} > k\|x\|^2.$$

Therefore T is neither strictly hemicontractive nor strongly pseudocontractive. Observe that for all $x, y \in K$,

$$\langle Tx - Tx, j(x-y)\rangle \le \frac{\|x-y\|^2 + \|x-y\|^3}{1+2\|x-y\|}$$
$$= \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

That is, T is ϕ-strongly pseudocontractive. Since $F(T) \neq \emptyset$, it follows that T is ϕ-hemicontractive. Set

$$a_n = 1 - (2+n)^{-1} - (2+n^2)^{-1}, \quad b_n = (2+n)^{-1}, \quad c_n = (2+n^2)^{-1},$$
$$a'_n = 1 - 2(2+n)^{-1/2}, \quad b'_n = c'_n = (2+n)^{-1/2},$$

for all $n \ge 0$. Then all the assumptions of Theorem 3.3 are fulfilled. However, the conditions of Theorem in [1], Theorem 2 in [16], Theorem 1 in [17], Theorem 2 in [18] and Theorem 2 in [20] are not all satisfied.

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