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REGULARITY OF THE GENERALIZED CENTROID OF SEMI-PRIME GAMMA RINGS

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This paper is dedicated to the memory of Prof. Dr. Mehmet Sapanci.

ABSTRACT. The aim of this note is to study properties of the generalized centroid of the semi-prime gamma rings. Main results are the following theorems: (1) Let M be a semi-prime Γ-ring and Q a quotient Γ-ring of M. If W is a non-zero submodule of the right(left) M-module Q, then $W\Gamma W \neq 0$. Furthermore Q is a semi-prime Γ-ring. (2) Let M be a semi-prime Γ-ring and C_Γ the generalized centroid of M. Then C_Γ is a regular Γ-ring. (3) Let M be a semi-prime Γ-ring and C_Γ the extended centroid of M. If C_Γ is a Γ-field, then the Γ-ring M is a prime Γ-ring.

1. Introduction

Nobusawa studied on Γ-ring for the first time in [6]. After his research, Barnes studied on this Γ-ring in [1]. But Barnes approached to Γ-ring in some different way from that of Nobusawa and he defined the concept of Γ-ring and related definitions. After these two papers were published, many mathematicians made good works on Γ-ring in the sense of Barnes and Nobusawa, which are parallel to the results in the ring theory (see [2, 6, 9]). On the other hand, the topic of "prime rings satisfying a generalized polynomial identity" is important to and essential source of many researchers containing Martindale [5]. In [7] and [8], some parts of the researches on them have been extended to Γ-ring. That is, the concept of "centroid of a prime Γ-ring" was defined and researched in [7] and [8]. Furthermore it is shown that the extended centroid is a Γ-field in [8]. The aim of this paper is to prove that the generalized centroid of a semi-prime Γ-ring is a regular Γ-ring.

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2. Preliminaries

The gamma ring is defined in [1] as follows : A Γ-*ring* is a pair (M, Γ) where M and Γ are (additive) abelian groups for which exists a $(-,-,-) : M \times \Gamma \times M \to M$ (the image of (a, α, b) being denoted by $a\alpha b$ for $a, b \in M$ and $\alpha \in \Gamma$) satisfying for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$:

- $(a+b)\alpha c = a\alpha c + b\alpha c$,
- $a(\alpha + \beta)b = a\alpha b + a\beta b$,
- $a\alpha(b+c) = a\alpha b + a\alpha c$,
- $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Let M be a Γ-ring. A *right* (resp. *left*) *ideal* of M is an additive subgroup U such that $U\Gamma M \subset U$ (resp. $M\Gamma U \subset U$). If U is both a right and left ideal, then we say that U is an *ideal*. For each $a \in M$ the smallest right (resp. left) ideal containing a is called the *principal right* (resp. *left*) *ideal* generated by a and is denoted by $|a\rangle$ (resp. $\langle a|$). Also, we define $\langle a \rangle$, the principal two-side (right and left) ideal generated by a. An ideal Q of M is *semi-prime* if, for any ideal U of M, $U\Gamma U \subseteq Q$ implies $U \subseteq Q$. A Γ-ring M is said to be *semi-prime* if the zero ideal is semi-prime.

REMARK 2.1. A Γ-ring M is semi-prime if and only if all of its non-zero ideals have a non-zero multiplication, i.e., for an ideal U the equality $U\Gamma U = \langle 0 \rangle$ implies $U = \langle 0 \rangle$.

THEOREM 2.2. [2] *If Q is an ideal of a Γ-ring M, then the following conditions are equivalent.*

(i) *Q is a semi-prime ideal.*
(ii) *If $a \in M$ such that $a\Gamma M\Gamma a \subseteq Q$, then $a \in Q$.*
(iii) *If $\langle a \rangle$ is a principal ideal in M such that $\langle a \rangle\Gamma\langle a \rangle \subseteq Q$, then $a \in Q$.*
(iv) *If U is a right ideal in M such that $U\Gamma U \subseteq Q$, then $U \subseteq Q$.*
(v) *If U is a left ideal in M such that $V\Gamma V \subseteq Q$, then $V \subseteq Q$.*

LEMMA 2.3. [2] *A Γ-ring M is semi-prime if and only if $a\Gamma M\Gamma a = \langle 0 \rangle$ implies $a = 0$.*

Let M be a Γ-ring. For a subset U of M,

$$Ann_l U = \{a \in M \mid a\Gamma U = \langle 0 \rangle\}$$

is called the *left annihilator* of U. A *right annihilator* $Ann_r U$ can be defined similarly. An ideal of M is said to be *essential* if it has non-zero intersection with any non-zero ideal of M.

LEMMA 2.4. [9] *Let M be a semi-prime Γ-ring and U a non-zero ideal of M. Then $Ann_l U = Ann_r U$, and in this case we will write $Ann_l U = Ann_r U = Ann U$.*

LEMMA 2.5. [9] *Let M be a semi-prime Γ-ring and U a non-zero ideal of M. Then*

(i) *$Ann U$ is an ideal of M,*
(ii) *$U \cap Ann U = \langle 0 \rangle$.*

DEFINITION 2.6. [8] Let M be a Γ-ring with unity. An element u in M is called a *unity* of M if it has a multiplicative inverse in M. If every nonzero element of M is a unity, we say that M is a Γ-*division ring*. A Γ-ring M is a Γ-*field* if it is a commutative Γ-division ring.

DEFINITION 2.7. [8] Let M be a Γ-ring and Q the quotient Γ-ring of M. The set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *extended centroid* of M.

LEMMA 2.8. [8] *If M is a Γ-ring, then the extended centroid C_Γ of M is a Γ-field.*

3. Main results

Let M be a semi-prime Γ-ring. Let us denote by F a set of all ideals of M which have zero annihilator in M. In this case, the set F is closed under multiplication by Lemma 2.4. Indeed, let U and V be in F. The equality $U\Gamma V\beta x = 0$ for $x \in M$ and all $\beta \in \Gamma$ yields $V\beta x \subseteq Ann_r U = \langle 0 \rangle$, i.e., $V\beta x = 0$ and so $x \in Ann_r V = \langle 0 \rangle$ which implies $x = 0$. Then we get that $U\Gamma V \in F$.

LEMMA 3.1. *Let M be a semi-prime Γ-ring and U a non-zero ideal of M. Then the direct sum $U + Ann U$ belongs to F.*

PROOF. We get that $U \cap Ann U = \langle 0 \rangle$ by Lemma 2.5(ii). Since U is a non-zero ideal of M, $Ann U$ is equal to zero. If $(U + Ann U)\Gamma x = \langle 0 \rangle$, where $x \in M$, then $U\Gamma x + Ann U\Gamma x = \langle 0 \rangle$ and since $Ann U = \langle 0 \rangle$, we have $U\Gamma x = \langle 0 \rangle$, where $x \in M$ and so $x \in Ann U = B$. On the other hand, B is an ideal of M by Lemma 2.5.(i). Therefore, we get that $B\Gamma x = \langle 0 \rangle$, because $U\Gamma x + Ann U\Gamma x = \langle 0 \rangle$ and $U\Gamma x = \langle 0 \rangle$. So $x \in Ann B$ and $x \in B \cap Ann B = \langle 0 \rangle$ implies $x = 0$ which is required proof. □

LEMMA 3.2. *Let M be a semi-prime Γ-ring and U a non-zero ideal of M. Then $U \in F$ if and only if U is essential.*

PROOF. ($\Rightarrow$) Let U be in F. Then $\langle 0 \rangle \neq U\Gamma V \subseteq U \cap V$ where V is a non-zero ideal of M. Thus U is essential.

($\Leftarrow$) Let U be essential. In this case, we get that $U \cap AnnU = \langle 0 \rangle$ by Lemma 2.5(ii). Since U is an essential, we get $AnnU = \langle 0 \rangle$ and so $U \in F$. □

REMARK 3.3. If $U, V \in F$, then $U \cap V \in F$.

Let M be a semi-prime Γ-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \left\{ (U,f) \;\middle|\; \begin{array}{l} f : U \to M \text{ is a right } M\text{-module} \\ \text{homomorphism for all } U \in F \end{array} \right\}$$

Define a relation, "$\sim$" on $\mathcal{M}$ by $(U,f) \sim (V,g) \Leftrightarrow \exists W \subset U \cap V$ such that $f = g$ on $W \in F$. Since the set F is closed under multiplication, it is possible to find such an ideal $W \in F$ and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of $\mathcal{M}$. We denote the equivalence class by $Cl(U,f) = \hat{f}$, where

$$\hat{f} := \{g : V \to M \mid (U,f) \sim (V,g)\},$$

and denote by Q the set of all equivalence classes. We define an addition "+" on Q as follows:

$$\hat{f} + \hat{g} := Cl(U,f) + Cl(V,g) = Cl(U \cap V, f+g)$$

where $f+g : U \cap V \to M$ is a right M-module homomorphism. Assume that $(U_1, f_1) \sim (U_2, f_2)$ and $(V_1, g_1) \sim (V_2, g_2)$. Then $\exists W_1 (\in F) \subset U_1 \cap U_2$ such that $f_1 = f_2$ and $\exists W_2 (\in F) \subset V_1 \cap V_2$ such that $g_1 = g_2$. Taking $W = W_1 \cap W_2$ and so $W \in F$. For any $w \in W$, we have $(f_1 + g_1)(w) = f_1(w) + g_1(w) = f_2(w) + g_2(w) = (f_2 + g_2)(w)$ and so $f_1 + g_1 = f_2 + g_2$ in W. Therefore, $(U_1 \cap V_1, f_1 + g_1) \sim (U_2 \cap V_2, f_2 + g_2)$, which means that the addition "+" in Q is well-defined.

Now we will prove that Q is additive abelian group. Let $\hat{f} := Cl(U,f)$, $\hat{g} := Cl(V,g)$ and $\hat{h} := Cl(W,h)$ be elements of Q. Then one can easily check $(\hat{f} + \hat{g}) + \hat{h} = \hat{f} + (\hat{g} + \hat{h})$ and $\hat{f} + \hat{g} = \hat{g} + \hat{f}$. Taking $\hat{0} := Cl(M, 0)$ where $0 : M \to M, x \mapsto 0$ for all $x \in M$ we have $\hat{f} + \hat{0} = Cl(U,f) + Cl(M,0) = Cl(U \cap M, f + 0) = Cl(U,f) = \hat{f}$ and similarly $\hat{0} + \hat{f} = \hat{f}$. $\hat{0}$ is the additive identity in Q. For any element $\hat{f} = Cl(U,f)$ of Q, it is easy to show that $-\hat{f} = Cl(U,-f)$ additive inverse of $\hat{f} = Cl(U,f)$. Therefore, $(Q,+)$ is an abelian group. Since $M\Gamma M \neq M$ and M is a semi-prime Γ-ring, $M\Gamma M (\neq 0)$ is an ideal of

M and so is $M\beta M$ for every $\beta(\neq 0) \in \Gamma$. $0 \neq M\beta M\Gamma U \subset M\beta M \cap U$ where U

is a non-zero ideal of M. Therefore $M\beta M$ is essential and so $M\beta M \in F$ for every $\beta(\neq 0) \in \Gamma$ by Lemma 3.2. We can take the homomorphism $1_{M\beta} : M\beta M \to M$ defined by $1_{M\beta}(m_1\beta m_2) = m_1\beta m_2$ as non-zero M-module homomorphism. Denote

$$\mathcal{N} := \{(M\beta M, 1_{M\beta}) \mid 0 \neq \beta \in \Gamma\}$$

and define a relation, "$\approx$" on $\mathcal{N}$ by 11$(M\beta M, 1_{M\beta}) \approx (M\gamma M, 1_{M\gamma}) \Leftrightarrow \exists W := M\alpha M(\in F) \subset M\beta M \cap M\gamma M$ such that $1_{M\beta} = 1_{M\gamma}$ on $W \in F$. We can easily check that "$\approx$" is an equivalence relation on $\mathcal{N}$. Denote by $Cl(M\beta M, 1_{M\beta}) = \hat{\beta}$, the equivalence class containing $(M\beta M, 1_{M\beta})$ and by $\hat{\Gamma}$ the set of all equivalence classes of $\mathcal{N}$ with respect to 11$\approx$", that is, $\hat{\Gamma} := \{\hat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition "+" on $\hat{\Gamma}$ as follows:

$$\begin{aligned}\hat{\beta} + \hat{\gamma} &:= Cl(M\beta M, 1_{M\beta}) + Cl(M\gamma M, 1_{M\gamma})\\ &= Cl(M\beta M \cap M\gamma M, 1_{M\beta} + 1_{M\gamma})\end{aligned}$$

for every $\beta(\neq 0), \gamma(\neq 0) \in \Gamma$. Then, $(\hat{\Gamma}, +)$ is an abelian group. Now we define a mapping

$$(-,-,-) : Q \times \hat{\Gamma} \times Q \to Q,\ (\hat{f}, \hat{\beta}, \hat{g}) \mapsto \hat{f}\hat{\beta}\hat{g},$$

as follows:

$$\hat{f}\hat{\beta}\hat{g} = Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(V, g) = Cl(V\Gamma M\beta M\Gamma U, f1_{M\beta}g)$$

where $V\Gamma M\beta M\Gamma U \in F$ and $f1_{M\beta}g : V\Gamma M\beta M\Gamma U \to M$, which is given by

$$(f1_{M\beta}g)(\sum v_i\gamma_i m_i\beta n_i\alpha_i u_i) = f(\sum g(v_i)\gamma_i m_i\beta n_i\alpha_i u_i)$$

is a right M-module homomorphism. Then it is routine to check that such mapping is well-defined. Now we will show that Q is a $\hat{\Gamma}$-ring with unity. Let $\hat{f}$, $\hat{g}$ and $\hat{h} \in Q$ and $\hat{\beta}, \hat{\gamma} \in \hat{\Gamma}$, i.e., $\hat{f} = Cl(U, f)$, $\hat{g} = Cl(V, g)$, $\hat{h} = Cl(W, h)$, $\hat{\beta} = Cl(M\beta M, 1_{M\beta})$, and $\hat{\gamma} = Cl(M\gamma M, 1_{M\gamma})$. Then

$$\begin{aligned}(\hat{f} + \hat{g})\hat{\beta}\hat{h} &= Cl(U \cap V, f + g)Cl(M\beta M, 1_{M\beta})Cl(W, h)\\ &= Cl(W\Gamma M\beta M\Gamma(U \cap V), (f + g)1_{M\beta}h)\\ &= Cl((W\Gamma M\beta M\Gamma)U \cap (W\Gamma M\beta M\Gamma V), f1_{M\beta}h + g1_{M\beta}h)\\ &= \hat{f}\hat{\beta}\hat{h} + \hat{g}\hat{\beta}\hat{h}.\end{aligned}$$

and the equalities $\hat{f}(\hat{\beta} + \hat{\gamma})\hat{g} = \hat{f}\hat{\beta}\hat{g} + \hat{f}\hat{\gamma}\hat{g}$, $\hat{f}\hat{\beta}(\hat{g} + \hat{h}) = \hat{f}\hat{\beta}\hat{g} + \hat{f}\hat{\beta}\hat{h}$, and $(\hat{f}\hat{\beta}\hat{g})\hat{\gamma}\hat{h} = \hat{f}\hat{\beta}(\hat{g}\hat{\gamma}\hat{h})$ are proved in a analogous way.

Next we will show that Q has a multiplicative identity. Let $\hat{f} \in Q$ and $\hat{\beta} \in \hat{\Gamma}$. Take $\hat{I} = Cl(M, I) \in Q$ } where $I : M \to M$, $x \mapsto x$, is a M-module homomorphism. Then

$$\begin{aligned} \hat{f}\hat{\beta}\hat{I} &= Cl(U, f)Cl(M\beta M, 1_{M\beta})Cl(M, I) \\ &= Cl(M\Gamma M\beta M\Gamma U, f1_{M\beta}I) = Cl(U, f) = \hat{f} \end{aligned}$$

and similarly we have $\hat{I}\hat{\beta}\hat{f} = \hat{f}$. Notice that the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$. Here the case of to be $0 \neq \beta$, it does not mean that image of zero of Γ under φ doesn't map to zero of $\hat{\Gamma}$. That is, $\varphi(0) = \hat{0} = Cl(M\Gamma M, 0_{M\Gamma})$. If $\beta = 0$, then $M\beta M = 0$. In this case, there is a contradiction with $M\beta M \neq 0$. For this reason, we get the mapping $\varphi : \Gamma \to \hat{\Gamma}$ defined by $\varphi(\beta) = \hat{\beta}$ for every $0 \neq \beta \in \Gamma$. Noticing that the mapping φ is an isomorphism, we know that the $\hat{\Gamma}$-ring Q is a Γ-ring.

For a fixed element a in M and every element γ in Γ, consider a mapping $\lambda_{a\gamma} : M \to M$ defined by $\lambda_{a\gamma}(x) = a\gamma x$ for all $x \in M$. It is easy to prove that the mapping $\lambda_{a\gamma}$ is a right M-module homomorphism and so $\lambda_{a\gamma}$ is an element of Q. Define a mapping $\psi : M \to Q$ by $\psi(a) = \hat{a} = Cl(M, \lambda_{a\gamma})$ for all $a \in M$ and $\gamma \in \Gamma$. It is easy to prove that the mapping ψ is a right M-module injective homomorphism and so M is a subring of Q, and in this case, we call Q the *right quotient* Γ*-ring* of M and will be denoted by $Q_r(M)$ (or, briefly Q). One can, of course, characterize $Q_l(M)$, the left quotient Γ-ring of M in a similar manner. For purposes of convenience, we use q instead of $\hat{q} \in Q$.

DEFINITION 3.4. Let M be a semi-prime Γ-ring and Q the quotient Γ-ring of M. Then the set

$$C_\Gamma := \{g \in Q \mid g\gamma f = f\gamma g \text{ for all } f \in Q \text{ and } \gamma \in \Gamma\}$$

is called the *generalized centroid* of M.

The following theorem characterizes the quotient Γ-ring Q of M. The proof is a minor modification of the proof of the corresponding theorem in ring theory, and we omit it.

THEOREM 3.5. *Let M be a semi-prime Γ-ring and Q the quotient Γ-ring of M. Then the Γ-ring Q satisfies the following properties:*

(i) *For any element $q \in Q$, there exists an ideal $U_q \in F$ which is an essential ideal with a right M-module homomorphism $q : U \to M$, such that $q(U_q) \subseteq M$ (or $q\gamma U_q \subseteq M$ for all $\gamma \in \Gamma$).*
(ii) *If $q \in Q$ and $q(U_q) = \langle 0 \rangle$ for a certain $U_q \in F$ (or $q\gamma U_q = \langle 0 \rangle$ for a certain $U_q \in F$ and for all $\gamma \in \Gamma$), then $q = 0$.*

(iii) *If $U \in F$ and $\Psi : U \to M$ is a right M-module homomorphism, then there exists an element $q \in Q$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = q\gamma u$ for all $u \in U$ and $\gamma \in \Gamma$).*

(iv) *Let W be a submodule (an (M,M)-sub-bimodule) in Q and $\Psi : W \to Q$ a right M-module homomorphism. If W contains the ideal U of the Γ-ring M such that $\Psi(U) \subseteq M$ and $AnnU = Ann_r W$, then there is an element $q \in Q$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = q\gamma b$ for any $b \in W$ and $\gamma \in \Gamma$) and $q(a) = 0$ for any $a \in Ann_r W$ (or $q\gamma a = 0$ for any $a \in Ann_r W$ and $\gamma \in \Gamma$).*

Let W be a non-zero submodule of the right M-module Q. Then we get that $(0 \neq) w \in W$ and U_w is an essential ideal of the Γ- ring M such that $w\gamma U_w \subseteq M$ (or $w(U_w) \subseteq M$) for any $\gamma \in \Gamma$ and so $\langle 0 \rangle \neq w\gamma U_w \Gamma w\gamma U_w \subseteq W\Gamma W$, for any $\gamma \in \Gamma$ by Theorem 3.5(i). Thus, the right and left annihilators of the (M,M)-submodule W in Q are the same, since the equality $Ann_r W = \langle 0 \rangle$ implies $(Ann_l W)\Gamma(Ann_l W) = \langle 0 \rangle$ or $(Ann_r W)\Gamma(Ann_r W) = \langle 0 \rangle$ and $Ann_l W = \langle 0 \rangle$ implies

$$(Ann_r W)\Gamma(Ann_r W) = \langle 0 \rangle.$$

This property shows that it is possible to obtain similar results for left or two-sided annihilators instead of right annihilators which are defined in case (iv) of Theorem 3.5.

Let W satisfy the case (iv) of Theorem 3.5. Let us denote an annihilator of an ideal U in M by L. Let us extend $\Psi : U \to M$ which is a right M-module homomorphism to $\Psi : L + U \to M$ such that $\Psi(L) = 0$, since $L + U$ is an ideal of M by Lemma 2.5(i). Since the annihilator of $L + U$ equals zero, in this case, we find an element q in Q such that $q(L) = 0$ (or $q\gamma L = \langle 0 \rangle$ for all $\gamma \in \Gamma$) and $\Psi(u) = q(u)$ (or $\Psi(u) = q\gamma u$ for all $\gamma \in \Gamma$), where $u \in U$, by Theorem 3.5(iii). Then, if $w \in W$ and $a \in U\Gamma U_w$, then $w\beta a \in U$ and hence $\Psi(w)\beta a = \Psi(w\beta a) = q(w\beta a)$ (or $\Psi(w\beta a) = q\gamma w\beta a$ for all $\gamma, \beta \in \Gamma$) for all $\beta \in \Gamma$ and so $(\Psi(w) - q(w))\beta a = 0$ for all $\beta \in \Gamma$ (or $(\Psi(w) - q\gamma w)\beta a = 0$ for all $\gamma, \beta \in \Gamma$). Therefore $(\Psi(w) - q(w))\Gamma U = \langle 0 \rangle$. Also, since $W\Gamma L = \langle 0 \rangle$ by Theorem 3.5(iv), i.e., $(\Psi(w) - q(w))\Gamma L = \langle 0 \rangle$, which implies $\Psi(w) = q(w)$ (or $\Psi(w) = q\gamma w = \langle 0 \rangle$ for all $\gamma \in \Gamma$). If $Ann_r W = \langle 0 \rangle$, then $Ann_r W\Gamma U_a = \langle 0 \rangle$ and so $b\Gamma U_a \subseteq L$, where $b \in Ann_r W$. Consequently, $q(b\beta U_a) = 0$ (or $q\gamma b\beta U_a = 0$ for all $\gamma, \beta \in \Gamma$) and so we get that $q(b) = 0$ (or $q\beta b = 0$ for all $\beta \in \Gamma$) by Theorem 3.5(ii), and so we give the following proposition.

PROPOSITION 3.6. *Let M be a semi-prime Γ-ring and Q the quotient Γ- ring of M. If W is a non-zero submodule of the right(left) M-module Q, then $W\Gamma W \neq 0$. Furthermore Q is a semi-prime Γ-ring.*

DEFINITION 3.7. A Γ-ring M is called *regular* if for any element $x \in M$, there exists an element $x' \in M$ such that $x'\beta x\gamma x = x$, where $\gamma, \beta \in \Gamma$.

THEOREM 3.8. *Let M be a semi-prime Γ-ring and C_Γ the generalized centroid of M. Then C_Γ is a regular Γ-ring.*

PROOF. Let a be an element of C_Γ. Then $a, a^2 \in Q$, and so we get that U_a and U_{a^2} are essential ideals of M and so $U_a \cap U_{a^2} \in F$. We consider a mapping $\psi : U_a \cap U_{a^2} \to M$ defined by $\psi(a^2\beta x) = a\beta x$ for all $\beta \in \Gamma$ and where x runs through the set $J = U_a \cap U_{a^2}$. Let $a^2\beta x = 0$. Then $(a\beta x)\Gamma M\Gamma(a\beta x) = 0$ implies $a\beta x = 0$. Therefore ψ is a right M-module homomorphism. Hence, there is an element $a_1 \in Q$ such that $a_1\alpha a^2\beta x = a\beta x$ for all $x \in J$ by Theorem 3.5(iii). We have that $a_1\alpha a^2 = a$ for all $\alpha \in \Gamma$ by Theorem 3.5(ii). Let us prove that the element a_1 in C_Γ. In this case, let q be an arbitrary element of Q. Then $[(a_1\alpha a^2)^2, q]_\beta = [a^2, q]_\beta$ where $[a^2, q]_\beta = a^2\beta q - q\beta a^2$. Since $a \in C_\Gamma$, we have $0 = [a_1^2, q]_\beta = [(a_1\alpha a^2)^2, q]_\beta = [a_1^2\alpha a^4, q]_\beta = a^4\alpha[a_1^2, q]_\beta$. Multiplying this equality from the left by a_1^3 ($a_1^3 = a_1\gamma_1 a_1\gamma_2 a_1$, γ_1, $\gamma_2 \in \Gamma$), we get $0 = a\alpha[a_1^2, q]_\beta = a\alpha[a_1, q]_\beta$. Thus, we get $[a_1, q]_\beta = 0$ by Proposition 3.6. This completes the proof. □

REMARK 3.9. We have shown that C_Γ is a regular Γ-ring. For any element $a \in C_\Gamma$, there exists an element $a' \in C_\Gamma$ such that $a'\beta a\gamma a = a$ for $\beta, \gamma \in \Gamma$. If $a'\beta a\gamma a = a'\beta a^2 = a$, then $(a'\beta a)^2 = (a'\beta a)\gamma(a'\beta a) = a'\beta(a'\gamma a\beta a) = a'\beta a$, i.e., $e = a'\beta a$ is an idempotent, and so we get $e\gamma a = a$, $\gamma \in \Gamma$. Therefore the C_Γ has a sufficient number of idempotents. Thus in the set E of all the central idempotents the relation $\leq$ defined by

$$e_1 \leq e_2 \iff e_2\gamma e_1 = e_1, \ \gamma \in \Gamma$$

is a partial order.

DEFINITION 3.10. Let M be a semi-prime Γ-ring, Q the quotient Γ-ring of M and let $S \subseteq Q$. The least of idempotent elements $e(S) = e \in C_\Gamma$ such that $e\gamma s = s$ for all $s \in S$, $\gamma \in \Gamma$ is called the *support* of the set S.

LEMMA 3.11. *Let M be a semi-prime Γ-ring, Q the quotient Γ-ring of M and $S \subseteq Q$. If S has a support $e(S) = e \in C_\Gamma$, then the equality $q\gamma M\Gamma S = 0$ for an element $q \in Q$ ($S\Gamma M\gamma q = 0$) is equivalent to $q\gamma e(S) = 0$.*

PROOF. Let V be a (two-sided) M-submodule in Q that is generated by the set S. In this case, $U = V \cap M$ is a (two-sided) ideal of the

Γ-ring M. We proved that its annihilator in the Γ-ring M coincides with the annihilator of V in M. Now, let $q\gamma U = 0$. If $v \in V$, then $v\beta U_v \in U$ and so $q\alpha v\beta U_v = 0$. We get $q\alpha v = 0$ by Theorem 3.5(iii). By Theorem 3.5(iv), we have that for the identical mapping $\psi : V \to V$, there exists an element $e \in Q$ such that $e\gamma v = v$ for all $v \in V$, $\gamma \in \Gamma$ and e annihilates the annihilator L of the set V in the Γ-ring Q. This implies that for any $1 \in L$, $v \in V$, $q \in Q$ and γ, $\beta \in \Gamma$ the following equalities are valid:

$$[e, q]_\beta \gamma(1 + v) = 0, \ (e^2 - e)\gamma(1 + v) = 0.$$

Since the annihilator of the sum $L + V$ has a zero multiplication, we have $e \in C_\Gamma$ and e is an idempotent by Proposition 3.6. If e_1 is a central idempotent such that $e_1\alpha s = s$ for all $s \in S$, $\alpha \in \Gamma$, then $e_1\alpha v = v$ for $v \in V$, $\alpha \in \Gamma$ and so $1 - e_1 \in L$, i.e.,

$$\begin{aligned} 0 = e\gamma(1 - e_1) = e - e\gamma e_1 \Rightarrow e\gamma e_1 = e \\ \Rightarrow e_1\gamma e = e \ (e_1 \in C_\Gamma) \Rightarrow e \leq e_1 \end{aligned}$$

by Remark 3.9. Finally, let $q\gamma M\Gamma S = 0$, where $\gamma \in \Gamma$. Then $q\beta M$ is in the annihilator of V and so $q\beta M\gamma e = 0$, where $\gamma, \beta \in \Gamma$, which implies $M\beta q\gamma e = 0$ and $q\gamma e = 0$. This completes the proof. □

LEMMA 3.12. *Let M be a semi-prime Γ-ring, Q the quotient Γ-ring of M and $S \subseteq Q$ that has a support $e(S) = e \in C_\Gamma$. If $0 \neq e_1 \leq e(S)$, then $e_1\Gamma S \neq 0$.*

PROOF. If $e_1\Gamma S = 0$, then the idempotent $f = 1 - e_1$ adjust to $f\gamma s = s$ for all $s \in S$, $\gamma \in \Gamma$. Therefore $f \geq e(S) \geq e_1$, i.e., $f\gamma e_1 = e_1 = 0$ which is a contradiction. □

Now we give the converse of Lemma 2.8 in the following.

PROPOSITION 3.13. *Let M be a semi-prime Γ-ring and C_Γ the extended centroid of M. If C_Γ is a Γ-field, then the Γ-ring M is a prime Γ-ring.*

PROOF. If $x\Gamma M\Gamma y = 0$, then we have $e(x)\gamma y = 0$ for all $\gamma \in \Gamma$ by Lemma 3.12. Therefore we get $e(x)\Gamma M\Gamma y = 0$ and so, since C_Γ is Γ-field, $M\Gamma y = 0$ which implies $y = 0$. Thus the proof is over. □

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