Commun. Korean Math. Soc. **19** (2004), No. 2, pp. 197–204

ON THE STRUCTURE OF ORTHOMODULAR LATTICES SATISFYING LOOP LEMMA

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Abstract. Every orthomodular lattice satisfying the loop lemma is the direct product of a Boolean algebra and an irreducible path-connected orthomodular lattice.

1. Preliminaries

An *orthomodular lattice*(abbreviated by OML) is an ortholattice L which satisfies *the orthomodular law*: if $x \leq y$, then $y = x \vee (x' \wedge y)$ [6]. A *Boolean algebra* B is an ortholattice satisfying the *distributive law*: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \forall x,\, y,\, z \in B$.

A *subalgebra* of an OML L is a nonempty subset M of L which is closed under the operations $\vee$, $\wedge$ and $'$. We write $M \leq L$ if M is a subalgebra of L. If $M \leq L$ and $a, b \in M$ with $a \leq b$, then the *relative interval sublattice* $M[a,b] = \{x \in M \mid a \leq x \leq b\}$ is an OML with the *relative orthocomplementation* $\sharp$ on $M[a,b]$ given by $c^\sharp = (a \vee c') \wedge b = a \vee (c' \wedge b) \quad \forall c \in M[a,b]$. In particular, $L[a,b]$ will be denoted by $[a,b]$ if there is no ambiguity.

The *commutator of* a *and* b of an OML L is denoted by $a * b$, and is defined by $a * b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')$. The set of all commutators of L is denoted by $ComL$ and L is said to be *commutator-finite* if $|ComL|$ is finite [5]. For elements a, b of an OML, we say a *commutes with* b, in symbols $a \,\mathbf{C}\, b$, if $a * b = 0$. If M is a subset of an OML L, the set $\mathbf{C}(M) = \{x \in L \,|\, x \,\mathbf{C}\, m \quad \forall m \in M\}$ is called the *commutant* of M in L and the set $\mathbf{C}en(M) = \mathbf{C}(M) \cap M$ is called the *center* of M. The set $\mathbf{C}(L)$ is called the center of L and then $\mathbf{C}(L) = \bigcap \{\mathbf{C}(a) \,|\, a \in L\}$.

Received August 22, 2003.
2000 Mathematics Subject Classification: 06C15.
Key words and phrases: orthomodular lattice, loop lemma, path-connected, Boolean algebra.

An OML L is called *irreducible* if $\mathbf{C}(L) = \{0,1\}$, and L is called *reducible* if it is not irreducible.

A *block* of an OML L is a maximal Boolean subalgebra of L. The set of all blocks of L is denoted by $\mathfrak{A}_L$. Note that $\bigcup \mathfrak{A}_L = L$ and $\bigcap \mathfrak{A}_L = \mathbf{C}(L)$. An OML L is said to be *block-finite* if $|\mathfrak{A}_L|$ is finite.

For any e in an OML L, the subalgebra $S_e = [0,e'] \cup [e,1]$ is called the *(principal) section generated by e*. Note that for $A, B \in \mathfrak{A}_L$, if $e \in (A \cap B)$ and $A \cap B = S_e \cap (A \cup B)$, then $A \cap B = S_e \cap A = S_e \cap B$.

DEFINITION 1.1. For blocks A, B of an OML L define $A \overset{wk}{\sim} B$ if and only if $A \cap B = S_e \cap (A \cup B)$ for some $e \in A \cap B$; $A \sim B$ if and only if $A \neq B$ and $A \cup B \leq L$; $A \approx B$ if and only if $A \sim B$ and $A \cap B \neq \mathbf{C}(L)$.

A *path* in L is a finite sequence B_0, B_1, ..., B_n $(n \geq 0)$ in $\mathfrak{A}_L$ satisfying $B_i \sim B_{i+1}$ whenever $0 \leq i < n$. The path is said to *join* the blocks B_0 and B_n. The number n is said to be the *length* of the path. A path is said to be *proper* if and only if $n = 1$ or $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$. A path is called to be *strictly proper* if and only if $B_i \approx B_{i+1}$ holds whenever $0 \leq i < n$ [1].

Let A, B be two blocks of an OML L. If $A \sim B$ holds, then there exists a unique element $e \in A \cap B$ satisfying $A \cap B = (A \cup B) \cap S_e$ [1]. Using this element e, we say that A and B are *linked at e* (*strongly linked at e*) if $A \sim B$ $(A \approx B)$, and use the notation $A \sim_e B$ $(A \approx_e B)$. The element e is called a *vertex* of L and it is the commutator of any $x \in A \setminus B$ and $y \in B \setminus A$ [1]. The set of all vertices of L is denoted by V_L and L is said to be *vertex-finite* if $|V_L|$ is finite.

Note that $A \approx B$ implies $A \sim B$, and $A \sim B$ implies $A \overset{wk}{\sim} B$. Some authors, for example Greechie, use the phrase "*A and B meet in the section* S_e" to describe $A \overset{wk}{\sim} B$ [3].

DEFINITION 1.2. Let L be an OML, and $A, B \in \mathfrak{A}_L$. We will say that A and B are *path-connected in L*, *strictly path-connected in L* if A and B are joined by a proper path, a strictly proper path, respectively. We will say A and B are *nonpath-connected* if there is no proper path joining A and B, and L is called *nonpath-connected* if there exist two blocks which are nonpath-connected. An OML L is called *path-connected in L*, *strictly path-connected in L* if any two blocks in L are joined by a proper path, a strictly proper path, respectively. An OML L is called *relatively path-connected* if each $[0,x]$ is path-connected for all $x \in L$.

Let L be an OML, and $A, B, C \in \mathfrak{A}_L$. If A and B are joined with a strictly proper path $A = B_0 \approx B_1 \approx \ldots \approx B_{m-1} \approx B_m = B$ and if B and C are joined with a strictly proper path $B = C_0 \approx C_1 \approx \ldots \approx C_{n-1} \approx C_n = C$ then A and C are strictly path-connected by the *concatenated path* $A = B_0 \approx B_1 \approx \ldots \approx B_{m-1} \approx B \approx C_1 \approx \ldots \approx C_{n-1} \approx C_n = C$.

The following lemma and three theorems are well known.

LEMMA 1.3. *If L_1, L_2 are OMLs, $L = L_1 \times L_2$, $A, B \in \mathfrak{A}_{L_1}$ and $C, D \in \mathfrak{A}_{L_2}$, then $A \times C \sim B \times D$ holds in L if and only if either $A = B$ and $C \sim D$ or $A \sim B$ and $C = D$. If A and B are linked at a then $A \times C$ and $B \times C$ are linked at $(a, 0)$. If C and D are linked at c then $A \times C$ and $A \times D$ are linked at $(0, c)$* [1].

THEOREM 1.4. *Every finite direct product of path-connected orthomodular lattices is path-connected* [8].

THEOREM 1.5. *Every infinite direct product of path-connected OMLs containing infinitely many non-Boolean factors is nonpath-connected* [7].

THEOREM 1.6. *Let L be an OML. Then the following are equivalent:*
(1) *L is relatively path-connected;*
(2) *$\mathbf{C}(x)$ is path-connected $\forall x \in L$;*
(3) *S_x is path-connected $\forall x \in L$* [8].

We need the following lemma to prove Theorem 1.8.

LEMMA 1.7. *Let L be an OML, and $A, B \in \mathfrak{A}_L$. If $A \cap B = \mathbf{C}(L)$ and $A \cup B \not\leq L$, then there exist $C, D \in \mathfrak{A}_L$ such that $A \cap C \neq \mathbf{C}(L)$, $C \cap D \neq \mathbf{C}(L)$ and $D \cap B \neq \mathbf{C}(L)$.*

PROOF. There exist c, d such that $c, d \in A \cup B$ and $c \vee d \notin A \cup B$ since $A \cup B \not\leq L$. Hence $c \vee d \notin \mathbf{C}(L) = \bigcap \mathfrak{A}_L$. We may assume that $c \in A \setminus B$ and $d \in B \setminus A$. Therefore there exist $C, D \in \mathfrak{A}_L$ such that $c, c \vee d \in C$ and $d, c \vee d \in D$. Then $c, d, c \vee d \notin \mathbf{C}(L)$ with $c \in A \cap C$, $c \vee d \in C \cap D$ and $d \in D \cap B$. This completes the proof. □

Let L be an OML. A subalgebra S of L is said to be a *full subalgebra* if every block of S is a block of L. Note that each $\mathbf{C}(x)$ is a full subalgebra of L for all $x \in L$ since $\mathfrak{A}_{\mathbf{C}(x)} = \{B \in \mathfrak{A}_L \mid x \in B\}$.

THEOREM 1.8. *Let L be an OML. If $[0, x]$ is path-connected $\forall x \in L \setminus \mathbf{C}(L)$, then L is path-connected.*

PROOF. Let $A, B \in \mathfrak{A}_L$. First, if $A \cap B \neq \mathbf{C}(L)$, then there exists $y \in A \cap B \setminus \mathbf{C}(L)$. Since $y, y' \notin \mathbf{C}(L)$, $[0, y]$ and $[0, y']$ are path-connected by hypothesis. Thus $\mathbf{C}(y)$ is path-connected by Theorem 1.4 since $\mathbf{C}(y) = [0, y] \oplus]0, y']$. Thus A and B are path-connected in $\mathbf{C}(y)$ and therefore in L since $\mathbf{C}(y)$ is a full subalgebra of L. Second, if $A \cap B = \mathbf{C}(L)$ and $A \cup B \leq L$, then A and B are path-connected. Finally, if $A \cap B = \mathbf{C}(L)$ and $A \cup B \not\leq L$, then there exist $C, D \in \mathfrak{A}_L$ such that $A \cap C \neq \mathbf{C}(L)$, $C \cap D \neq \mathbf{C}(L)$ and $D \cap B \neq \mathbf{C}(L)$ by Lemma 1.7. Thus A and B are path-connected by a concatenated path by the first case. This completes the proof. □

2. Orthomodular lattices satisfying the atomistic Loop Lemma

Roddy presented an extension of the **(Atomic) Loop Lemma** [4] using the following convention [9].

CONVENTION (*). Let $\mathcal{B}$ be a nonempty set of Boolean algebras with the following properties:

(1) for all $B, C \in \mathcal{B}$, if $B \subseteq C$ then $B = C$;

(2) for all $B, C \in \mathcal{B}$, $0_B = 0_C$ and $1_B = 1_C$ (hence we define $0 = 0_C$, $1 = 1_C$);

(3) for any distinct $B, C \in \mathcal{B}$, either $B \cap C = \{0, 1\}$ or there exists $a \in (B \cap C) \setminus \{0, 1\}$ such that $B \cap C = B[0, a'] \cup B[a, 1] = C[0, a'] \cup C[a, 1]$ in the latter case we write $B \cap C = S_a^{BC}$; and $\forall x, y \in B \cap C$, $x'^B = x'^C$, and $x \leq_B y$ if and only if $x \leq_C y$;

(4) for any pairwise distinct $B, C, D \in \mathcal{B}$ such that $B \cap C = S_a^{BC}$ and $C \cap D = S_b^{CD}$, either $a = b$ or $a'^C <_C b$.

Let $\mathcal{B}$ be a set of Boolean algebras satisfying the above convention. A subscript on an interval, operation or partial ordering indicates the Boolean algebra in which the interval, operation or partial ordering is taken. If $y = x'^B$ and $z = x'^C$, then $x \in B \cap C$ and $y = x'^B = x'^C = z$. Thus the subscripts on the orthocomplementation are unnecessary.

Let $\leq$ and $'$ be the ordering and orthocomplementation induced on $\bigcup \mathcal{B}$ as follows:

(1) $x \leq y$ if and only if there exists $B \in \mathcal{B}$ such that $x \leq_B y$, i.e. $\leq = \bigcup \{\leq_B \mid B \in \mathcal{B}\}$;

(2) the map $' : \bigcup \mathcal{B} \longrightarrow \bigcup \mathcal{B}$ is defined by $x' = x'^B$ if $x \in B$, i.e. $' = \bigcup \{'^B \mid B \in \mathcal{B}\}$.

We call the elements of $\mathcal{B}$ the *initial blocks* of $\bigcup\mathcal{B}$.

DEFINITION 2.1. Let $\mathcal{B}$ be a set of Boolean algebras satisfying the above convention $(*)$. A *loop of order* $n(n \geq 3)$ in $\mathcal{B}$ is a sequence of initial blocks $(B_0, B_1, ..., B_{n-1})$ satisfying the following:

(1) for any distinct $i, j \in \{0, 1, 2, ..., n-1\}$ $B_i \cap B_j = \{0, 1\}$ if $|i - j| \neq 1$, and there exists $a_i \in L \setminus \{0, 1\}$ with $B_i \cap B_{i+1} = S_{a_i}^{B_i B_{i+1}}$ where the computation of i, j is modulo n;

(2) for all pairwise distinct $i, j, k \in \{0, 1, 2, ..., n-1\}$, $B_i \cap B_j \cap B_k = \{0, 1\}$.

EXTENDED LOOP LEMMA. *Let $\mathcal{B}$ be a set of Boolean algebras satisfying the convention, and let $L = (\bigcup\mathcal{B}, \leq, ', 0, 1)$. Then L has the following properties:*

(1) L is an orthomodular poset if and only if $\mathcal{B}$ admits no loop of order less than 4;

(2) L is an orthomodular lattice if and only if $\mathcal{B}$ admits no loop of order less than 5;

(3) if L is an orthomodular lattice, then the blocks of L are precisely the initial blocks [9].

We will show that every OML satisfying the extended loop lemma is path-connected using the following lemma.

LEMMA 2.2. *Let L be an OML satisfying the extended loop lemma, and B, C, D be distinct blocks of L. If $B \cap C \cap D \neq \{0, 1\}$, then there exist $a \in C$ such that $B \cap C = S_a^{BC}$ and $C \cap D = S_a^{CD}$. Moreover, if $B \cap C = S_a^{BC}$, then $B \cap C = S_a$.*

PROOF. Let us prove the first part of this lemma. Since $B \cap C \cap D \neq \{0, 1\}$, there exist elements $x, a, b \in L \setminus \{0, 1\}$ such that $a \in B \cap C$, $b \in C \cap D$, $B \cap C = S_a^{BC}$, $C \cap D = S_b^{CD}$ and $x, x' \in (B \cap C \cap D) \setminus \{0, 1\} = (S_a^{BC} \cap S_b^{CD}) \setminus \{0, 1\}$. We will prove that $a'^C \not<_C b$. Suppose $a'^C <_C b$. If $x \in C[0, b']$, then $x \leq b' < a$ in C. Then $x \notin C[a, 1]$ so that $x \in C[0, a']$, since $x \in B \cap C = S_a^{BC} = C[0, a'] \cup C[a, 1]$. Thus $x \leq_C a' \wedge_C a = 0$ which contradicts that $x \neq 0$. Similarly, if $x \in C[b, 1]$ then $x' \in C[0, b']$ and $x' = 0$ which is a contradiction. Hence $a'^C \not<_C b$. Thus $a = b$ by (4) of the convention $(*)$.

Let us prove the second part of lemma. It is sufficient to show that $S_a \subseteq S_a^{BC} = B \cap C$ since $S_a^{BC} \subseteq S_a$ is clear. Let $x \in [0, a']$, and let $D \in \mathfrak{A}_L$ with $x \leq_D a'$. We may assume that $x \neq 0$ and D is distinct from B and C. Thus $a \in (B \cap C \cap D) \setminus \{0, 1\}$ and hence $B \cap D = S_a^{BD}$ by the first part of this lemma. Therefore $x \in D[0, a'] = B[0, a'] = C[0, a']$ and hence $x \in S_a^{BC}$. If $x \in [a, 1]$, then $x' \in [0, a']$; so by the above argument $x' \in B \cap C$, and hence $x \in B \cap C$. Thus $B \cap C = S_a$. This completes the proof. □

An OML L is called the *horizontal sum* of a family $(L_i)_{i \in I}$ (denoted by $\circ(L_i)_{i \in I}$) of at least two subalgebras, if $\bigcup L_i = L$, and $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$, and one of the following equivalent conditions is satisfied:

(1) if $x \in L_i \setminus L_j$ and $y \in L_j \setminus L_i$, then $x \vee y = 1$;

(2) every block of L belongs to some L_i;

(3) if S_i is a subalgebra of L_i, then $\bigcup S_i$ is a subalgebra of L [2].

An OML L is said to be the *weak horizontal sum* of a family $(L_i)_{i \in I}$ of subalgebras if and only if there exists an isomorphism f of L onto a product of $L_0 \times L'$ of a Boolean algebra L_0 and an OML L' such that the subalgebra L_i of L correspond via f to subalgebras of the form $L_0 \times L'_i$ and L' is the horizontal sum of the family $(L'_i)_{i \in I}$ [1].

LEMMA 2.3. *Every OML L with only two blocks is isomorphic with an OML of the form $\mathbf{B} \times (\mathbf{A} \circ \mathbf{C})$ where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are Boolean algebras and $\mathbf{A} \circ \mathbf{C}$ is the horizontal sum of $\mathbf{A}$ and $\mathbf{C}$. In other words, every OML with only two blocks is the weak horizontal sum of its blocks* [1].

Let L be an OML and $B_1, B_2 \in \mathfrak{A}_L$. If $b_1 \in B_1 \setminus B_2$ and $b_2 \in B_2 \setminus B_1$, then we say that (b_1, b_2) *distinguishes* (B_1, B_2) and we write $(b_1, b_2)\, \delta\, (B_1, B_2)$.

We are ready to prove the following theorem.

THEOREM 2.4. *Let L be an OML satisfying the extended loop lemma. Then L is path-connected and there exists $a \in L$ such that $L = [0, a'] \oplus [0, a]$ where $[0, a']$ is a Boolean algebra and $[0, a]$ is an irreducible path-connected OML.*

PROOF. We may assume that $|\mathfrak{A}_L| \geq 3$ by Lemma 2.3. Let us show that L is path-connected. Let $B_2, B_3 \in \mathfrak{A}_L$. Assume first $B_2 \cap B_3 \neq \{0, 1\}$. Then $B_2 \cap B_3 = S_{m_2}$ for some element $m_2 \in L \setminus \{0, 1\}$. We will show that $B_2 \cup B_3 \leq L$. Suppose that $B_2 \cup B_3 \not\leq L$. Then there exist $b_2, b_3 \in L$ such that $(b_2, b_3)\delta(B_2, B_3)$ and $b_2 \vee b_3 \notin B_2 \cup B_3$. Note that $0 < b_2, b_3 < b_2 \vee b_3 < 1$.

If $b_2\, \mathbf{C}\, b_3$, then there exist $B_1 \in \mathfrak{A}_L$ and $m_1, m_3 \in L \setminus \{0, 1\}$ with $b_2, b_3, b_2 \vee b_3 \in B_1$, $B_1 \cap B_2 = S_{m_1}$ and $B_3 \cap B_1 = S_{m_3}$. We see that $m_1 \neq m_2$ since $b_2 \in S_{m_1} \setminus S_{m_2}$, $m_2 \neq m_3$ since $b_3 \in S_{m_3} \setminus S_{m_2}$, and $m_3 \neq m_1$ since $b_3 \in S_{m_3} \setminus S_{m_1}$. Thus $m_i \neq m_j$ for all distinct $i, j \in \{1, 2, 3\}$. Therefore $B_1 \cap B_2 \cap B_3 = \{0, 1\}$ by Lemma 2.2. Then (B_1, B_2, B_3) is a loop of order 3 contradicting the extended loop lemma.

If $b_2 \not\mathcal{C} b_3$, then there exist $B_1, B_4 \in \mathfrak{A}_L$ and $m_1, m_3, m_4 \in L \setminus \{0, 1\}$ such that $b_2, b_2 \vee b_3 \in B_1$, $b_3, b_2 \vee b_3 \in B_4$, $B_1 \cap B_2 = S_{m_1}$, $B_3 \cap B_4 = S_{m_3}$ and $B_4 \cap B_1 = S_{m_4}$ since $0 < b_2, b_3 < b_2 \vee b_3 < 1$. We see that $m_1 \neq m_2$ since $b_2 \in S_{m_1} \setminus S_{m_2}$, $m_2 \neq m_3$ since $b_3 \in S_{m_3} \setminus S_{m_2}$, $m_3 \neq m_4$ since $b_2 \vee b_3 \in S_{m_4} \setminus S_{m_3}$, $m_4 \neq m_1$ since $b_2 \vee b_3 \in S_{m_4} \setminus S_{m_1}$, $m_2 \neq m_4$ since $b_2 \vee b_3 \in S_{m_4} \setminus S_{m_2}$, and $m_1 \neq m_3$ since $b_2 \in S_{m_1} \setminus S_{m_3}$. Thus $m_i \neq m_j$ for all distinct $i, j \in \{1, 2, 3, 4\}$. Therefore $B_i \cap B_j \cap B_k = \{0, 1\}$ for all pairwise distinct $i, j, k \in \{1, 2, 3, 4\}$, otherwise without loss of generality we may assume that $B_1 \cap B_2 \cap B_3 \neq \{0, 1\}$ and hence $m_1 = m_2$ by Lemma 2.2 contradicting $m_1 \neq m_2$. Then (B_1, B_2, B_3, B_4) is a loop of order of 4 contradicting the extended loop lemma. Therefore $B_2 \cup B_3 \leq L$.

Assume finally $B_2 \cap B_3 = \{0, 1\}$. If $B_2 \cup B_3 \leq L$, then B_2 and B_3 are path-connected. If $B_2 \cup B_3 \not\leq L$, then there exist B_1, B_4 such that $B_2 \cap B_1 \neq \{0, 1\}$, $B_1 \cap B_4 \neq \{0, 1\}$ and $B_4 \cap B_3 \neq \{0, 1\}$ by Lemma 1.7. Thus B_2 and B_3 are path-connected by a concatenated path by the first case.

Let us show that $L = [0, a'] \oplus [0, a]$ for some $a \in L$ where $[0, a']$ is a Boolean algebra and $[0, a]$ is irreducible path-connected OML. We may assume that $|\mathfrak{A}_L| \geq 3$ by Lemma 2.3. If there exist $A, C \in \mathfrak{A}_L$ with $A \cap C = \{0, 1\}$, then L is irreducible; hence the conclusion holds with $[0, a'] = \{0\}$ and $[0, a] = [0, 1] = L$. Thus we may assume that $A \cap C \neq \{0, 1\}$ $\forall A, C \in \mathfrak{A}_L$. Let B_1, B_2, B_3 be three distinct blocks in L. Then $B_1 \cap B_2 = S_x$, $B_2 \cap B_3 = S_y$ and $B_1 \cap B_3 = S_z$ for some $0 < x, y, z < 1$ by our assumption. If $B_1 \cap B_2 \cap B_3 = \{0, 1\}$, then (B_1, B_2, B_3) is a loop of order 3 contradicting the Extended Loop Lemma. Thus we may assume $B_1 \cap B_2 \cap B_3 \neq \{0, 1\}$. Then $B_1 \cap B_2 = B_2 \cap B_3 = S_a$ for some $a \in B_2$ by Lemma 2.2. Since B_1, B_2 and B_3 was arbitrary, $\mathbf{C}(L) = \bigcap \mathfrak{A}_L = A \cap C = S_a$ $\forall A, C \in \mathfrak{A}_L$. Then $L = [0, a'] \oplus [0, a]$ and $[0, a']$ is Boolean and $[0, a]$ is a horizontal sum of Boolean algebras, otherwise there exist two blocks $D, E \in \mathfrak{A}_{[0,a]}$ such that $D \cap E \neq \{0, a'\}$ and hence $([0, a'] \oplus D) \cap ([0, a'] \oplus E) \neq S_a$. Thus $[0, a]$ is irreducible. Moreover $[0, a]$ is path-connected. This completes the proof. □

We have the following corollary as a special case of Theorem 2.4.

Corollary 2.5. *Every OML L satisfying the Loop Lemma* [4] *is a path-connected OML and $L = [0, a'] \oplus [0, a]$ where $[0, a']$ is a Boolean* *algebra with* $|[0, a']| \leq 2$ *and* $[0, a]$ *is an irreducible path-connected OML.*

PROOF. Let L be an OML satisfying the Loop Lemma. Then for distinct blocks A, B of L $|A \cap B| \leq 4$ and L satisfies the Extended Loop Lemma. This completes the proof by Theorem 2.4. □

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