Bull. Korean Math. Soc. **41** (2004), No. 2, pp. 387–391

A NOTE ON THE LOCAL HOMOLOGY

S. Rasoulyar

Abstract. Let A be Noetherian ring, $\mathfrak{a} = (r_1, \ldots, r_n)$ an ideal of A and $\mathcal{C}_A$ be category of A-modules and A-homomorphisms. We show that the connected left sequences of covariant functors $\{\varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, -))\}_{i\geq 0}$ and $\{\varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^A(\frac{A}{\mathfrak{a}^t}, -)\}_{i\geq 0}$ are isomorphic from $\mathcal{C}_A$ to itself, where $\mathfrak{r}^t = r_1^t, \ldots, r_n^t$.

Introduction

In this paper A is a commutative ring with identity and $\mathfrak{a}$ is an ideal of A. We use $\mathbb{N}$ (resp. $\mathbb{N}_0$) to denote the set of positive (resp. non-negative) integers.

The theory of local homology is first studied by E. Matlis. We denote the covariant $\mathfrak{a}$-adic completion functor by $(-)^\wedge$. He defined i-th local homology functor as the i-th left derived functor of $(-)^\wedge$, where $\mathfrak{a}$ is the ideal generated by finite regular A-sequence (see [2, 3]). A.M. Simon extended Matlis's definition for arbitrary ideal and she denoted i-th local homology of an A-module M with respect to $\mathfrak{a}$ by $U_i^{\mathfrak{a}}(M)$ (See [6, 7]). Also in [9] Z. Tang has introduced a definition of local homology for Artinian A-modules. Let M be a non-zero Artinian A-module, and $\mathfrak{r} = r_1, r_2, \ldots, r_n$ a sequence of elements of A. He defined i-th local homology $H_i^{\mathfrak{r}}(M)$ as $\varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, M))$, where $H_i(K_\bullet(\mathfrak{r}^t, M))$ denotes the i-th homology module of the Koszul complex of M with respect to $\mathfrak{r}^t = r_1^t, r_2^t, \ldots, r_n^t$. Let A be a Noetherian ring, it was shown for each $i \in \mathbb{N}_0$, $H_i^{\mathfrak{r}}(M) \simeq \varprojlim_{t\in\mathbb{N}} \mathrm{Tor}_i^A(\frac{A}{\mathfrak{a}^t}, M)$, where M be an Artinian A-module (See [1, Theorem 4.3]).

In this paper, we prove that if A be a Noetherian ring, then the connected left sequences of covariant functors $\{\varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, -))\}_{i\geq 0}$ and

Received November 14, 2002.
2000 Mathematics Subject Classification: 13D07, 13D25, 13E05.
Key words and phrases: local homology modules, Koszul complex, Noetherian ring, inverse limit, completion.

$\{\varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t},-)\}_{i\geq 0}$ are isomorphic from $\mathcal{C}_A$ to itself. Therefore by using Tang's notation and without Artinian assumpotion on M, we have $H_i^{\mathfrak{r}}(M) \simeq \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t}, M)$ for all $i \in \mathbb{N}_0$.

Let $\mathfrak{a}$ be an ideal of a commutative ring A. For any A-module M, there exists natural epimorphism $U_0^{\mathfrak{a}}(M) \longrightarrow \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_0^A(\frac{A}{\mathfrak{a}^t}, M) \simeq \hat{M}$ (See [6, 5.1]). In an example, we give the a Noetherian ring A and the an A-module M, such that this natural epimorphism is not isomorphism. It follows that, in general the connected left sequences of covariant functors $\{U_i^{\mathfrak{a}}(-)\}_{i\geq 0}$ and $\{\varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t},-)\}$ are not isomorphism from $\mathcal{C}_A$ to itself, although A be a Notherian ring.

REMARK 1. Let $s \in \mathbb{N}_0$ and $t \geq 1$, for each $p \geq 1$ define $\theta_p^{t,s}$: $K_p^{t+s} \longrightarrow K_p^t$ where $\theta_p^{t,s}(e_\alpha) = r_\alpha^s e_\alpha$, when $\alpha = (i_1, \ldots, i_p)$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_p$ and e_α is a basis element of $K_p(\mathfrak{r})$, here we have writen r_a for product $r_{i_1} r_{i_2} \ldots r_{i_p}$ (See [8, 4.2 and 4.3]). Let A be a Noetherian ring, and $t \geq 1$ a given integer. There exists $s_0 \geq 0$ such that, for $s \geq s_0$, the map $H_i(\theta_\bullet^{t,s}) : H_i(K_\bullet(\mathfrak{r}^{t+s}, A)) \longrightarrow H_i(K_\bullet(\mathfrak{r}^t, A))$ is null morphism for all $i \geq 1$ (see [8, 4.3.3 Lemma]). Since every projective A-module P is flat, hence $H_i(K_\bullet(\mathfrak{r}^u, A)) \otimes_A P \simeq H_i(K_\bullet(\mathfrak{r}^u, P))$ for all $u \in \mathbb{N}$ and $i \geq 0$. It follows that for $t \geq 1$, there exists $s_0 \geq 0$ such that for $s \geq s_0$, the map $H_i(\theta_\bullet^{t,s}) \otimes id_P : H_i(K_\bullet(\mathfrak{r}^{t+s}, P)) \longrightarrow H_i(K_\bullet(\mathfrak{r}^t, P))$ is null morphism for all $i \geq 1$. Thus inverse system $\{H_i(K_\bullet(\mathfrak{r}^t, P))\}_{t\geq 1}$ statisfies the trivial Mittag-Leffler condition for all $i \geq 1$ (See [10, Definition 3.5.6]). Now by [10, Proposition 3.5.7], we have $\varprojlim_{t\in\mathbb{N}}{}^1 H_i(K_\bullet(\mathfrak{r}^t, P)) = 0$ for all $i \geq 1$, and also from definition of inverse limit, it is easy to show that $\varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, P)) = 0$ for all $i \geq 1$.

THEOREM 2. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Then the connected left sequences of covariant functors $\{\varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t,-))\}_{i\geq 0}$ and $\{\varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t},-)\}_{i\geq 0}$ are isomorphic from $\mathcal{C}_A$ to itself.*

Proof. Let $\mathfrak{a} = (r_1, \ldots, r_n)$ be an ideal of A, and M an A-module. Then for any $t \in \mathbb{N}$, $\mathfrak{a}^{nt} M \subseteq \mathfrak{a}_t M \subseteq \mathfrak{a}^t M$,where $\mathfrak{a}_t = (r_1^t, \ldots, r_n^t)$. It follows that $\{\mathfrak{a}^t M\}_{t\geq 1}$ and $\{\mathfrak{a}_t M\}_{t\geq 1}$ give same topology on M. Now by [4, p.55], we have the natural isomorphism $\varprojlim_{t\in\mathbb{N}} H_0(K_\bullet(\mathfrak{r}^t, M)) =$

$\varprojlim_{t\in\mathbb{N}} \frac{M}{(r_1^t,\ldots,r_n^t)M} \simeq \varprojlim_{t\in\mathbb{N}} \frac{M}{\mathfrak{a}^t M} \simeq \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_0^A(\frac{A}{\mathfrak{a}^t}, M)$. Let M be an A-module and $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ $(*)$ the exact sequence, where P is projective A-module. For each $i \in \mathbb{N}_0$, this gives rise to exact sequences

$$0 \longrightarrow \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_{i+1}^A(\frac{A}{\mathfrak{a}^t}, M) \longrightarrow \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t}, L) \longrightarrow \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t}, P).$$

Furthermore from $(*)$, for each $i \geq 0$, we have the long exact sequences

$$(**) \qquad \begin{array}{l} \cdots \longrightarrow H_{i+1}(K_\bullet(\mathfrak{r}^t, P)) \xrightarrow{f_{i+1,t}} H_{i+1}(K_\bullet(\mathfrak{r}^t, M)) \\ \xrightarrow{h_{i,t}} H_i(K_\bullet(\mathfrak{r}^t, L)) \xrightarrow{g_{i,t}} H_i(K_\bullet(\mathfrak{r}^t, P)) \end{array}$$

Since $\operatorname{Im}(f_{i+1,t}) \simeq \frac{H_{i+1}(K_\bullet(\mathfrak{r}^t,P))}{\operatorname{Ker}(f_{i+1,2})}$ for all $i \geq 0$, hence by Remark 1, the inverse system $\{\operatorname{Im}(f_{i+1,t})\}_{t\geq 1}$ satisfies the trivial Mittag-Leffler condition for all $i \geq 0$. It follows that

$$\varprojlim_{t\in\mathbb{N}}(\operatorname{Im}(f_{i+1,t})) = \varprojlim_{t\in\mathbb{N}}{}^1 \quad (\operatorname{Im}(f_{i+1,t})) = 0 \text{ for all } i \geq 0.$$

Now, from $(**)$ for each $i \geq 0$, we have exact sequences

$$0 \longrightarrow \operatorname{Im}(h_{i,t}) \xrightarrow{\subseteq} H_i(K_\bullet(\mathfrak{r}^t, L)) \xrightarrow{g_{i,t}} H_i(K_\bullet(\mathfrak{r}^t, P))$$

for all $t \in \mathbb{N}$. Since inverse limit functor is left exact, hence we have exact sequences

$$0 \longrightarrow \varprojlim_{t\in\mathbb{N}}(\operatorname{Im}(h_{i,t})) \longrightarrow \varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, L)) \longrightarrow \varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, P))$$

for all $i \in \mathbb{N}_0$. Also for each $t \geq 1$, from the exact seqcuences

$$0 \longrightarrow \operatorname{Im}(f_{i+1,t}) \xrightarrow{\subseteq} H_{i+1}(K_\bullet(\mathfrak{r}^t, M)) \xrightarrow{h_{i,t}} \operatorname{Im}(h_{i,t}) \longrightarrow 0$$

for all $i \in \mathbb{N}_0$ and by using [10, vista 3.5.12], we have long exact sequences

$$\begin{array}{l} 0 \longrightarrow \varprojlim_{t\in\mathbb{N}}(\operatorname{Im}(f_{i+1,t})) \longrightarrow \varprojlim_{t\in\mathbb{N}} H_{i+1}(K_\bullet(\mathfrak{r}^t, M)) \longrightarrow \varprojlim_{t\in\mathbb{N}}(\operatorname{Im}(h_{i,t})) \\ \qquad \longrightarrow \varprojlim_{t\in\mathbb{N}}{}^1(\operatorname{Im}(f_{i+1,t})) \longrightarrow \varprojlim_{t\in\mathbb{N}}{}^1(H_{i+1}(K_\bullet(\mathfrak{r}^t, M)) \longrightarrow \ldots \end{array}$$

for all $i \in \mathbb{N}_0$. It follows that $\varprojlim_{t\in\mathbb{N}}(\operatorname{Im}(h_{i,t})) \simeq \varprojlim_{t\in\mathbb{N}} H_{i+1}(K_\bullet(\mathfrak{r}^t, M))$ for all $i \in \mathbb{N}_0$. Whence, we have the exact sequences

$$0 \longrightarrow \varprojlim_{t\in\mathbb{N}} H_{i+1}(K_\bullet(\mathfrak{r}^t, M)) \longrightarrow \varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, L) \longrightarrow \varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, P))$$

for all $i \in \mathbb{N}_0$. Now the required result follows from [5, Corollary p.120]. □

COROLLARY 3. *Let A be a Noetherian ring, $\mathfrak{a} = (r_1, \ldots, r_n)$ an ideal of A and M an A-module. If for each $i \geq 0$, $H_i^{\mathfrak{r}}(M)$ as $\varprojlim_{t\in\mathbb{N}} H_i(K_\bullet(\mathfrak{r}^t, M))$, then $H_i^{\mathfrak{r}}(M) \simeq \varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}^t}, M)$.*

Let $\mathfrak{a}$ be an ideal of a commutative ring A. There exists natural epimorphism $U_0^{\mathfrak{a}}(M) \longrightarrow \hat{M}$ for any A-module M. In following, we give an example such that this natural epimorphism is not isomorphism.

EXAMPLE 4. (A.M. Simon).

Let K be a filed and A be a formal power series ring in one variable t over K, so that A is a complete local domain with maximal tA. For each $i \in \mathbb{N}$ we write $t(i)$, for i power of t. Let M be direct sum of the modules $\frac{A}{t(i)A}$, $i > 0$.

Let L be the direct sum of countably many copies of A and $u : L \longrightarrow L$ is given by $u(X)(i) = t(i)X(i)$ for all $i \in \mathbb{N}$. M has an obvious free resolution $0 \longrightarrow L \xrightarrow{u} L \xrightarrow{\rho} M \longrightarrow 0$, where $\rho : L \longrightarrow M$ is natural epimorphism. This exact sequence, gives rise to 0-sequence

$$\hat{L} \xrightarrow{\hat{u}} \hat{L} \xrightarrow{\hat{\rho}} \hat{M} \longrightarrow 0.$$

In view of [6, 9.4], completion of L is

$$\hat{L} = \{(m_i)_{i\in\mathbb{N}} \in \pi\hat{A} \simeq \pi A : \text{for all } n, \text{ all but finitely many } m_i \text{ belong to } (t^n)\hat{A} \simeq (t^n)A\}.$$

Observe that the completion u is still injective. Let $X = (t(i))_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$, $t(i) \in (t^n)$ for all $i \in \mathbb{N}$ but finitely many i. Hence $(t(i))_{i\in\mathbb{N}} \in \hat{L}$ and we have $\hat{\rho}((t(i))) = 0$, where $\hat{\rho}$ is completion of ρ, hence $(t(i))_{i\in\mathbb{N}}$ is an element of $\operatorname{Ker}\hat{\rho}$. We show that element $(t(i))_{i\in\mathbb{N}}$ dose not belong to the image of the completion of u.

Suppose that there exists $(m_i)_{i\in\mathbb{N}} \in \hat{L}$, such that $\hat{u}((m_i))_{i\in\mathbb{N}} = (t(i))_{i\in\mathbb{N}}$. It follows that $t(i)m_i = t(i)$ for all $i \in \mathbb{N}_0$. Hence $m_i = 1$ for all $i \in \mathbb{N}$, so that $(1)_{i\in\mathbb{N}} \in \hat{L}$. This is a contradiction to the description of the elements of $\hat{L}$.

$(t(i))_{i\in\mathbb{N}} + \operatorname{Im}\hat{u}$ is non-zero element of

$$U_0^{\mathfrak{a}}(M) = \frac{\hat{L}}{\operatorname{Im}\hat{u}}$$

and

$$(\hat{\rho})^*((t(i)_{i\in\mathbb{N}}) + \operatorname{Im}\hat{u}) = 0$$

where $(\hat{\rho})^* : U_0^{\mathfrak{a}}(M) \longrightarrow \hat{M}$ is natural induced homomorphism. Thus the natural map from $U_0^{\mathfrak{a}}(M)$ to $\hat{M}$ is not isomorphism.

In view of the above example, we can deduce in general, the connected left sequences of covariant functors $\{U_i^{\mathfrak{a}}(-)\}_{i\geq 0}$ and $\{\varprojlim_{t\in\mathbb{N}} \operatorname{Tor}_i^A(\frac{A}{\mathfrak{a}_t}, -)\}_{i\geq 0}$ are not isomorphic, although A is a Noetherian ring.

ACKNOWLEDGMENTS. The author would like to thank Prof. A. M. Simon for example and Dr.M. Tousi for usefull discussions and helpfull suggestions.

References

[1] M. T. Dibaei and K. Divaani-Aazar, *The theory of local homology for Artinian modules*, Southeast Asian Bull. Math. **24** (2000), no. 1, 31–40.
[2] E. Matlis, *The Koszul complex and duality*, Comm. Algebra **1** (1974), 87–144.
[3] ———, *The higher properties of R-Sequences*, J. Algebra **50** (1978), no. 1, 77–122.
[4] H. Matsumora, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
[5] D. G. Northcott, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1960.
[6] A. M. Simon, *Some homological properties of complete modules*, Math. Proc. Cambridge Philos. Soc. **108** (1990), no. 2, 231–246.
[7] A. M. Simon, *Adic completion and some dual homological results*, Publications Mathematiques **36** (1992), no. 2B, 965–979.
[8] J. R. Strooker, *Homological questions in local algebra*, London Math. Soc.; Lect. Note Ser. **145**, Cambridge University Press, Cambridge, 1990.
[9] Z. Tang, *Local homology theory for Artinian modules*, Comm. Algebra **22** (1994), no. 5, 1675–1684.
[10] C. A. Weibel, *An introduction to homological Algebra*, Cambridge University Press, Cambridge, 1994.

THE DEPARTMENT OF MATHEMATICS OF FACULTY OF SCIENCE, UNIVERSITY OF KURDISTAN, BULVAR PASDARAN, SANANDIJ, IRAN
E-mail: srasoulyar@uok.ac.ir