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BOUNDS OF CORRELATION DIMENSIONS FOR SNAPSHOT ATTRACTORS

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ABSTRACT. In this paper, we reformulate a snapshot attractor([5]), $(K, \bar{\mu}_l)$ generated by a random baker's map with a sequence of probability measures $\{\bar{\mu}_l\}$ on K. We obtain bounds of the correlation dimensions of $(K, \bar{\mu}_l)$ for all $l \geq 1$.

1. Introduction

Ledrappier and Young [4] have theoretically studied about the dynamics generated by random maps. And they proved that, with probability one, a given realization of the random process turns out to be a sequence of patterns each having an information dimension given by the Kaplan-Yorke formula ([6]) in the two dimensional cases.

In fluid mechanics, to explain the particle motion on the surface of a temporally irregular fluid, many authors have experimentally dealt with strange attractors induced by random maps ([9]). In [5], they considered a model random map given in the section 2, the given model illustrate the evolution of the pattern of scum floating on a fluid surface. And they considered a finite number of realizations of a random sequence and experimentally defined snapshot attractors. Also they obtained bounds of information dimensions of the snapshot attractors by the numerical calculations.

On the other hand, it was emphasized by Grossberger, Hentschel and Procaccia that the correlation dimension is particularly suitable because of the relatively easy experimental computation ([2], [3], [6]). It is well-known that the correlation dimension is less than or equal to the information dimension with respect to any probability measure ([7], [8]).

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In this paper, in order to obtain the theoretical results about attractors induced by random maps, we modify a model random map to a solvable random map and reformulate the snapshot attractors generated by a random baker's map in connection with each random sequence in $(0,1)$. For each snapshot attractor, we have a lower bound and an upper bound for the correlation dimension via calculations of energy (see Theorem 3.3). Also, we show that the Kaplan-Yorke dimension, the information dimension and the correlation dimension have the same value with respect to the natural measure ([3], [6]) (see Remark 3.6).

2. Preliminaries

Let us introduce a model random map [5] given by

$$\tag{1} \begin{cases} x_{n+1} = x_n + (1-e^{-\alpha})y_n/\alpha & \pmod{2\pi} \\ y_{n+1} = \kappa \sin(x_{n+1}+c_n) + e^{-\alpha}y_n \end{cases}$$

where c_n is chosen randomly in $[0,2\pi]$ at each iteration and κ and α fixed positive constants.

To obtain the theories about the random map(1), we consider a solvable random baker's map M on the set $X = [0,1]\times[0,1]$ in $\mathbb{R}^2$ defined as follows : for $n = 1,2,3,...$, $(x_{n+1}, y_{n+1}) = M(x_n, y_n)$ such that

$$x_{n+1} = \begin{cases} \lambda_1 x_n & (\text{if } y_n \le c_n) \\ 1-\lambda_2+\lambda_2 x_n & (\text{if } y_n > c_n) \end{cases}$$

and

$$y_{n+1} = \begin{cases} \dfrac{y_n}{c_n} & (\text{if } y_n \le c_n) \\ \dfrac{y_n - c_n}{1-c_n} & (\text{if } y_n > c_n) \end{cases}$$

where $0 < \lambda_1, \lambda_2$ and $\lambda_1 + \lambda_2 < 1$, and $c_n \in (0,1)$ is chosen randomly at each iteration. Then at the first iteration, X is split into the left strip($\equiv X_1$) with the width λ_1 and the right strip ($\equiv X_2$) with the width λ_2. At the second iteration, $X_i (i = 1,2)$ is split into the left substrip ($\equiv X_{i,1}$) of X_i with the width $\lambda_i \cdot \lambda_1$ and the right substrip ($\equiv X_{i,2}$) of X_i with the width $\lambda_i \cdot \lambda_2$. Continuing this way, we can obtain after n-th iteration 2^n strips $X_{i_1,i_2,\cdots,i_n}$ which have the width $\lambda_{i_1}\cdot\lambda_{i_2}\cdots\lambda_{i_n}$ for $i_j \in \{1,2\}$ $(j = 1,2,...,n)$. Denote

$$\{1,2\}^m \equiv \{(i_1, i_2, \cdots, i_m) : \ i_j \in \{1,2\},\ j = 1,...,m\}.$$

Set

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1,i_2,\cdots,i_n)\in\{1,2\}^n} X_{i_1,i_2,\cdots,i_n}.$$

We call K *the random baker's set* generated by the random baker's map M.

Now we are in a position to define a probability measure on K. For any $n \geq 1$ and all $(i_1, i_2, \cdots, i_n)$, we define a map p as follows:

$$p(X_{i_1,i_2,\cdots,i_n}) = \prod_{j=1}^{n} p_{i_j} = p_{i_1} \cdot p_{i_2} \cdots p_{i_n}$$

where $p_{i_k} = c_k$ if $i_k = 1$ and $p_{i_k} = 1 - c_k$ if $i_k = 2$ for $k = 1, 2, ..., n$. For any Borel subset A of K, let

$$\bar{\mu}_p(A) = \inf\left\{ \sum_{(i_1,\cdots,i_n)\in\, \mathcal{C}} p(X_{i_1,\cdots,i_n}) : A \subset \bigcup_{(i_1,\cdots,i_n)\in\, \mathcal{C}\subset\{1,2\}^n} X_{i_1,\cdots,i_n} \right\}.$$

Then $\bar{\mu}_p$ is a probability measure on K.

We denote $\hat{E} = \text{proj}(E)$ for $E \subset \mathbb{R}^2$, where $\text{proj}(x, y) = x$ for all $(x, y) \in \mathbb{R}^2$. Let μ_p be the induced measure on $\hat{K}$, *i.e.* $\mu_p(\hat{E}) = \bar{\mu}_p(E)$ for any Borel subset E of K. Put

$$\Sigma_2 \equiv \{1,2\}^{\mathbb{N}} = \{(i_1, i_2, i_3, \cdots) : \ i_j \in \{1,2\}, j = 1, 2, ... \}.$$

Then Σ_2 is a compact set with the metric ρ given by $\rho(\mathbf{i}, \mathbf{j}) = 2^{-k}$, where $k = \min\{n : i_n \neq j_n \text{ for } n \geq 1\}$ for $\mathbf{i} = (i_1, i_2, \cdots)$ and $\mathbf{j} = (j_1, j_2, j_3, \cdots) \in \Sigma_2$. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map defined by $\sigma(i_1, i_2, i_3, \cdots) = (i_2, i_3, \cdots)$. Write $\sigma^j = \sigma \circ \sigma^{j-1}$ for all $j \geq 1$. We define a bijective map Π from $\hat{K}$ to Σ_2 as $\Pi(x) = (i_1, i_2, \cdots) \in \Sigma_2$ for any $x = \cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} \in \hat{K}$. Let ν_p be the probability measure on Σ_2 satisfying $\nu_p = \mu_p \circ \Pi^{-1}$. Then the measure ν_p is a σ-invariant measure on Σ_2. For the randomly chosen number $c_1 \in (0, 1)$, define a continuous function $\phi : \Sigma_2 \to \mathbb{R}$ as $\phi(i_1, i_2, i_3, \cdots) = \log p_{i_1}$, where $p_{i_1} = c_1$ if $i_1 = 1$ and $p_{i_1} = 1 - c_1$ if $i_1 = 2$.

In the special case, $p_{i_n} = c_1(i_n = 1)$ and $p_{i_n} = 1 - c_1(i_n = 2)$ for all $n \geq 1$, it is well-known that the σ-invariant measure ν_p on Σ_2 becomes an ergodic measure ([1], [6]). However, in general case, we do not know whether the σ-invariant measure ν_p becomes an ergodic measure on Σ_2.

PROPOSITION 2.1. *Let $\hat{K}$, Σ_2, μ_p and ν_p and ϕ be defined as above. If ν_p is an ergodic measure on Σ_2, then*

$$\frac{1}{n} \log \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n}) \longrightarrow c_1 \log c_1 + (1 - c_1) \log(1 - c_1) \ \text{as } n \to \infty$$

for ν_p-almost all $\mathbf{i} = (i_1, i_2, i_3, \cdots) \in \Sigma_2$.

Proof. Note that the measure ν_p is σ-invariant and an ergodic measure on Σ_2, and the function $\phi \in L^1(\mu_p)$. We obtain, using the ergodic Theorem ([1], [6]), for ν_p-almost all $\mathbf{i} = (i_1, i_2, \cdots) \in \Sigma_2$,

$$\begin{aligned}
\frac{1}{n}\log \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n}) &= \frac{1}{n}\log \bar{\mu}_p(X_{i_1,i_2,\cdots,i_n}) \\
&= \frac{1}{n}\log \prod_{j=1}^{n} p_{i_j} \\
&= \frac{1}{n}\sum_{j=1}^{n}\log p_{i_j} = \frac{1}{n}\sum_{j=0}^{n-1}\phi(\sigma^j(\mathbf{i})) \\
&\to \frac{1}{\nu_p(\Sigma_2)}\int_{\Sigma_2}\phi \, d\nu_p \\
&= c_1\log c_1 \; + \; (1-c_1)\log(1-c_1).
\end{aligned}$$

□

For each fixed $l \in \mathbb{N}$, denote $\{c_n^l\}(n = 1, 2, 3, ...)$ a random sequence in $(0, 1)$. Then we obtain a probability measure $\bar{\mu}_{p(l)}$ on K such that for any $n \geq 1$ and all $(i_1, i_2, ..., i_n)$,

$$\bar{\mu}_{p(l)}(X_{i_1,i_2,\cdots,i_n}) = \prod_{j=1}^{n} p_{i_j} = p_{i_1} \cdot p_{i_2} \cdots p_{i_n}$$

where $p_{i_k} = c_k^l$ for $i_k = 1$ and $p_{i_k} = 1 - c_k^l$ for $i_k = 2$. Write $\bar{\mu}_l$ for $\bar{\mu}_{p(l)}$ being the generated measure from each random sequence $\{ c_n^l \}_{n=1}^{\infty}$ in $(0, 1)$, for each $l \in \mathbb{N}$. We call $(K, \bar{\mu}_l)$ *a snapshot attractor* for each $l = 1, 2, 3, ...$

Denote μ_l for $\mu_{p(l)}$ being the induced probability measure on $\hat{K}$ from $\bar{\mu}_{p(l)}$ for each $l \geq 1$. From Proposition 2.1, we have the following.

PROPOSITION 2.2. *For each $l \in \mathbb{N}$, let $\hat{K}$, Σ_2, μ_l, ν_l and ϕ be defined as above. If ν_l is an ergodic measure on Σ_2, then*

$$\frac{1}{n}\log \mu_l(\hat{X}_{i_1,i_2,\cdots,i_n}) \longrightarrow c_1^l \log c_1^l + (1-c_1^l)\log(1-c_1^l) \text{ as } n \to \infty$$

for ν_l-almost all $\mathbf{i} = (i_1, i_2, i_3, \cdots) \in \Sigma_2$.

REMARK 2.3. $f(x) = x\log x + (1-x)\log(1-x)$ is a concave upward function on $(0, 1)$ and the expectation of $f(x)$ is $\int_0^1 f(x)dx = -\frac{1}{2}$.

3. Main results

We denote $|A|$ for the diameter of a set A in $\mathbb{R}^d$.

LEMMA 3.1. *If $E \subseteq \hat{K}$ satisfies*

$$\alpha_p \leq \liminf_{n\to\infty} \frac{\log \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n})}{\log |\hat{X}_{i_1,i_2,\cdots,i_n}|}$$

for any $\cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} (\equiv x) \in E$, then for some constant $a_1 > 0$, we have $\mu_p(B(x,r) \cap E) \leq a_1 r^{\alpha_p}$, where $B(x,r)$ is the closed ball of radius r with center x.

Proof. Let $r > 0$ be given. For each $x \in E$, there exists a unique sequence $(i_1, i_2, i_3, \cdots)$ satisfying $\cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} = x$. We can find a large number $n \in \mathbb{N}$ satisfying $x \in \hat{X}_{i_1,i_2,\cdots,i_{n+1}}$ and

$$(2) \qquad |\hat{X}_{i_1,i_2,\cdots,i_{n+1}}| \leq r < |\hat{X}_{i_1,i_2,\cdots,i_n}|.$$

For $y(\neq x) \in B(x,r) \cap E$, there exists a unique $(j_1, j_2, j_3, \cdots)$ such that $\cap_{k=1}^{\infty} \hat{X}_{j_1,j_2,\cdots,j_k} = y$, and we also can find $k \in \mathbb{N}$ such that $|\hat{X}_{j_1,\cdots,j_{k+1}}| \leq r < |\hat{X}_{j_1,\cdots,j_k}|$ and $y \in \hat{X}_{j_1,\cdots,j_{k+1}}$. If $\hat{X}_{i_1,\cdots,i_{n+1}} \cap \hat{X}_{j_1,\cdots,j_{k+1}} = \phi$, then $B(x,r)$ meets at most two sets $\hat{X}_{i_1,\cdots,i_{n+1}}$ and $\hat{X}_{j_1,\cdots,j_{k+1}}$. Otherwise, $(B(x,r) \cap E) \subset \hat{X}_{i_1,\cdots,i_n}$ or $(B(x,r) \cap E) \subset \hat{X}_{j_1,\cdots,j_k}$. Using the hypothesis, we get for large number l,

$$(3) \qquad \mu_p(\hat{X}_{i_1,i_2,\cdots,i_l}) \leq |\hat{X}_{i_1,i_2,\cdots,i_l}|^{\alpha_p}.$$

If we take r sufficiently small, then the numbers n and k become to satisfy (2) and (3). Put $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. Hence for all $x \in E$, using the facts (2) and (3),

$$\begin{aligned} \mu_p(B(x,r) \cap E) &\leq \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n}) + \mu_p(\hat{X}_{j_1,j_2,\cdots,j_k}) \\ &\leq |\hat{X}_{i_1,i_2,\cdots,i_n}|^{\alpha_p} + |\hat{X}_{j_1,j_2,\cdots,j_k}|^{\alpha_p} \\ &\leq 2\, \lambda_0^{-\alpha_p}\, r^{\alpha_p} \equiv a_1 r^{\alpha_p}. \end{aligned}$$

□

LEMMA 3.2. *If $E \subseteq \hat{K}$ satisfies*

$$\limsup_{n\to\infty} \frac{\log \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n})}{\log |\hat{X}_{i_1,i_2,\cdots,i_n}|} \leq \beta_p$$

for any $\cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} (\equiv x) \in E$, then for all $x \in E$ and $r > 0$, we have $\mu_p(B(x,r) \cap E) \geq a_2 r^{\beta_p}$, where some constant $a_2 > 0$.

Proof. Let $r > 0$ be given. For each $x = \cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} \in E$, there exists a large number $n_0 \in \mathbb{N}$ such that $x \in \hat{X}_{i_1,i_2,\cdots,i_{n_0+1}}$ and $|\hat{X}_{i_1,i_2,\cdots,i_{n_0+1}}| \le r < |\hat{X}_{i_1,i_2,\cdots,i_{n_0}}|$. By the hypothesis, there exists a large number $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$, $\mu_p(\hat{X}_{i_1,i_2,\cdots,i_n}) \ge |\hat{X}_{i_1,i_2,\cdots,i_n}|^{\beta_p}$. For a sufficiently small number $0 < r < 1$, we may assume that $n_0 \ge N_0$. Therefore, for all $x = \cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} \in E$, we have $B(x,r) \supset \hat{X}_{i_1,i_2,\cdots,i_{n_0+1}}$ and

$$\begin{aligned}
\mu_p(B(x,r) \cap E) &\ge \mu_p(\hat{X}_{i_1,i_2,\cdots,i_{n_0+1}}) \\
&\ge |\hat{X}_{i_1,i_2,\cdots,i_{n_0+1}}|^{\beta_p} \\
&\ge \frac{|\hat{X}_{i_1,i_2,\cdots,i_{n_0+1}}|^{\beta_p}}{|\hat{X}_{i_1,i_2,\cdots,i_{n_0}}|^{\beta_p}} \cdot r^{\beta_p} \\
&\ge \lambda_0^{\beta_p} \cdot r^{\beta_p} \equiv a_2 \cdot r^{\beta_p}.
\end{aligned}$$

□

We recall the following definition of *the correlation dimension* of $A(\subset \mathbb{R}^d)$ with respect to a probability measure η on A([8]) ;

$$D_2(A,\eta) \equiv \sup\{s \ge 0 : I_s(\eta) < \infty\} = \inf\{s \ge 0 : I_s(\eta) = \infty\},$$

where $I_s(\eta) = \int_A \int_A |x-y|^{-s} d\eta(x) d\eta(y)$ is the s-energy of A with respect to η.

THEOREM 3.3. *If $E \subseteq \hat{K}$ satisfies*

$$(4) \quad \alpha_p \le \liminf_{n\to\infty} \frac{\log \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n})}{\log |\hat{X}_{i_1,i_2,\cdots,i_n}|} \le \limsup_{n\to\infty} \frac{\log \mu_p(\hat{X}_{i_1,i_2,\cdots,i_n})}{\log |\hat{X}_{i_1,i_2,\cdots,i_n}|} \le \beta_p$$

for any $\cap_{n=1}^{\infty} \hat{X}_{i_1,i_2,\cdots,i_n} \in E$, then we have $\alpha_p \le D_2(E,\mu_p) \le \beta_p$.

Proof. (i): In order to obtain the lower bound of the correlation dimension of E, we calculate the energy $I_t(\mu_p)$ on E with respect to the measure μ_p. We put $\phi_t(x) = \int_E |x-y|^{-t} d\mu_p(y)$. Using the Lemma 3.1, we have,

$$\begin{aligned}
\phi_t(x) &= \int_0^{\infty} \mu_p(\{y \in E : |x-y|^{-t} \ge r\}) dr \\
&= \int_0^{\infty} \mu_p(B(x, r^{-1/t}) \cap E) dr \\
&= t \int_0^{\infty} \epsilon^{-t-1} \, \mu_p(B(x,\epsilon) \cap E) d\epsilon
\end{aligned}$$

$$< t\left[\int_0^1 \epsilon^{-t-1}\mu_p(B(x,\epsilon)\cap E)d\epsilon \;+\; \int_1^\infty \epsilon^{-t-1}\mu_p(E)d\epsilon\right]$$
$$\leq a_1 t\int_0^1 \epsilon^{\alpha_p - t-1}\, d\epsilon \;+\; \mu_p(E) \;<\infty,$$

for all $0 \leq t < \alpha_p$. Hence $I_t(\mu_p) = \int_E \phi_t(x)\, d\mu_p(x) < \infty$ for all $t < \alpha_p$, which implies $D_2(E,\mu_p) \geq \alpha_p$.

(ii): In order to obtain the upper bound, we analogously calculate $\phi_t(x)$ as follows. Using Lemma 3.2, for all $t > \beta_p$,

$$\phi_t(x) = t\int_0^\infty \epsilon^{-t-1}\; \mu_p(B(x,\epsilon))d\epsilon$$
$$\geq t\int_0^\infty \epsilon^{-t-1}\cdot a_2 \cdot \epsilon^{\beta_p} d\epsilon = \infty.$$

Therefore $I_t(\mu_p) = \int_E \phi_t(x)\, d\mu_p(x) = \infty$ for all $t > \beta_p$, which implies $D_2(E,\mu_p) \leq \beta_p$. □

COROLLARY 3.4. *For each $\bar{\mu}_l$, if $E \subseteq K$ satisfies (4), then we have*

$$1 + \alpha_{p(l)} \leq D_2(E,\bar{\mu}_l) \leq 1 + \beta_{p(l)}.$$

Let $c_1 \in (0,1)$ be fixed and let $p_1 = c_1$ and $p_2 = 1 - c_1$. Let $p_{i_n} = p_1$ if $i_n = 1$ and $p_{i_n} = p_2$ if $i_n = 2$ for all $n = 1,2,3,\ldots$. We consider the Borel subset $\hat{K}(p_1,p_2)$ of $\hat{K}$:

$$\hat{K}(p_1,p_2) \equiv \left\{\Pi(\mathbf{i}) \in \hat{K} : \frac{\#\{j : i_j = k,\; 1 \leq j \leq n\}}{n} \to p_k (n\to\infty), k = 1,2\right\}.$$

Then the probability measure $\mu_{(p_1,p_2)}$ on $\hat{K}(p_1,p_2)$ satisfies that for any $\cap_{n=1}^\infty \hat{X}_{i_1,i_2,\cdots,i_n} \in \hat{K}(p_1,p_2)$,

$$\mu_{(p_1,p_2)}(\hat{X}_{i_1,i_2,\cdots,i_n}) = \prod_{j=1}^n p_{i_j} = p_1^r \cdot p_2^{n-r}.$$

COROLLARY 3.5. *Let $\bar{\mu}_{(p_1,p_2)}$ be the induced probability measure by $\mu_{(p_1,p_2)}$ on a subset $K(p_1,p_2)$ of K, for a given (p_1,p_2). Then, we have*

$$D_2(K(p_1,p_2),\bar{\mu}_{(p_1,p_2)}) = 1 + \frac{p_1\log p_1 + p_2 \log p_2}{p_1 \log\lambda_1 + p_2\log\lambda_2}.$$

Proof. By the definition of $\hat{K}(p_1,p_2)$, we have $|\hat{X}_{i_1,\cdots,i_n}| = \lambda_1^m \cdot \lambda_2^{n-m}$ for all $\cap_{n=1}^{\infty}\hat{X}_{i_1,i_2,\cdots,i_n} \in \hat{K}(p_1,p_2)$. Then

$$\lim_{n\to\infty} \frac{\log \mu_{(p_1,p_2)}(\hat{X}_{i_1,\cdots,i_n})}{\log |\hat{X}_{i_1,\cdots,i_n}|} = \lim_{n\to\infty} \frac{m\log p_1 + (n-m)\log p_2}{m\log\lambda_1 + (n-m)\log\lambda_2}$$
$$= \frac{p_1\log p_1 + p_2\log p_2}{p_1\log\lambda_1 + p_2\log\lambda_2} \equiv \gamma.$$

Since $\bar{\mu}_{(p_1,p_2)}(K(p_1,p_2)) = 1$, we get $D_2(K(p_1,p_2), \bar{\mu}_{(p_1,p_2)}) = 1+\gamma$. □

REMARK 3.6. For the set $K(p_1,p_2)$ with natural measures p_1 and p_2, it is known that the Kaplan-Yorke formula($\equiv \mathrm{D}_{KY}$) in the two-dimensional cases is the same as the information dimension($\equiv \mathrm{D}_1$) ([4], [6]). And we note that D_1 is the same as the correlation dimension because of self-similarity ([3], [6]). Therefore we have

$$D_2(K(p_1,p_2), \bar{\mu}_{(p_1,p_2)}) = 1+\gamma = \mathrm{D}_1 = \mathrm{D}_{KY}.$$

If each probability p_i is related with the contraction ratio λ_i $(i=1,2)$, then we have the following result.

COROLLARY 3.7. *Let s be the number satisfying $\lambda_1^s + \lambda_2^s = 1$ and let $p_i = \lambda_i^s$ $(i=1,2)$. Then $D_2(K(p_1,p_2), \bar{\mu}_{(p_1,p_2)}) = 1+s$.*

Proof. In Corollary 3.5, substituting λ_i^s for p_i $(i=1,2)$, we get this Corollary. □

4. Example

In the following example, we introduce a simple random baker's map on $\mathbb{R}^1$ with a probability sequence.

EXAMPLE 4.1. Let $M_1(x) = \frac{1}{27}x$ and $M_2(x) = \frac{1}{27}(8x+19)$ for all $x \in [0,1]$. Set $K_{i_1,\cdots,i_n} = (M_{i_1} \circ \cdots \circ M_{i_n})([0,1])$ for each $i_j \in \{1,2\}$ $(j = 1,2,...,n)$. Put $K = \cap_{n=1}^{\infty} \cup_{(i_1,\cdots,i_n)\in\{1,2\}^n} K_{i_1,\cdots,i_n}$. Fix each $l \in \mathbb{N}$, let $\{c_n^l : n = 1,2,...\}$ be a random sequence in $(0,1)$, *i.e.* c_j^l is the probability of contractive map M_{i_j} for each $i_j \in \{1,2\}$ and $j = 1,2,....$ Define a probability measure $\mu_{p(l)}$ on K such that for each $l \geq 1$ and all $n \geq 1$, $\mu_{p(l)}(K_{i_1,\cdots,i_n}) = p_{i_1}p_{i_2}\cdots p_{i_n}$ where $p_{i_k} = c_k^l(i_k = 1)$ and $p_{i_k} = 1 - c_k^l(i_k = 2)$.

(a) For the set $K(p_1,p_2)$ where $p_1 + p_2 = 1$, we have

$$D_2(K(p_1,p_2),\ \mu_{(p_1,p_2)}) = \frac{p_1\log p_1 + p_2\log p_2}{3(p_1\log\frac{1}{3} + p_2\log\frac{2}{3})}.$$

(b) In particular, if we take $p_1 = \left(\frac{1}{27}\right)^s$ and $p_2 = \left(\frac{8}{27}\right)^s$ where s is the number satisfying $\left(\frac{1}{27}\right)^s + \left(\frac{8}{27}\right)^s = 1$, then we have $D_2\left(K(\frac{1}{3}, \frac{2}{3}),\ \mu_{(\frac{1}{3}, \frac{2}{3})}\right) = s = \frac{1}{3}$.

Proof. By Corollary 3.5 and 3.7, we can obtain (a) and (b). □

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