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WEAK LAWS FOR WEIGHTED SUMS OF RANDOM VARIABLES

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Abstract. Let $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of constants. Let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be $\{a_{ni}\}$-uniformly integrable random variables. Weak laws for the weighted sums $\sum_{i=u_n}^{v_n} a_{ni}X_{ni}$ are obtained.

1. Introduction

Let $\{u_n, n \ge 1\}$ and $\{v_n, n \ge 1\}$ be two sequences of integers (not necessarily positive or finite), and let $\{k_n, n \ge 1\}$ be a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. Consider an array of constants $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ and an array of random variables $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. When $u_n = 1, v_n = k_n, n \ge 1$, weak laws of large numbers for the array $\{X_{ni}\}$ have been established by several authors (see, Gut [2], Hong and Lee [3], Hong and Oh [4], and Sung [8]). An array $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ is said to be Cesàro uniformly integrable if

$$\lim_{a\to\infty} \sup_{n\ge 1} \frac{1}{k_n} \sum_{i=1}^{k_n} E|X_{ni}|I(|X_{ni}| > a) = 0. \tag{1}$$

This condition was introduced by Chandra [1].

Ordonez Cabrera [5] extended the notion of Cesàro uniform integrability to $\{a_{ni}\}$-uniform integrability as follows.

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An array of random variables $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is said to be $\{a_{ni}\}$-uniformly integrable if

$$\lim_{a\to\infty} \sup_{n\ge 1} \sum_{i=u_n}^{v_n} |a_{ni}| E|X_{ni}| I(|X_{ni}| > a) = 0. \tag{2}$$

Note that $\{a_{ni}\}$-uniform integrability reduces to Cesàro uniform integrability when $a_{ni} = k_n^{-1}$ for $1 \le i \le k_n$ and 0 elsewhere. Ordonez Cabrera [5] obtained weak laws for weighted sums of $\{a_{ni}\}$-uniformly integrable random variables, where the weights satisfy $\lim_{n\to\infty} \sup_{u_n \le i \le v_n} |a_{ni}| = 0$ and $\sup_{n\ge 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r < \infty$ for some $0 < r \le 1$.

In this paper, we prove the results of Ordonez Cabrera [5] under a weaker condition on the weights.

2. Main result

Throughout this section, let $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of constants, and let $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ be an array of random variables.

To prove the main result, we will need the following lemmas.

LEMMA 1. (Sung [9]). *Suppose that $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of random variables satisfying the following conditions.*

$$\sup_{n\ge 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r < \infty \tag{3}$$

and

$$\lim_{a\to\infty} \sup_{n\ge 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r I(|X_{ni}|^r > a) = 0, \tag{4}$$

where $r > 0$ and $\{k_n, n \ge 1\}$ is a sequence of positive numbers such that $k_n \to \infty$ as $n \to \infty$. If $\beta > r$, then

$$\sum_{i=u_n}^{v_n} E|X_{ni}|^\beta I(|X_{ni}|^r \le k_n) = o(k_n^{\beta/r}).$$

LEMMA 2. (Sung [9]). *Suppose that* $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ *is an array of random variables satisfying* (3) *and* (4) *for some* $0 < r < 1$ *and* $\{k_n, n \ge 1\}$. *Then*

$$\frac{\sum_{i=u_n}^{v_n} X_{ni}}{k_n^{1/r}} \to 0$$

in L^r *and, hence, in probability as* $n \to \infty$.

LEMMA 3. *Let* $r > 0$. *Let* $\{|X_{ni}|^r, u_n \le i \le v_n, n \ge 1\}$ *be* $\{|a_{ni}|^r\}$-*uniformly integrable random variables, where* $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ *is an array of constants satisfying* $\lim_{n\to\infty} \sup_{u_n \le i \le v_n} |a_{ni}| = 0$. *Assume that*

$$\sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r < \infty. \tag{5}$$

If $\beta > r$, *then*

$$\sum_{i=u_n}^{v_n} |a_{ni}|^\beta E|X_{ni}|^\beta I(|X_{ni}|^r \le k_n) = o(1),$$

where $k_n = 1/sup_{u_n \le i \le v_n} |a_{ni}|^r$.

Proof. Take $k_n^{1/r} a_{ni} X_{ni}$ instead of X_{ni} in Lemma 1. Then we have by (5) that

$$\sup_{n\ge1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n^{1/r} a_{ni} X_{ni}|^r < \infty.$$

Since $k_n |a_{ni}|^r \le 1$ for $u_n \le i \le v_n$, it follows that

$$\begin{aligned}
&\lim_{a\to\infty} \sup_{n\ge1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|k_n^{1/r} a_{ni} X_{ni}|^r I(|k_n^{1/r} a_{ni} X_{ni}|^r > a) \\
&\le \lim_{a\to\infty} \sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > a) = 0.
\end{aligned}$$

Thus we obtain by Lemma 1 that

$$\frac{1}{k_n^{\beta/r}} \sum_{i=u_n}^{v_n} E|k_n^{1/r} a_{ni} X_{ni}|^\beta I(|k_n^{1/r} a_{ni} X_{ni}|^r \le k_n) = o(1).$$

So the result follows since $k_n \le 1/|a_{ni}|^r$ for $u_n \le i \le v_n$. □

Now, we state and prove our main result which generalizes some results in the literature in this area. See the corollaries and example following Theorem 1.

THEOREM 1. *Let* $0 < r \leq 1$. *Let* $\{a_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *and* $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ *be as in Lemma 3. When* $r = 1$, *we assume further that* $\{X_{ni}\}$ *is an array of (rowwise) pairwise independent random variables with* $EX_{ni} = 0$, *i.e., for each fixed* n, $X_{n,u_n}, \cdots, X_{n,v_n}$ *are pairwise independent. Then*

$$\sum_{i=u_n}^{v_n} a_{ni}X_{ni} \to 0$$

in L^r *and, hence, in probability as* $n \to \infty$.

Proof. Take $k_n = 1/\sup_{u_n \leq i \leq v_n} |a_{ni}|^r$ and $k_n^{1/r} a_{ni}X_{ni}$ instead of X_{ni} in Lemma 2. When $0 < r < 1$, the result follows from Lemma 2.

We now prove the result when $r = 1$. Define $X'_{ni} = X_{ni}I(|X_{ni}| \leq k_n)$ and $X''_{ni} = X_{ni} - X'_{ni}$ for $u_n \leq i \leq v_n$ and $n \geq 1$. Since $X'_{n,u_n}, \cdots, X'_{n,v_n}$ are pairwise independent random variables, we have by Lemma 3 with $r = 1$ and $\beta = 2$ that

$$E|\sum_{i=u_n}^{v_n} a_{ni}(X'_{ni} - EX'_{ni})|^2 \leq \sum_{i=u_n}^{v_n} a_{ni}^2 E|X'_{ni}|^2 \to 0$$

as $n \to \infty$. Also we obtain by the definition of $\{a_{ni}\}$-uniform integrability that

$$E|\sum_{i=u_n}^{v_n} a_{ni}(X''_{ni} - EX''_{ni})| \leq 2\sum_{i=u_n}^{v_n} |a_{ni}|E|X''_{ni}| \to 0$$

as $n \to \infty$, since $k_n \to \infty$ as $n \to \infty$. Thus we have

$$\begin{aligned} &E|\sum_{i=u_n}^{v_n} a_{ni}X_{ni}| \\ &\leq E|\sum_{i=u_n}^{v_n} a_{ni}(X'_{ni} - EX'_{ni})| + E|\sum_{i=u_n}^{v_n} a_{ni}(X''_{ni} - EX''_{ni})| \\ &\leq (E|\sum_{i=u_n}^{v_n} a_{ni}(X'_{ni} - EX'_{ni})|^2)^{1/2} + E|\sum_{i=u_n}^{v_n} a_{ni}(X''_{ni} - EX''_{ni})| \to 0 \end{aligned}$$

as $n \to \infty$. □

COROLLARY 1. *Let $0 < r < 1$. Let $\{|X_{ni}|^r, u_n \le i \le v_n, n \ge 1\}$ be $\{|a_{ni}|^r\}$-uniformly integrable random variables, where $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ is an array of constants satisfying* $\lim_{n\to\infty} \sup_{u_n \le i \le v_n} |a_{ni}| = 0$ *and for some constant $C > 0$*

$$\sum_{i=u_n}^{v_n} |a_{ni}|^r < C \text{ for all } n. \tag{6}$$

Then

$$\sum_{i=u_n}^{v_n} a_{ni} X_{ni} \to 0$$

in L^r and, hence, in probability as $n \to \infty$.

Proof. From Theorem 1, it is enough to show that (5) holds. Since $\{|X_{ni}|^r\}$ is $\{|a_{ni}|^r\}$-uniformly integrable, there exists $a > 0$ such that

$$\sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > a) \le 1.$$

Then

$$\begin{aligned} E|X_{ni}|^r &= E|X_{ni}|^r I(|X_{ni}|^r \le a) + E|X_{ni}|^r I(|X_{ni}|^r > a) \\ &\le a + E|X_{ni}|^r I(|X_{ni}|^r > a), \end{aligned}$$

which implies by (6) that

$$\begin{aligned} &\sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r \\ &\le a \cdot \sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r + \sup_{n\ge1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r I(|X_{ni}|^r > a) \\ &\le a \cdot C + 1. \end{aligned}$$

Hence (5) is satisfied. □

The above corollary has been proved by Ordonez Cabrera [5]. Rohatgi [7] established a weaker result (convergence in probability) under the stronger condition that $\{X_n, n \ge 1\}$ is a sequence of independent random variables which is uniformly bounded by a random variable X with $E|X|^r < \infty$. Wang and Rao [10] extended Rohatgi's result to L^r-convergence under the uniform integrability (without independent condition).

Corollary 2. *Let* $\{X_{ni}, u_n \le i \le v_n, n \ge 1\}$ *be* $\{a_{ni}\}$*-uniformly integrable (rowwise) pairwise independent random variables with* $EX_{ni} = 0$ *for* $u_n \le i \le v_n$ *and* $n \ge 1$*, where* $\{a_{ni}, u_n \le i \le v_n, n \ge 1\}$ *is an array of constants satisfying* $\lim_{n\to\infty} \sup_{u_n \le i \le v_n} |a_{ni}| = 0$ *and for some constant* $C > 0$

$$\sum_{i=u_n}^{v_n} |a_{ni}| < C \text{ for all } n. \tag{7}$$

Then

$$\sum_{i=u_n}^{v_n} a_{ni} X_{ni} \to 0$$

in L^1 *and, hence, in probability as* $n \to \infty$.

Proof. By Theorem 1, it is enough to show that (5) holds when $r = 1$. The proof of the rest is similar to that of Corollary 1 and is omitted. □

The above corollary has been proved by Ordonez Cabrera [5]. Pruitt [6] established a weaker result (convergence in probability) under the stronger condition that $\{X_n, n \ge 1\}$ is a sequence of independent and identically distributed random variables with $EX_n = 0$ for $n \ge 1$. Rohatgi [7] extended Pruitt's result to a sequence of independent random variables which is uniformly bounded by a random variable X with $E|X| < \infty$. Wang and Rao [10] extended Rohatgi's result to L^1-convergence for uniformly integrable pairwise independent random variables.

The following example shows that the conditions of Theorem 1 are weaker than the conditions of Corollary 2.

Example 1. Let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables such that $X_n = \pm 1/\log n$ with probability 1/2 if n is not a perfect cube, and $X_n = \pm n^{1/3}$ with probability 1/2 if n is a perfect cube (i.e., $n = j^3$ for some positive integer j). Define an array of constants $\{a_{ni}, i \ge 1, n \ge 1\}$ as follows.

$$a_{ni} = \begin{cases} \log n/n & \text{if } 1 \le i \le n, \\ 0 & \text{if } i > n. \end{cases}$$

Since $\sum_{i=1}^{\infty} |a_{ni}| = \log n$, we can not apply this example to Corollary 2. Observe that

$$\sum_{i \ne j^3, i \le n} E|X_i| = \sum_{i \ne j^3, i \le n} 1/\log i = O(n/\log n)$$

and

$$\sum_{i=j^3, i\leq n} E|X_i| = \sum_{i=j^3, i\leq n} i^{1/3} = \sum_{j^3\leq n} j = \frac{j_0(j_0+1)}{2} \leq \frac{n^{1/3}(n^{1/3}+1)}{2},$$

where $j_0 = \max\{j : j^3 \leq n\}$. Thus we have

$$\sum_{i=1}^{\infty} |a_{ni}| E|X_i| = \frac{\log n}{n} \sum_{i=1}^{n} E|X_i|$$
$$\leq \frac{\log n}{n}(O(\frac{n}{\log n}) + \frac{n^{1/3}(n^{1/3}+1)}{2}) = O(1).$$

If $a > 1$, then

$$\sum_{i=1}^{\infty} |a_{ni}| E|X_i| I(|X_i| > a) = \frac{\log n}{n} \sum_{i=j^3, a^3<i\leq n} E|X_i|.$$

Hence, for $a > 1$, we have

$$\sup_{n\geq 1} \sum_{i=1}^{\infty} |a_{ni}| E|X_i| I(|X_i| > a) = \sup_{n>a^3} \frac{\log n}{n} \sum_{i=j^3, a^3<i\leq n} E|X_i|$$
$$\leq \sup_{n>a^3} \frac{n^{1/3}(n^{1/3}+1)\log n}{2n} \to 0$$

as $a \to \infty$. Therefore the conditions of Theorem 1 with $r = 1$ are satisfied. By Theorem 1, we obtain

$$\sum_{i=1}^{\infty} a_{ni} X_i \to 0$$

in L^1 and, hence, in probability as $n \to \infty$.

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