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ON MCSHANE-STIELTJES INTEGRALS OF INTERVAL-VALUED FUNCTIONS AND FUZZY-NUMBER-VALUED FUNCTIONS

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ABSTRACT. In this paper we introduce the concept of the McShane-Stieltjes integrals of interval-valued functions and fuzzy-number-valued functions and investigate some of their properties.

1. Introduction

The Henstock integral of real-valued functions is the special case of the McShane integral of real-valued functions [2]. Congxin Wu and Zengtai Gong [8] introduced the concept of the Henstock integrals of interval-valued functions and fuzzy-number-valued functions and obtained some of their properties. J. H. Yoon [9] introduced the concept of the McShane-Stieltjes integral of real-valued functions which is a generalization of the McShane integral and obtained its properties.

In this paper we introduce the concept of the McShane-Stieltjes integrals of interval-valued functions and fuzzy-number-valued functions which are generalizations of the Henstock integrals of interval-valued functions and fuzzy-number-valued functions [8] and investigate some of their properties.

2. Preliminaries

DEFINITION 2.1 [3]. A McShane partition of $[a, b]$ is a finite collection $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ such that $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a, b]$ covering $[a, b]$ and $t_i \in$

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$[a,b]$ for each $i \leq n$. A gauge on $[a,b]$ is a function $\delta : [a,b] \to (0,\infty)$. A McShane partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is subordinate to a gauge δ if $[c_i,d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$ for every $i \leq n$. If $f : [a,b] \to \mathbf{R}$ and if $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a,b]$, we will denote $f(\mathcal{P})$ for $\sum_{i=1}^{n} f(t_i)(d_i - c_i)$. A function $f : [a,b] \to \mathbf{R}$ is McShane integrable on $[a,b]$, with McShane integral $L \in \mathbf{R}$, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that $|f(\mathcal{P}) - L| < \varepsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a,b]$ subordinate to δ. We write $(M)\int_a^b f(x)dx = L$ and $f \in M[a,b]$.

Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function. If $f : [a,b] \to \mathbf{R}$ and if $\mathcal{P}$ $= \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a,b]$, we will denote $f_\alpha(\mathcal{P})$ for $\sum\limits_{i=1}^{n} f(t_i)\,[\alpha(d_i) - \alpha(c_i)]$.

Definition 2.2 [9]. Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function. A function $f : [a,b] \to \mathbf{R}$ is McShane-Stieltjes integrable with respect to α on $[a,b]$, with McShane-Stieltjes integral $L \in \mathbf{R}$, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that $|f_\alpha(\mathcal{P}) - L| < \varepsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a,b]$ subordinate to δ. We write $(MS)\int_a^b f d\alpha = L$ and $f \in MS_\alpha[a,b]$. The function f is McShane-Stieltjes integrable with respect to α on a set $E \subset [a,b]$ if $f\chi_E$ is McShane-Stieltjes integrable with respect to α on $[a,b]$.

Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function. A function $f : [a,b] \to \mathbf{R}$ is McShane-Stieltjes integrable with respect to α on $[a,b]$ if and only if for each $\epsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that $|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are McShane partitions of $[a,b]$ subordinate to δ.

Theorem 2.3 [7]. *Let $\alpha : [a,b] \to \mathbf{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to \mathbf{R}$ be a bounded function. Then f is McShane-Stieltjes integrable with respect to α on $[a,b]$ if and only if $\alpha' f$ is McShane integrable on $[a,b]$.*

3. McShane-Stieltjes integral of interval-valued functions

In this section, we introduce the concept of the McShane-Stieltjes integral of interval-valued functions and investigate some of their properties.

DEFINITION 3.1 [8]. Let $I_{\mathbf{R}} = \{I = [I^-, I^+] : I$ is the closed bounded interval on the real line $\mathbf{R}\}$. For $A, B, C \in I_{\mathbf{R}}$, we define $A \leq B$ if $A^- \leq B^-$ and $A^+ \leq B^+$, $A+B = C$ if $C^- = A^- + B^-$ and $C^+ = A^+ + B^+$, and $A \cdot B = \{a \cdot b : a \in A, b \in B\}$, where $(A \cdot B)^- = \min\{A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+\}$ and $(A \cdot B)^+ = \max\{A^- \cdot B^-, A^- \cdot B^+, A^+ \cdot B^-, A^+ \cdot B^+\}$. Define $d(A, B) = \max\{|A^- - B^-|, |A^+ - B^+|\}$ as the distance between A and B.

DEFINITION 3.2. An interval-valued function $F : [a, b] \to I_{\mathbf{R}}$ is McShane integrable on $[a, b]$, with McShane integral $I_0 \in I_{\mathbf{R}}$, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that $d(\sum_{i=1}^n F(t_i)(d_i - c_i), I_0) < \epsilon$ whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a, b]$ subordinate to δ. We write $(IM) \int_a^b F(x)dx = I_0$ and $F \in IM[a, b]$. The interval-valued function F is McShane integrable on a set $E \subset [a, b]$ if $F\chi_E$ is McShane integrable on $[a, b]$.

DEFINITION 3.3. Let $\alpha : [a, b] \to \mathbf{R}$ be an increasing function. An interval-valued function $F : [a, b] \to I_{\mathbf{R}}$ is McShane-Stieltjes integrable with respect to α on $[a, b]$, with McShane-Stieltjes integral $I_0 \in I_{\mathbf{R}}$, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that $d\left(\sum_{i=1}^n F(t_i)[\alpha(d_i) - \alpha(c_i)], I_0\right) < \epsilon$ whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a, b]$ subordinate to δ. We write $(IMS) \int_a^b F d\alpha = I_0$ and $F \in IMS_\alpha[a, b]$. The interval-valued function F is McShane-Stieltjes integrable with respect to α on a set $E \subset [a, b]$ if $F\chi_E$ is McShane-Stieltjes integrable with respect to α on $[a, b]$.

THEOREM 3.4. *Let $\alpha : [a, b] \to \mathbf{R}$ be an increasing function. Then an interval-valued function $F \in IMS_\alpha[a, b]$ if and only if $F^-, F^+ \in MS_\alpha[a, b]$ and*

$$(IMS) \int_a^b F d\alpha = \left[(MS) \int_a^b F^- d\alpha, (MS) \int_a^b F^+ d\alpha\right].$$

Proof. Let $F \in IMS_\alpha[a, b]$. Then there exists an interval $I_0 = [I_0^-, I_0^+]$ with the property that for each $\varepsilon > 0$ there exists a gauge $\delta : [a, b] \to (0, \infty)$ such that $d\left(\sum_{i=1}^n F(t_i)[\alpha(d_i) - \alpha(c_i)], I_0\right) < \epsilon$ whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a, b]$ subordinate to δ. Since $\alpha(d_i) - \alpha(c_i) \geq 0$ for $1 \leq i \leq n$, we have

$$d\left(\sum_{i=1}^n F(t_i)[\alpha(d_i) - \alpha(c_i)], I_0\right)$$

$$= \max\left\{\left|\left(\sum_{i=1}^{n} F(t_i)[\alpha(d_i)-\alpha(c_i)]\right)^{-} - I_0^{-}\right|,\right.$$

$$\left.\left|\left(\sum_{i=1}^{n} F(t_i)[\alpha(d_i)-\alpha(c_i)]\right)^{+} - I_0^{+}\right|\right\}$$

$$= \max\left\{\left|\sum_{i=1}^{n} F^{-}(t_i)[\alpha(d_i)-\alpha(c_i)] - I_0^{-}\right|,\right.$$

$$\left.\left|\sum_{i=1}^{n} F^{+}(t_i)[\alpha(d_i)-\alpha(c_i)] - I_0^{+}\right|\right\}.$$

Hence $\left|\sum_{i=1}^{n} F^{-}(t_i)[\alpha(d_i)-\alpha(c_i)] - I_0^{-}\right| < \epsilon$ and $\left|\sum_{i=1}^{n} F^{+}(t_i)[\alpha(d_i)-\alpha(c_i)] - I_0^{+}\right| < \epsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$ subordinate to δ. Thus $F^{-}, F^{+} \in MS_\alpha[a,b]$ and $(IMS)\int_a^b F d\alpha = I_0 = [I_0^{-}, I_0^{+}] = [(MS)\int_a^b F^{-} d\alpha, (MS)\int_a^b F^{+} d\alpha]$.

Conversely, let $F^{-}, F^{+} \in MS_\alpha[a,b]$. Then there exists $M_1 \in \mathbf{R}$ with the property that given $\varepsilon > 0$ there exists a gauge $\delta_1 : [a,b] \to (0,\infty)$ such that $\left|\sum_{i=1}^{n} F^{-}(t_i)[\alpha(d_i)-\alpha(c_i)] - M_1\right| < \epsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$ subordinate to δ_1. Similarly, there exists $M_2 \in \mathbf{R}$ with the property that given $\varepsilon > 0$ there exists a gauge $\delta_2 : [a,b] \to (0,\infty)$ such that $\left|\sum_{i=1}^{n} F^{+}(t_i)[\alpha(d_i)-\alpha(c_i)] - M_2\right| < \epsilon$ whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$ subordinate to δ_2. Since $F^{-}(x) \le F^{+}(x)$ for all $x \in [a,b]$, $M_1 \le M_2$. Let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ for $x \in [a,b]$ and $I_0 = [M_1, M_2]$. Then we have

$$d\left(\sum_{i=1}^{n} F(t_i)[\alpha(d_i)-\alpha(c_i)], I_0\right)$$

$$= \max\left\{\left|\left(\sum_{i=1}^{n} F(t_i)[\alpha(d_i)-\alpha(c_i)]\right)^{-} - M_1\right|,\right.$$

$$\left.\left|\left(\sum_{i=1}^{n} F(t_i)[\alpha(d_i)-\alpha(c_i)]\right)^{+} - M_2\right|\right\}$$

$$= \max\left\{\left|\sum_{i=1}^{n} F^{-}(t_i)[\alpha(d_i)-\alpha(c_i)] - M_1\right|,\right.$$

$$\left|\sum_{i=1}^{n} F^{+}(t_i)[\alpha(d_i)-\alpha(c_i)] - M_2\right|\Bigg\}$$
$$< \epsilon$$

whenever $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ is a McShane partition of $[a, b]$ subordinate to δ. Hence $F \in IMS_\alpha[a, b]$. □

THEOREM 3.5. *Let $\alpha : [a, b] \to \mathbf{R}$ be an increasing function. If $F, G \in IMS_\alpha[a, b]$ and $p, q \in \mathbf{R}$, then $pF+qG \in IMS_\alpha[a, b]$ and $(IMS)\int_a^b (pF+qG)d\alpha = p(IMS)\int_a^b Fd\alpha + q(IMS)\int_a^b Gd\alpha$.*

Proof. If $F, G \in IMS_\alpha[a, b]$, then $F^-, F^+, G^-, G^+ \in MS_\alpha[a, b]$ by Theorem 3.4. Hence $pF^- + qG^-, pF^- + qG^+, pF^+ + qG^-, pF^+ + qG^+ \in MS_\alpha[a, b]$.

(i) If $p > 0$ and $q > 0$, then

$$\begin{aligned}
&(MS)\int_a^b (pF+qG)^- d\alpha \\
&= (MS)\int_a^b (pF^- + qG^-)d\alpha \\
&= p(MS)\int_a^b F^- d\alpha + q(MS)\int_a^b G^- d\alpha \\
&= p\left((IMS)\int_a^b Fd\alpha\right)^- + q\left((IMS)\int_a^b Gd\alpha\right)^- \\
&= \left(p(IMS)\int_a^b Fd\alpha + q(IMS)\int_a^b Gd\alpha\right)^- .
\end{aligned}$$

(ii) If $p < 0$ and $q < 0$, then

$$\begin{aligned}
&(MS)\int_a^b (pF+qG)^- d\alpha \\
&= (MS)\int_a^b (pF^+ + qG^+)d\alpha \\
&= p(MS)\int_a^b F^+ d\alpha + q(MS)\int_a^b G^+ d\alpha \\
&= p\left((IMS)\int_a^b Fd\alpha\right)^+ + q\left((IMS)\int_a^b Gd\alpha\right)^+
\end{aligned}$$

$$= \left(p(IMS) \int_a^b F d\alpha + q(IMS) \int_a^b G d\alpha \right)^- .$$

(iii) If $p > 0$ and $q < 0$ (or $p < 0$ and $q > 0$), then

$$\begin{aligned}
&(MS) \int_a^b (pF + qG)^- d\alpha \\
&= (MS) \int_a^b (pF^- + qG^+) d\alpha \\
&= p(MS) \int_a^b F^- d\alpha + q(MS) \int_a^b G^+ d\alpha \\
&= p \left((IMS) \int_a^b F d\alpha \right)^- + q \left((IMS) \int_a^b G d\alpha \right)^+ \\
&= \left(p(IMS) \int_a^b F d\alpha + q(IMS) \int_a^b G d\alpha \right)^- .
\end{aligned}$$

Similarly, for four cases above we have

$$(MS) \int_a^b (pF + qG)^+ d\alpha = \left(p(IMS) \int_a^b F d\alpha + q(IMS) \int_a^b G d\alpha \right)^+ .$$

Hence by Theorem 3.4, $pF + qG \in IMS_\alpha[a,b]$ and $(IMS) \int_a^b (pF + qG) d\alpha = p(IMS) \int_a^b F d\alpha + q(IMS) \int_a^b G d\alpha$. □

THEOREM 3.6. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function and let $c \in (a,b)$. If $F \in IMS_\alpha[a,c]$ and $F \in IMS_\alpha[c,b]$, then $F \in IMS_\alpha[a,b]$ and $(IMS) \int_a^c F d\alpha + (IMS) \int_c^b F d\alpha = (IMS) \int_a^b F d\alpha$.*

Proof. If $F \in IMS_\alpha[a,c]$ and $F \in IMS_\alpha[c,b]$, then by Theorem 3.4 $F^-, F^+ \in MS_\alpha[a,c]$ and $F^-, F^+ \in MS_\alpha[c,b]$. Hence $F^-, F^+ \in MS_\alpha[a,b]$ and

$$\begin{aligned}
(MS) \int_a^b F^- d\alpha &= (MS) \int_a^c F^- d\alpha + (MS) \int_c^b F^- d\alpha \\
&= \left((IMS) \int_a^c F d\alpha + (IMS) \int_c^b F d\alpha \right)^- .
\end{aligned}$$

Similarly, $(MS)\int_a^b F^+d\alpha = \left((IMS)\int_a^c Fd\alpha + (IMS)\int_c^b Fd\alpha\right)^+$. Hence by Theorem 3.4 $F \in IMS_\alpha[a,b]$ and

$$(IMS)\int_a^c Fd\alpha + (IMS)\int_c^b Fd\alpha$$
$$= (IMS)\int_a^b Fd\alpha.$$

□

THEOREM 3.7. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function such that $\alpha \in C^1([a,b])$. If $F = G$ nearly everywhere on $[a,b]$ and $F, G \in IMS_\alpha[a,b]$, then $(IMS)\int_a^b Fd\alpha = (IMS)\int_a^b Gd\alpha$.*

Proof. Let $F = G$ nearly everywhere on $[a,b]$ and $F, G \in IMS_\alpha[a,b]$. Then $F^-, F^+, G^-, G^+ \in MS_\alpha[a,b]$ and $F^- = G^-, F^+ = G^+$ nearly everywhere on $[a,b]$. By Theorem 2.2 [9], $(MS)\int_a^b F^-d\alpha = (MS)\int_a^b G^-d\alpha$ and $(MS)\int_a^b F^+d\alpha = (MS)\int_a^b G^+d\alpha$. Hence

$$(IMS)\int_a^b Fd\alpha = (IMS)\int_a^b Gd\alpha$$

by Theorem 3.4. □

THEOREM 3.8. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function such that $\alpha \in C^1([a,b])$. If $F \leq G$ nearly everywhere on $[a,b]$ and $F, G \in IMS_\alpha[a,b]$, then $(IMS)\int_a^b Fd\alpha \leq (IMS)\int_a^b Gd\alpha$.*

Proof. Let $F \leq G$ nearly everywhere on $[a,b]$ and $F, G \in IMS_\alpha[a,b]$. Then $F^-, F^+, G^-, G^+ \in MS_\alpha[a,b]$ and $F^- \leq G^-, F^+ \leq G^+$ nearly everywhere on $[a,b]$. By Theorem 2.7 [9], $(MS)\int_a^b F^-d\alpha \leq (MS)\int_a^b G^-d\alpha$ and $(MS)\int_a^b F^+d\alpha \leq (MS)\int_a^b G^+d\alpha$. Hence

$$(IMS)\int_a^b Fd\alpha \leq (IMS)\int_a^b Gd\alpha$$

by Theorem 3.4. □

DEFINITION 3.9. An interval-valued function $F : [a,b] \to I_{\mathbf{R}}$ is bounded if there exist $I_0, I_1 \in I_{\mathbf{R}}$ such that $I_0 \leq F(x) \leq I_1$ for all $x \in [a,b]$.

Note that an interval-valued function $F : [a,b] \to I_{\mathbf{R}}$ is bounded if and only if $F^-,\ F^+ : [a,b] \to \mathbf{R}$ are bounded.

THEOREM 3.10. *Let $\alpha : [a,b] \to \mathbf{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $F : [a,b] \to I_{\mathbf{R}}$ be a bounded interval-valued function. Then $F \in IMS_\alpha[a,b]$ if and only if $\alpha' F \in IM[a,b]$.*

Proof. From Theorem 3.4 we have $F \in IMS_\alpha[a,b]$ if and only if $F^-, F^+ \in MS_\alpha[a,b]$. Since $F : [a,b] \to I_{\mathbf{R}}$ is bounded if and only if $F^-,\ F^+ : [a,b] \to \mathbf{R}$ are bounded, from Theorem 2.3 we have $F^-, F^+ \in MS_\alpha[a,b]$ if and only if $\alpha' F^-, \alpha' F^+ \in M[a,b]$. Since α is strictly increasing on $[a,b]$, $\alpha' > 0$ on $[a,b]$. Hence $\alpha' F^- = (\alpha' F)^-$ and $\alpha' F^+ = (\alpha' F)^+$ on $[a,b]$. From Theorem 2.1 [8] we have $\alpha' F \in IM[a,b]$ if and only if $(\alpha' F)^-, (\alpha' F)^+ \in M[a,b]$. Thus $F \in IMS_\alpha[a,b]$ if and only if $\alpha' F \in IM[a,b]$. □

4. McShane-Stieltjes integral of fuzzy-number-valued functions

In this section, we introduce the concept of the McShane-Stieltjes integral of fuzzy-number-valued functions and investigate some of their properties.

DEFINITION 4.1 [6]. Let $\widetilde{A} \in F(\mathbf{R})$ be a fuzzy subset on $\mathbf{R}$. If $A_\lambda = [A_\lambda^-, A_\lambda^+]$ for any $\lambda \in [0,1]$ and $A_1 \neq \phi$, where $A_\lambda = \{x \in \mathbf{R} : \widetilde{A}(x) \geq \lambda\}$, then $\widetilde{A}$ is called a fuzzy number. If $\widetilde{A}$ is convex, normal, upper semicontinuous and has the compact support, then $\widetilde{A}$ is called a compact fuzzy number. $\widetilde{R}$ denotes the set of all fuzzy numbers and $\widetilde{R}^C$ denotes the set of all compact fuzzy numbers.

DEFINITION 4.2 [6]. For $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \widetilde{R}$, we define $\widetilde{A} \leq \widetilde{B}$ if $A_\lambda \leq B_\lambda$ for any $\lambda \in (0,1]$, $\widetilde{A} + \widetilde{B} = \widetilde{C}$ if $A_\lambda + B_\lambda = C_\lambda$ for any $\lambda \in (0,1]$, $\widetilde{A} \cdot \widetilde{B} = \widetilde{C}$ if $A_\lambda \cdot B_\lambda = C_\lambda$ for any $\lambda \in (0,1]$. For $\widetilde{A}, \widetilde{B} \in \widetilde{R}^C$, $D(\widetilde{A}, \widetilde{B}) = \sup_{\lambda \in [0,1]} d(A_\lambda, B_\lambda)$ is called the distance of $\widetilde{A}$ and $\widetilde{B}$.

LEMMA 4.3 [5]. *If a function $H : [0,1] \to I_{\mathbf{R}}, \lambda \to H(\lambda) = [m_\lambda, n_\lambda]$, satisfies $[m_{\lambda_1}, n_{\lambda_1}] \supset [m_{\lambda_2}, n_{\lambda_2}]$ when $\lambda_1 < \lambda_2$, then $\widetilde{A} = \cup_{\lambda \in (0,1]} \lambda H(\lambda) \in \widetilde{\mathbf{R}}$ and $A_\lambda = \cap_{n=1}^{\infty} H(\lambda_n)$, where $\lambda_n = [1 - 1/(n+1)]\lambda$.*

DEFINITION 4.4 [8]. Let $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$. If the interval-valued function $F_\lambda(x) = [F_\lambda^-(x), F_\lambda^+(x)]$ is McShane integrable on $[a,b]$ for any $\lambda \in (0,1]$, then $\widetilde{F}$ is called McShane integrable on $[a,b]$ and the integral value is defined by

$$(FM)\int_a^b \widetilde{F}(x)dx = \cup_{\lambda\in(0,1]}\ \lambda(IM)\int_a^b F_\lambda(x)dx$$
$$= \cup_{\lambda\in(0,1]}\ \lambda\left[(M)\int_a^b F_\lambda^-(x)dx, (M)\int_a^b F_\lambda^+(x)dx\right].$$

We write $\widetilde{F} \in FM[a,b]$.

DEFINITION 4.5. Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function and let $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$. If the interval-valued function $F_\lambda(x) = [F_\lambda^-(x), F_\lambda^+(x)]$ is McShane-Stieltjes integrable with respect to α on $[a,b]$ for any $\lambda \in (0,1]$, then $\widetilde{F}$ is called McShane-Stieltjes integrable with respect to α on $[a,b]$ and the integral value is defined by

$$(FMS)\int_a^b \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b F_\lambda d\alpha$$
$$= \cup_{\lambda\in(0,1]}\ \lambda\left[(MS)\int_a^b F_\lambda^- d\alpha, (MS)\int_a^b F_\lambda^+ d\alpha\right].$$

We write $\widetilde{F} \in FMS_\alpha[a,b]$.

THEOREM 4.6. *Let* $\alpha : [a,b] \to \mathbf{R}$ *be an increasing function such that* $\alpha \in C^1([a,b])$ *and let* $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$. *If* $\widetilde{F} \in FMS_\alpha[a,b]$, *then* $(FMS)\int_a^b \widetilde{F}d\alpha \in \widetilde{\mathbf{R}}$ *and* $[(FMS)\int_a^b \widetilde{F}d\alpha]_\lambda = \cap_{n=1}^{\infty}(IMS)\int_a^b F_{\lambda_n}d\alpha$, *where* $\lambda_n = [1 - 1/(n+1)]\lambda$.

Proof. Let $H : (0,1] \to I_{\mathbf{R}}, H(\lambda) = [(MS)\int_a^b F_\lambda^- d\alpha, (MS)\int_a^b F_\lambda^+ d\alpha]$. Since $F_\lambda^-(x)$ and $F_\lambda^+(x)$ are increasing and decreasing on λ respectively, $F_{\lambda_1}^-(x) \le F_{\lambda_2}^-(x)$ and $F_{\lambda_1}^+(x) \ge F_{\lambda_2}^+(x)$ on $[a,b]$ when $0 < \lambda_1 \le \lambda_2 \le 1$. From Theorem 3.8 we have

$$\left[(MS)\int_a^b F_{\lambda_1}^- d\alpha, (MS)\int_a^b F_{\lambda_1}^+ d\alpha\right]$$
$$\supset \left[(MS)\int_a^b F_{\lambda_2}^- d\alpha, (MS)\int_a^b F_{\lambda_2}^+ d\alpha\right].$$

From Theorem 3.4 and Lemma 4.3 we have

$$(FMS)\int_a^b \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]}\ \lambda\left[(MS)\int_a^b F_\lambda^- d\alpha, (MS)\int_a^b F_\lambda^+ d\alpha\right] \in \widetilde{\mathbf{R}}$$

and for any $\lambda \in (0,1]$, $[(FMS)\int_a^b \widetilde{F}d\alpha]_\lambda = \cap_{n=1}^{\infty}(IMS)\int_a^b F_{\lambda_n} d\alpha$, where $\lambda_n = [1-1/(n+1)]\lambda$. □

THEOREM 4.7. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function. If $\widetilde{F}, \widetilde{G} \in FMS_\alpha[a,b]$ and $p, q \in \mathbf{R}$, then $p\widetilde{F} + q\widetilde{G} \in FMS_\alpha[a,b]$ and $(FMS)\int_a^b (p\widetilde{F}+q\widetilde{G})d\alpha = p(FMS)\int_a^b \widetilde{F}d\alpha + q(FMS)\int_a^b \widetilde{G}d\alpha$.*

Proof. If $\widetilde{F}, \widetilde{G} \in FMS_\alpha[a,b]$, then the interval-valued functions $F_\lambda(x) = [F_\lambda^-(x), F_\lambda^+(x)]$ and $G_\lambda(x) = [G_\lambda^-(x), G_\lambda^+(x)]$ are McShane-Stieltjes integrable with respect to α on $[a,b]$ for any $\lambda \in (0,1]$ and $(FMS)\int_a^b \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b F_\lambda d\alpha$ and $(FMS)\int_a^b \widetilde{G}d\alpha = \cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b G_\lambda d\alpha$. From Theorem 3.5 we have $pF_\lambda + qG_\lambda \in IMS_\alpha[a,b]$ and $(IMS)\int_a^b (pF_\lambda + qG_\lambda)d\alpha = p(IMS)\int_a^b F_\lambda d\alpha + q(IMS)\int_a^b G_\lambda d\alpha$ for any $\lambda \in (0,1]$. Hence $p\widetilde{F} + q\widetilde{G} \in FMS_\alpha[a,b]$ and

$$\begin{aligned}
&(FMS)\int_a^b (p\widetilde{F}+q\widetilde{G})d\alpha \\
&= \cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b (pF_\lambda + qG_\lambda)d\alpha \\
&= \cup_{\lambda\in(0,1]}\ \lambda\left(p(IMS)\int_a^b F_\lambda d\alpha + q(IMS)\int_a^b G_\lambda d\alpha\right) \\
&= p\cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b F_\lambda d\alpha + q\cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b G_\lambda d\alpha \\
&= p(FMS)\int_a^b \widetilde{F}d\alpha + q(FMS)\int_a^b \widetilde{G}d\alpha.
\end{aligned}$$

□

THEOREM 4.8. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function and let $c \in (a,b)$. If $\widetilde{F} \in FMS_\alpha[a,c]$ and $\widetilde{F} \in FMS_\alpha[c,b]$, then $\widetilde{F} \in FMS_\alpha[a,b]$ and $(FMS)\int_a^c \widetilde{F}d\alpha + (FMS)\int_c^b \widetilde{F}d\alpha = (FMS)\int_a^b \widetilde{F}d\alpha$.*

Proof. If $\widetilde{F} \in FMS_\alpha[a,c]$ and $\widetilde{F} \in FMS_\alpha[c,b]$, then the interval-valued function $F_\lambda(x) = [F_\lambda^-(x), F_\lambda^+(x)]$ is McShane-Stieltjes integrable with respect to α on $[a,c]$ and $[c,b]$ for any $\lambda \in (0,1]$ and $(FMS)\int_a^c \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^c F_\lambda d\alpha$ and $(FMS)\int_c^b \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]} \lambda(IMS)\int_c^b F_\lambda d\alpha$. From Theorem 3.6 we have $F_\lambda \in IMS_\alpha[a,b]$ and $(IMS)\int_a^b F_\lambda d\alpha = (IMS)\int_a^c F_\lambda d\alpha + (IMS)\int_c^b F_\lambda d\alpha$ for any $\lambda \in (0,1]$.

Hence $\widetilde{F} \in FMS_\alpha[a,b]$ and

$$
\begin{aligned}
&(FMS)\int_a^b \widetilde{F}d\alpha \\
&= \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^b F_\lambda d\alpha \\
&= \cup_{\lambda\in(0,1]} \lambda\left((IMS)\int_a^c F_\lambda d\alpha + (IMS)\int_c^b F_\lambda d\alpha\right) \\
&= \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^c F_\lambda d\alpha + \cup_{\lambda\in(0,1]} \lambda(IMS)\int_c^b F_\lambda d\alpha \\
&= (FMS)\int_a^c \widetilde{F}d\alpha + (FMS)\int_c^b \widetilde{F}d\alpha.
\end{aligned}
$$

□

THEOREM 4.9. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function such that $\alpha \in C^1([a,b])$. If $\widetilde{F} = \widetilde{G}$ nearly everywhere on $[a,b]$ and $\widetilde{F}, \widetilde{G} \in FMS_\alpha[a,b]$, then $(FMS)\int_a^b \widetilde{F}d\alpha = (FMS)\int_a^b \widetilde{G}d\alpha$.*

Proof. If $\widetilde{F} = \widetilde{G}$ nearly everywhere on $[a,b]$ and $\widetilde{F}, \widetilde{G} \in FMS_\alpha[a,b]$, then $F_\lambda = G_\lambda$ nearly everywhere on $[a,b]$ for any $\lambda \in (0,1]$ and F_λ and G_λ are McShane-Stieltjes integrable with respect to α on $[a,b]$ for any $\lambda \in (0,1]$ and $(FMS)\int_a^b \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^b F_\lambda d\alpha$ and $(FMS)\int_a^b \widetilde{G}d\alpha = \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^b G_\lambda d\alpha$. From Theorem 3.7 we have $(IMS)\int_a^b F_\lambda d\alpha = (IMS)\int_a^b G_\lambda d\alpha$ for any $\lambda \in (0,1]$. Hence $(FMS)\int_a^b \widetilde{F}d\alpha = \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^b F_\lambda d\alpha = \cup_{\lambda\in(0,1]} \lambda(IMS)\int_a^b G_\lambda d\alpha = (FMS)\int_a^b \widetilde{G}d\alpha$. □

THEOREM 4.10. *Let $\alpha : [a,b] \to \mathbf{R}$ be an increasing function such that $\alpha \in C^1([a,b])$. If $\widetilde{F} \leq \widetilde{G}$ nearly everywhere on $[a,b]$ and $\widetilde{F}, \widetilde{G} \in FMS_\alpha[a,b]$, then $(FMS)\int_a^b \widetilde{F}d\alpha \leq (FMS)\int_a^b \widetilde{G}d\alpha$.*

Proof. If $\widetilde{F} \leq \widetilde{G}$ nearly everywhere on $[a,b]$ and $\widetilde{F}, \widetilde{G} \in FMS_\alpha[a,b]$, then $F_\lambda \leq G_\lambda$ nearly everywhere on $[a,b]$ for any $\lambda \in (0,1]$ and F_λ and G_λ are McShane-Stieltjes integrable with respect to α on $[a,b]$ for any $\lambda \in (0,1]$ and $(FMS)\int_a^b \widetilde{F} d\alpha = \cup_{\lambda\in(0,1]} \lambda (IMS)\int_a^b F_\lambda d\alpha$ and $(FMS)\int_a^b \widetilde{G} d\alpha = \cup_{\lambda\in(0,1]} \lambda (IMS)\int_a^b G_\lambda d\alpha$. From Theorem 3.8 we have $(IMS)\int_a^b F_\lambda d\alpha \leq (IMS)\int_a^b G_\lambda d\alpha$ for any $\lambda \in (0,1]$. Hence

$$\begin{aligned} &(FMS)\int_a^b \widetilde{F} d\alpha \\ &= \cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b F_\lambda d\alpha \\ &\leq \cup_{\lambda\in(0,1]}\ \lambda(IMS)\int_a^b G_\lambda d\alpha \\ &= (FMS)\int_a^b \widetilde{G} d\alpha. \end{aligned}$$

□

DEFINITION 4.11. A fuzzy-number-valued function $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$ is bounded if there exist $\widetilde{A}, \widetilde{B} \in \widetilde{\mathbf{R}}$ such that $\widetilde{A} \leq \widetilde{F}(x) \leq \widetilde{B}$ for all $x \in [a,b]$.

Note that a fuzzy-number-valued function $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$ is bounded if and only if the interval-valued function $F_\lambda : [a,b] \to I_{\mathbf{R}}$ is bounded for any $\lambda \in (0,1]$.

THEOREM 4.12. *Let $\alpha : [a,b] \to \mathbf{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$ be a bounded fuzzy-number-valued function. Then $\widetilde{F} \in FMS_\alpha[a,b]$ if and only if $\alpha'\widetilde{F} \in FM[a,b]$.*

Proof. From Definition 4.5 we have $\widetilde{F} \in FMS_\alpha[a,b]$ if and only if $F_\lambda \in IMS_\alpha[a,b]$ for any $\lambda \in (0,1]$. Since $\widetilde{F} : [a,b] \to \widetilde{\mathbf{R}}$ is bounded if and only if $F_\lambda : [a,b] \to I_{\mathbf{R}}$ is bounded for any $\lambda \in (0,1]$, from Theorem 3.10 we have $F_\lambda \in IMS_\alpha[a,b]$ for any $\lambda \in (0,1]$ if and only if $\alpha' F_\lambda \in IM[a,b]$ for any $\lambda \in (0,1]$. Since α is strictly increasing on $[a,b]$, $\alpha' > 0$ on $[a,b]$. Hence $\alpha' F_\lambda^- = (\alpha' F_\lambda)^-$ and $\alpha' F_\lambda^+ = (\alpha' F_\lambda)^+$ on $[a,b]$ for any $\lambda \in (0,1]$. Hence $\alpha' F_\lambda = (\alpha' F)_\lambda$ on $[a,b]$ for any $\lambda \in (0,1]$. From Definition 4.4 we have $\alpha'\widetilde{F} \in FM[a,b]$ if and only if $\alpha' F_\lambda \in IM[a,b]$ for any $\lambda \in (0,1]$. Hence $\widetilde{F} \in FMS_\alpha[a,b]$ if and only if $\alpha'\widetilde{F} \in FM[a,b]$. □

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