J. Korean Math. Soc. **41** (2004), No. 3, pp. 567–589

FINSLER SPACES WITH INFINITE SERIES (α, β)-METRIC

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ABSTRACT. In the present paper, we treat an infinite series (α, β)-metric $L = \beta^2/(\beta - \alpha)$. First, we find the conditions that a Finsler metric F^n with the metric above be a Berwald space, a Douglas space, and a projectively flat Finsler space, respectively. Next, we investigate the condition that a two-dimensional Finsler space with the metric above be a Landsbeg space. Then the differential equations of the geodesics are also discussed.

1. Introduction

A Finsler metric $L(\alpha, \beta)$ in a differentiable manifold M^n is called an (α, β)-metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = (a_{ij}(x)y^iy^j)^{1/2}$ and a one-form $\beta = b_i(x)y^i$ on M^n. The interesting and important examples of an (α, β)-metric are Randers metric $\alpha + \beta$, Kropina metric α^2/β and Matsumoto metric $\alpha^2/(\alpha - \beta)$. The notion of an (α, β)-metric was introduced by M. Matsumoto (cf. [14]) and has been studied by many authors.

A Finsler space is called a Berwald space if the Berwald connection is linear. Berwald spaces are specially interesting and important, because the connection is linear, and many examples of Berwald spaces have been known.

The notion of a Douglas space has been introduced by S. Bácsó and M. Matsumoto [4] as a generalization of a Berwald space from the viewpoint of geodesic equations. It is remarkable that a Finsler space is a Douglas space, if and only if the Douglas tensor vanishes identically. Recently, M. Matsumoto [16] has found the conditions that the Finseler spaces

Received September 18, 2003.
2000 Mathematics Subject Classification: 53B40.
Key words and phrases: Berwald space, Douglas space, differential equations of geodesics, Finsler space, (α, β)-metric, Landsberg space, projectively flat.
This was supported by Kyungsung University Research Grant in 2003.

with some (α,β)-metrics $\alpha+\beta^2/\alpha$ etc. be Douglas spaces and Berwald spaces, respectively.

A Finsler space $F^n=(M^n,L)$ is called projectively flat if for any point p of M^n there exists a local coordinate neighborhood (U,x^i) of p in which the geodesics can be represented by $(n-1)$ linear equations of x^i. Such a coordinate system is called rectilinear. The condition that a Finsler space with an (α,β)-metric be projectively flat was studied by M. Matsumoto [12]. Aikou, Hashiguchi and Yamauchi [1] gave interesting results on the projective flatness of Matsumoto space.

Now, we consider the Cartan connection $C\Gamma$ for a Finsler space. If the covariant derivative $C_{hij|k}$ of the C-torsion tensor of $C\Gamma$ satisfies $C_{hij|k}y^k=0$, then the Finsler space is called a Landsberg space. A Berwald space is characterzied by $C_{hij|k}=0$. On the other hand, if a Finsler space is a Landsberg space and satisfies some additional conditions, then it is merely a Berwald space (cf. [3]). In the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x,y)$ satisfies $I_{|i}y^i=0$ (cf. [7]).

The geodesics of a two-dimensional Finsler space $F^2=(M^2,L)$ with an (α,β)-metric are regarded as the curves of the associated Riemannian space $R^2=(M^2,\alpha)$ which are bent by the differential 1-form β (cf. [15]). M. Matsumoto and H. S. Park [17] have expressed the differential equations of geodesics in two-dimensional Finsler spaces with a Randers metric and a Kropina metric in the most clean form $y''=f(x,y,y')$, respectively.

The first part of the present paper is devoted to finding the conditions that the Finsler space F^n with an (α,β)-metric $L=\beta^2/(\beta-\alpha)$ be a Berwald space (Theorem 3.1), a Douglas space (Theorem 4.1, 4.2) and a projectively flat Finsler space (Theorem 5.1). The second part is devoted to investigating the two-dimensional case. A condition that a Finsler space F^2 with the metric above be a Landsberg space is derived, and it is shown that if F^2 is a Landsberg space, then it is a Berwald space (Theorem 6.1). Lastly, by referring an isothermal coordinate system, the differential equations of the geodesics are discussed (Theorem 7.1).

2. Preliminaries

Let us consider the *r-th series (α,β)-metric*

$$L(\alpha,\beta)=\beta\sum_{k=0}^{r}\left(\frac{\alpha}{\beta}\right)^k, \tag{2.1}$$

where we assume $\alpha<\beta$.

If $r=1$, then $L=\alpha+\beta$ is a Randers metric. The condition that the Randers space be a Berwlad space, and a Douglas space are found in [16], respectively. If $r=2$, then $L=\alpha+\beta+\dfrac{\alpha^2}{\beta}$ is treated in [13] as an (α,β)-metric that a locally Minkowski space is flat-parallel. If $r=\infty$, then this metric (2.1) is expressed as the form

$$L(\alpha,\beta)=\frac{\beta^2}{\beta-\alpha}. \tag{2.2}$$

Then the metric above is called an *infinite series (α,β)-metric.* We have not at all investigated the geometrical meaning about the metric above by this time. But this metric (2.2) is remarkable as the difference between a Randers metric and a Matsumoto metric. In the present paper, we want to deal with every geometric prosperty possible of a Finsler space with this metric (2.2).

On the other hand, the geodesics of a Finsler space $F^n=(M^n,L)$ are given by the system of differential equations including the function

$$4G^i(x,y)=g^{ij}(y^r\dot{\partial}_j\partial_r L^2-\partial_j L^2).$$

For an (α,β)-metric $L(\alpha,\beta)$ the space $R^n=(M^n,\alpha)$ is called the associated Riemannian space with $F^n=(M^n,L(\alpha,\beta))$ ([2], [9]). The covariant differentiation with respect to the Levi-Civita connection $\gamma_j{}^i{}_k(x)$ of R^n is denoted by (;). We put $(a^{ij})=(a_{ij})^{-1}$, and use the symbols as follows:

$$r_{ij}=\frac{1}{2}(b_{i;j}+b_{j;i}),\ \ s_{ij}=\frac{1}{2}(b_{i;j}-b_{j;i}),\ \ r^i{}_j=a^{ir}r_{rj},\ \ s^i{}_j=a^{ir}s_{rj},$$
$$r_j=b_r r^r{}_j, s_j=b_r s^r{}_j,\ \ b^i=a^{ir}b_r,\ \ b^2=a^{rs}b_r b_s.$$

According to [11], if $\beta^2L_\alpha+\alpha\gamma^2L_{\alpha\alpha}\neq 0$, where $\gamma^2=b^2\alpha^2-\beta^2$, then the function $G^i(x,y)$ of F^n with an (α,β)-metric is written in the form

$$\begin{aligned}&2G^i=\gamma_0{}^i{}_0+2B^i,\\ &B^i=\frac{\alpha L_\beta}{L_\alpha}s^i{}_0+C^*\left\{\frac{\beta L_\beta}{\alpha L}y^i-\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha}y^i-\frac{\alpha}{\beta}b^i\right)\right\},\end{aligned} \tag{2.3}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$ and $L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha$, the subscript 0 means the contraction by y^i and we put

$$C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2s_0\alpha L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}. \tag{2.4}$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, $\gamma_0{}^i{}_0$ is $hp(2)$.

From the former of (2.3) the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$ of F^n with an (α,β)-metric is given by

$$G^i{}_j = \dot\partial_j G^i = \gamma_0{}^i{}_j + B^i{}_j,$$
$$G_j{}^i{}_k = \dot\partial_k G^i{}_j = \gamma_j{}^i{}_k + B_j{}^i{}_k,$$

where we put $B^i{}_j = \dot\partial_j B^i$ and $B_j{}^i{}_k = \dot\partial_k B^i{}_j$. $B^i(x,y)$ is called the *difference vector* ([11]). On account of [11], $B_j{}^i{}_k$ is determined by

$$L_\alpha B_j{}^t{}_i y^j y_t + \alpha L_\beta (B_j{}^t{}_i b_t - b_{j;i}) y^j = 0, \tag{2.5}$$

where $y_k = a_{ik} y^i$.

A Finsler space F^n with an (α,β)-metric is a Douglas space, if and only if $B^{ij} \equiv B^i y^j - B^j y^i$ is $hp(3)$ [4]. From the latter of (2.3) B^{ij} is written as follows:

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i). \tag{2.6}$$

On the other hand, by [12, Theorem 1] a Finsler space F^n with an (α,β)-metric is projectively flat if and only if the space is covered by coordinate neighborhoods in which $\gamma_j{}^i{}_k(x)$ satisfies

$$\begin{aligned}&(\gamma_0{}^i{}_0 - \gamma_{000} y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0\\ &+ (L_{\alpha\alpha}/L_\alpha)C^*(\alpha^2 b^i/\beta - y^i) = 0,\end{aligned} \tag{2.7}$$

where $\gamma_{000} = \gamma_0{}^k{}_0 y_k$, and $\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$ is assumed. Since $\alpha^2 L_{\alpha\alpha} = \beta^2 L_{\beta\beta}$, we have

$$\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}) = \beta^2(\alpha L_\alpha + \gamma^2 L_{\beta\beta}).$$

Hence C^* is rewritten in the form

$$C^* = \frac{\alpha^2(r_{00}L_\alpha - 2s_0\alpha L_\beta)}{2\beta(\alpha L_\alpha + \gamma^2 L_{\beta\beta})}. \tag{2.8}$$

If $\alpha L_\alpha + \gamma^2 L_{\beta\beta} \neq 0$, then from (2.7) and (2.8), we get

$$\begin{aligned}&(\alpha L_\alpha + \gamma^2 L_{\beta\beta})\{(\gamma_0{}^i{}_0 - \gamma_{000} y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0\}\\ &+ (L_{\alpha\alpha}/L_\alpha)(\alpha^2/2\beta)(r_{00}L_\alpha - 2s_0\alpha L_\beta)(\alpha^2 b^i/\beta - y^i) = 0.\end{aligned} \tag{2.9}$$

Thus we have the following

THEOREM 2.1. *If* $\alpha L_\alpha + \gamma^2 L_{\beta\beta} \neq 0$, *then a Finsler space with an* (α, β)*-metric is projectively flat if and only if* (2.9) *is satisfied.*

We shall state the following Lemmas for later:

LEMMA 2.2 ([5]). *If* $\alpha^2 \equiv 0(\mathrm{mod}\,\beta)$, *that is,* $a_{ij}(x)y^i y^j$ *contains* $b_i(x)y^i$ *as a factor, then the dimension is equal to two and* b^2 *vanishes. In this case we have* $\delta = d_i(x)y^i$ *satisfying* $\alpha^2 = \beta\delta$ *and* $d_i b^i = 2$.

LEMMA 2.3 ([7]). *We consider the two-dimensional case.*

(1) *If* $b^2 \neq 0$, *then there exist a sign* $\varepsilon = \pm 1$ *and* $\delta = d_i(x)y^i$ *such that* $\alpha^2 = \beta^2/b^2 + \varepsilon\delta^2$ *and* $d_i b^i = 0$.

(2) *If* $b^2 = 0$, *then there exists* $\delta = d_i(x)y^i$ *such that* $\alpha^2 = \beta\delta$ *and* $d_i b^i = 2$.

If there are two functions $f(x)$ and $g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$, then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradition $\alpha^2 = (-g/f)\beta^2$.

Next, we shall consider the two-dimensional case. Let us denote by $R(C) = 0$ the differential equation of the Weierstrass form of a geodesic C of R^2. $R(C)$ is given by

$$R(C) = \alpha_{1(2)} - \alpha_{2(1)} + (y^1\dot{y}^2 - y^2\dot{y}^1)W_r,$$

where $\alpha_i = \partial\alpha/\partial x^i$ and $\alpha_{(i)} = \partial\alpha/\partial y^i$, $y^i = dx^i/dt$ and $\dot{y}^i = dy^i/dt$ and W_r is the Weierstrass invariant of R^2 (cf. [17]).

By putting $y^i{}_{;0} = \dot{y}^i + \gamma_0{}^i{}_0$, $R(C)$ can be written in the form

$$(2.10) \qquad R(C) = (y^1 y^2_{;0} - y^2 y^1_{;0})W_r, \quad W_r = \{a_{11}a_{22} - (a_{12})^2\}/\alpha^3.$$

Then we have

LEMMA 2.4 ([17]). *In a two-dimensional Finsler space with* (α, β)*-metric* $L(\alpha, \beta)$, *the geodesics are given by the differential equation*

$$(L_\alpha + w\alpha\gamma^2)R(C) + \beta_{;r}y^r\delta\omega - L_\beta(b_{1;2} - b_{2;1}) = 0,$$

where w *is the intrinsic Weierstrass invariant,* $R(C)$ *is defined by* (2.10) *and* $\delta = (a_{1r}b_2 - a_{2r}b_1)y^r$.

Suppose that the Riemannian metric α be positive-definite. Then we may refer to an isothermal coordinate system $(x^i, y^i) = (x, y, \dot{x}, \dot{y})$ ([6]) such that

$$\alpha = aE, \quad a = a(x, y) > 0, \quad E = \sqrt{\dot{x}^2 + \dot{y}^2}.$$

Then $R(C)$ is of the form $R_i(C)$, where $R_i(C) = \dfrac{a}{E^3}(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \dfrac{1}{E}(a_x\dot{y} - a_y\dot{x})$. Next $\gamma^2 = (b_1\dot{y} - b_2\dot{x})^2$, and hence we may put $\gamma = b_1\dot{y} - b_2\dot{x}$ ([6]) and $\delta = -a^2\gamma$. Therefore, we have

LEMMA 2.5 ([17]). *For the Finsler space of Lemma 2.4, if α is positive-definite and we refer to an isothermal coordinate system (x,y) such that $\alpha = aE$, then the differential equation of a geodesic is of the form:*

$$\begin{aligned}(2.11)\quad &\{L_\alpha + aE\omega(b_1\dot{y} - b_2\dot{x})^2\}\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\} \\ &-E^3L_\beta(b_{1y} - b_{2x}) - E^3a^2\omega(b_1\dot{y} - b_2\dot{x})b_{0;0} = 0,\end{aligned}$$

where

$$\begin{aligned}(2.12)\quad b_{0;0} &= (b_{1x}\dot{x} + b_{1y}\dot{y})\dot{x} + (b_{2x}\dot{x} + b_{2y}\dot{y})\dot{y} \\ &+ \frac{1}{a}\{(\dot{x}^2 + \dot{y}^2)(a_xb_1 + a_yb_2) - 2(b_1\dot{x} + b_2\dot{y})(a_x\dot{x} + a_y\dot{y})\}\end{aligned}$$

and we put $b_{ix} = \partial b_i/\partial x$, $b_{iy} = \partial b_i/\partial y$, $a_x = \partial a/\partial x$ and $a_y = \partial a/\partial y$.

3. Berwald space

In the present section, we find the condition that a Finsler space F^n with an (α,β)-metric (2.2) be a Berwald space. In the n-dimensional Finsler space F^n with an (α,β)-metric (2.2), we have

$$(3.1)\quad \begin{aligned} L_\alpha &= \frac{\beta^2}{(\beta-\alpha)^2}, \quad L_\beta = \frac{\beta(\beta-2\alpha)}{(\beta-\alpha)^2}, \\ L_{\alpha\alpha} &= \frac{2\beta^2}{(\beta-\alpha)^3}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\beta-\alpha)^3}. \end{aligned}$$

Substituting (3.1) into (2.5), we have

$$(3.2)\quad \{\beta B_j{}^t{}_i y^j y_t - 2\alpha^2(B_j{}^t{}_i - b_{j;i})y^j\} + \alpha\{\beta(B_j{}^t{}_i b_t - b_{j;i})y^j\} = 0.$$

Assume that F^n is a Berwald space, that is, $G_j{}^i{}_k = G_j{}^i{}_k(x)$. Then we have $B_j{}^t{}_i = B_j{}^t{}_i(x)$. Since α is irrational in (y^i), from (3.2) we have

$$\beta B_j{}^t{}_i y^j y_t - 2\alpha^2(B_j{}^t{}_i b_t - b_{j;i})y^j = 0, \quad \beta(B_j{}^t{}_i b_t - b_{j;i})y^j = 0.$$

The former yields $B_j{}^t{}_i y^j y_t = 0$ from the latter. Thus we have

$$B_j{}^t{}_i y^j y_t = 0 \quad \text{and} \quad (B_j{}^t{}_i b_t - b_{j;i})y^j = 0,$$

which show

$$B_j{}^t{}_i a_{th} + B_h{}^t{}_i a_{tj} = 0 \quad \text{and} \quad B_j{}^t{}_i b_t - b_{j;i} = 0.$$

The former yields $B_j{}^t{}_i = 0$ by the well-known Christoffel process. Therefore we have

THEOREM 3.1. *The Finsler space F^n with an (α,β)-metric (2.2) is a Berwald space if and only if $b_{j;i} = 0$, and then the Berwald connection is essentially Riemannian $(\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0)$.*

4. Douglas space

In the present section, we consider the condition that a Finsler space F^n with an (α, β)-metric (2.2) be a Douglas space. Substituting (3.1) into (2.6), we obtain

$$\begin{aligned}(4.1)\quad & \beta\{\beta^2(\beta-\alpha)+2\alpha\gamma^2\}B^{ij}+\alpha(2\alpha-\beta)\{\beta^2(\beta-\alpha)+2\alpha\gamma^2\} \\ & (s^i{}_0y^j-s^j{}_0y^i)\ -\alpha^3\{\beta r_{00}-2\alpha(\beta-2\alpha)s_0\}(b^iy^j-b^jy^i)=0.\end{aligned}$$

It is noted that $\beta^2 L_\alpha + \gamma^2 L_{\beta\beta} \neq 0$.

Suppose that F^n is a Douglas space, that is, B^{ij} are $hp(3)$. Separating (4.1) in the rational and irrational terms of y^i because α is irrational in (y^i), we have

$$\begin{aligned}(4.2)\quad & \beta^4B^{ij}+\alpha^2\beta(3\beta^2-2\gamma^2)(s^i{}_0y^j-s^j{}_0y^i)+2\alpha^4\beta s_0(b^iy^j-b^jy^i) \\ & +\alpha\Big[\beta(2\gamma^2-\beta^2)B^{ij}+\{2\alpha^2(2\gamma^2-\beta^2)-\beta^4\}(s^i{}_0y^j-s^j{}_0y^i) \\ & -\alpha^2(\beta r_{00}+4\alpha^2s_0)(b^iy^j-b^jy^i)\Big]=0.\end{aligned}$$

Hence the equation (4.2) is divided into two equations as follows:

$$(4.3)\quad \beta^3B^{ij}+\alpha^2(3\beta^2-2\gamma^2)(s^i{}_0y^j-s^j{}_0y^i)+2\alpha^4s_0(b^iy^j-b^jy^i)=0$$

and

$$\begin{aligned}(4.4)\quad & \beta(2\gamma^2-\beta^2)B^{ij}+\{2\alpha^2(2\gamma^2-\beta^2)-\beta^4\}(s^i{}_0y^j-s^j{}_0y^i) \\ & -\alpha^2(\beta r_{00}+4\alpha^2s_0)(b^iy^j-b^jy^i)=0.\end{aligned}$$

Eliminating B^{ij} from (4.3) and (4.4), we obtain

$$(4.5)\quad P(s^i{}_0y^j-s^j{}_0y^i)+\alpha^2Q(b^iy^j-b^jy^i)=0,$$

where

$$\begin{aligned}(4.6)\quad & P=\alpha^2(2\gamma^2-\beta^2)(3\beta^2-2\gamma^2)-\beta^2\{2\alpha^2(2\gamma^2-\beta^2)-\beta^4\}, \\ & Q=2\alpha^2(2\gamma^2-\beta^2)s_0+\beta^2(\beta r_{00}+4\alpha^2s_0).\end{aligned}$$

Transvection of (4.5) by b_iy_j leads to

$$(4.7)\quad Ps_0+Q\gamma^2=0.$$

The term of (4.7) which does not contain α^2 is found in $\beta^5(\beta s_0 - r_{00})$. Hence there exists $hp(5)$: V_5 such that

$$\beta^5(\beta s_0 - r_{00}) = \alpha^2 V_5. \tag{4.8}$$

Then it will be better to divide our consideration into three cases as follows:

(A) $V_5 = 0$, (B) $V_5 \neq 0$, $\alpha^2 \not\equiv 0 \pmod \beta$, (C) $V_5 \neq 0$, $\alpha^2 \equiv 0 \pmod \beta$.

First, the case of (A) leads to $r_{00} = \beta s_0$, that is, $2r_{ij} = b_i s_j + b_j s_i$. Therefore, substituting $r_{00} = \beta s_0$ into (4.7), we have

$$s_0(P + \gamma^2 Q_1) = 0, \tag{4.9}$$

where

$$Q_1 = 2\alpha^2(2\gamma^2 - \beta^2) + \beta^2(\beta^2 + 4\alpha^2).$$

If $P + \gamma^2 Q_1 = 0$ in (4.9), then we obtain

$$\begin{aligned} P + \gamma^2 Q_1 &= b^2\alpha^2 Q_1 + P - \beta^2 Q_1 \\ &= b^2\alpha^2 Q_1 + (\alpha^2 P_1 + \beta^6) - (\alpha^2\beta^2 Q_2 + \beta^6) \\ &= \alpha^2(b^2 Q_1 + P_1 - \beta^2 Q_2) = 0, \end{aligned}$$

where

$$\begin{aligned} P_1 &= 2(2\gamma^2 - \beta^2)(\beta^2 - \gamma^2), \\ Q_2 &= 2(2\gamma^2 + \beta^2). \end{aligned}$$

Thus the term of $b^2 Q_1 + P_1 - \beta^2 Q_2 = 0$ which does not contain α^2 is $(b^2 - 10)\beta^4$. Thus there exists $hp(2)$: V_2 such that

$$(b^2 - 10)\beta^4 = \alpha^2 V_2,$$

where we assume $b^2 \neq 10$. Hence we have $V_2 = 0$, which leads to a contradiction, that is, $P + \gamma^2 Q_1 \neq 0$. Therefore, we have $s_0 = 0$ from (4.9) and we obtain $r_{00} = 0$ easily. Substituting $s_0 = 0$ and $r_{00} = 0$ into (4.5), we have

$$P(s^i{}_0 y^j - s^j{}_0 y^i) = 0. \tag{4.10}$$

If $P = 0$, then from $(4.6)_1$, we have

$$\alpha^2(2\gamma^2 - \beta^2)(3\beta^2 - 2\gamma^2) - \beta^2\{\alpha^2(2\gamma^2 - \beta^2) - \beta^4\} = 0. \tag{4.11}$$

The term of (4.11) which does not contain α^2 is β^6. Thus there exists $hp(4)$: V_4 such that

$$\beta^6 = \alpha^2 V_4,$$

from which we have $V_4 = 0$. It is a contradiction, that is, $P \neq 0$. Therefore we obtain $s^i{}_0 y^j - s^j{}_0 y^i = 0$ in (4.10). Transvection of this equation by y_j gives $s^i{}_0 = 0$, which imply $s_{ij} = 0$. Consequently, we have $r_{ij} = s_{ij} = 0$, that is, $b_{i;j} = 0$ are obtained.

Next, we treat the case (B). The equation (4.8) shows that there exists a function $k = k(x)$ satisfying

$$\beta s_0 - r_{00} = k(x)\alpha^2. \tag{4.12}$$

Substituting (4.12) into (4.7) and using (4.6), we have

$$Qb^2 + P_1 s_0 - Q_3 \beta^2 = 0, \tag{4.13}$$

where

$$Q_3 = 2(2\gamma^2 - \beta^2)s_0 + \beta^2(4s_0 - k\beta).$$

The term of (4.13) which seemingly does not contain α^2 is included in the term: $\{(b^2 - 10)s_0 + k\beta\}\beta^4$. Thus there exists $hp(3)$: V_3 such that

$$\beta^4\{(b^2 - 10)s_0 + k\beta\} = \alpha^2 V_3.$$

From $\alpha^2 \not\equiv 0 \pmod{\beta}$, it follows that V_3 must vanish and hence we have

$$s_0 = -\frac{k(x)}{b^2 - 10}\beta. \tag{4.14}$$

From (4.14), we have $s_i = -k(x)b_i/(b^2 - 10)$. Transvection of the above by b^i leads to $k(x)b^2 = 0$. Hence we get $k(x) = 0$. Substituting $k(x) = 0$ into (4.12) and (4.14), we obtain $s_0 = 0$ and $r_{00} = 0$. From (4.10), we have $P(s^i{}_0 y^j - s^j{}_0 y^i) = 0$. If $P = 0$, then it is a contradiction. Hence $P \neq 0$. Therefore, we obtain $s^i{}_0 y^j - s^j{}_0 y^i = 0$. Transvection of this equation by y_j gives $s^i{}_0 = 0$. Hence both the case (A) and (B) lead to $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{i;j} = 0$.

Conversely if $b_{i;j} = 0$, then F^n is a Berwald space, so F^n is a Douglas space.

Finally, we deal with the case (C). Lemma 2.2 shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. (4.8) is of the form $\beta^4(\beta s_0 - r_{00}) = \delta V_5$, which must be reduced to $\beta s_0 - r_{00} = \delta V$, $V = V_i(x)y^i$. From (4.14) we obtain $s_0 = k(x)\beta/10$ easily. From (4.7) we have $(\beta - 12\delta)s_0 - \{(\beta - 2\delta)s_0 - \delta V\} = 0$, which implies $10s_0 = V$. Thus we get $V = k\beta$.

Consequently we obtain

$$r_{00} = k\beta(\beta - 10\delta)/10, \quad s_0 = k\beta/10. \tag{4.15}$$

Then (4.5) is written as

$$10(s^i{}_0 y^j - s^j{}_0 y^i) + k\delta(b^i y^j - b^j y^i) = 0.$$

Transvection by y_j leads to

$$s^i{}_0 = k(y^i - \delta b^i)/10. \tag{4.16}$$

Differentiating (4.16) with respect to y^m and using $\alpha^2 = \beta\delta$; $2a_{ij} = b_i d_j + b_j d_i$, we have

$$s_{ij} = k(b_j d_i - b_i d_j)/20. \tag{4.17}$$

Conversely, (4.3) gives B^{ij} of the form

$$B^{ij} = 3k\delta^2(b^i y^j - b^j y^i)/10,$$

which is $hp(3)$, that is, F^2 is a Douglas space. (4.15) and (4.17) lead to $b_{i;j}$ of the form

$$b_{i;j} = k(2b_i b_j - 9b_j d_i - 11b_i d_j)/20. \tag{4.18}$$

Thus we have the following

THEOREM 4.1. *An n-dimensional Finsler space F^n with an (α, β)-metric (2.2) provided $b^2 \neq 10$ is a Douglas space, if and only if*

(1) $\alpha^2 \not\equiv 0 \ (mod\, \beta) : b_{i;j} = 0$,

(2) $\alpha^2 \equiv 0 \ (mod\, \beta) : n = 2$ *and $b_{i;j}$ is written in the form (4.18), where $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$ and $k = k(x)$.*

From Theorem 3.1 and Theorem 4.1, we have

THEOREM 4.2. *If an n-dimensional Finsler space F^n $(n > 2)$with an (α, β)-metric (2.2) provided $b^2 \neq 10$ is a Douglas space, then it is a Berwald space.*

5. Projectively flat Finsler space

In the present section, we deal with the condition that a Finsler space with an (α, β)-metric (2.2) be projectively flat Finsler space.

First, it is noted that $\alpha L_\alpha + \gamma^2 L_{\beta\beta} \neq 0$. Hence making use of Theorem 2.1 and substiuting (3.1) into (2.9), we have

$$\begin{aligned}(2b^2\alpha^3 - 3\alpha\beta^2 + \beta^3)(-4\alpha^4 s^i{}_0 + 2\alpha^3\beta s^i{}_0 + \alpha^2\beta\gamma_0{}^i{}_0 - \gamma_{000}\beta y^i) \\ + (8\alpha^5 s_0 - 4\alpha^4\beta s_0 + 2\alpha^3\beta r_{00})(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{5.1}$$

The equation (5.1) is rewritten as a polynomial of the seventh degree in α as follows:

$$(a_7\alpha^6 + a_5\alpha^4 + a_3\alpha^2 + a_1)\alpha + a_6\alpha^6 + a_4\alpha^4 + a_2\alpha^2 + a_0 = 0,$$

where

$$\begin{aligned}&a_7 = 8(s_0 b^i - b^2 s^i{}_0), \quad a_6 = 4(b^2 s^i{}_0 - s_0 b^i)\beta,\\ &a_5 = 2(b^2\gamma_0{}^i{}_0 + 6\beta s^i{}_0 + r_{00}b^i - 4s_0 y^i)\beta, \quad a_4 = 2(s_0 y^i - 5\beta s^i{}_0)\beta^2,\\ &a_3 = (2\beta^3 s^i{}_0 - 2b^2\gamma_{000}y^i - 3\beta^2\gamma_0{}^i{}_0 - 2\beta r_{00}y^i)\beta,\\ &a_2 = \beta^4\gamma_0{}^i{}_0, \quad a_1 = 3\beta^3\gamma_{000}y^i, \quad a_0 = -\gamma_{000}\beta^4 y^i.\end{aligned}$$

Since $a_7\alpha^6 + a_5\alpha^4 + a_3\alpha^2 + a_1$ and $a_6\alpha^6 + a_4\alpha^4 + a_2\alpha^2 + a_0$ are polynomials and α is irrational in y^i, we have

$$a_7\alpha^6 + a_5\alpha^4 + a_3\alpha^2 + a_1 = 0, \tag{5.2}$$

$$a_6\alpha^6 + a_4\alpha^4 + a_2\alpha^2 + a_0 = 0. \tag{5.3}$$

Consequently, a_1 and a_0 must contain α^2. From a_1 or a_0 there exists $hp(1) : v_0 = v_i(x)y^i$ such that

$$\gamma_{000} = v_0\alpha^2. \tag{5.4}$$

From (5.4) the term of (5.3) which does not contain α^4 is $(\gamma_0{}^i{}_0 - v_0 y^i)\beta^4$. Therefore there exists $hp(4) : V_4$ such that

$$(\gamma_0{}^i{}_0 - v_0 y^i)\beta^4 = \alpha^2 V_4. \tag{5.5}$$

We suppose that $\alpha^2 \not\equiv 0 \pmod{\beta}$. Thus from (5.5) there exists a function $u^i = u^i(x)$ such that

$$\gamma_0{}^i{}_0 - v_0 y^i = u^i\alpha^2. \tag{5.6}$$

Transvecting (5.6) by y_i, from (5.4) we have $u^i y_i = 0$, which imply $u^i = 0$. Thus we have

$$\gamma_0{}^i{}_0 = v_0 y^i. \tag{5.7}$$

That is to say,

$$2\gamma_j{}^i{}_k = v_k \delta^i_j + v_j \delta^i_k, \tag{5.8}$$

which shows that the associated Riemannian space is projectively flat.

Next, substituting (5.4) and (5.7) into (5.1), we have

$$\begin{aligned}&(2b^2\alpha^3 - 3\alpha\beta^2 + \beta^3)(-4\alpha^2 + 2\alpha\beta)s^i{}_0\\ &+ (8\alpha^3 s_0 - 4\alpha^2\beta s_0 + 2\alpha\beta r_{00})(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{5.9}$$

Transvecting (5.9) by b_i, we get

$$\{2\beta(2\beta s_0 + b^2 r_{00})\alpha^2 + 2\beta^3(\beta s_0 - r_{00})\}\alpha - 6\beta^3 s_0\alpha^2 = 0, \tag{5.10}$$

which implies

$$2\beta(2\beta s_0 + b^2 r_{00})\alpha^2 + 2\beta^3(\beta s_0 - r_{00}) = 0, \tag{5.11}$$

$$s_0 = 0. \tag{5.12}$$

Substituting (5.12) into (5.11), we have $\gamma^2 r_{00} = 0$, that is, $r_{00} = 0$ because of $\gamma^2 \neq 0$.

Further, the term of (5.1) which does not contain β is $-8b^2 s^i{}_0\alpha^7$. Therefore there exists a function $\lambda^i = \lambda^i(x)$ satisfying $b^2 s^i{}_0 = \lambda^i\beta$. Transvection of the above by y_i leads to $\lambda^i y_i = 0$, that is, $\lambda^i = 0$. Thus we have $s^i{}_0 = 0$, that is, $s_{ij} = 0$. Therefore, from $r_{ij} = 0$ and $s_{ij} = 0$, we obtain $b_{i;j} = 0$.

Conversely, it is easy to see that (5.1) is a consequence of $b_{i;j} = 0$. Thus we have

THEOREM 5.1. *A Finsler space F^n $(n > 2)$ with an (α, β)-metric (2.2) projectively flat if and only if $b_{i;j} = 0$ is satisfied and the associated Riemannian space (M^n, α) is projectively flat. Then F^n is a Berwald space.*

6. Landsberg space of dimension two

In the present section, we investigate the condition that a two-dimensional Finsler space F^2 with an (α, β)-metric (2.2) be a Landsberg space.

Let $F^n = (M^n, L(\alpha, \beta))$ be an n-dimensional Finsler space with an (α, β)-metric given by (2.2). The difference vector B^i of the space has been first given in [21] (*cf.* [10], [11], [22]). Here, by means of (2.3) and (3.1) we have

$$2B^i = \frac{2\alpha^3 A}{\beta \Omega}\left\{b^i + \frac{\beta(\beta - 4\alpha)}{2\alpha^3} y^i\right\} + \frac{2\alpha}{\beta}(\beta - 2\alpha) s^i{}_0, \tag{6.1}$$

where we put

$$A = \{\beta r_{00} - 2\alpha(\beta - 2\alpha)s_0\},$$
$$\Omega = 2\alpha^3 b^2 - 3\alpha\beta^2 + \beta^3.$$

It is noted that $\Omega \neq 0$.

It follows from (6.1) that

$$r_{00} - 2b_r B^r = \frac{\alpha\beta A}{\Omega}. \tag{6.2}$$

Now we deal with the necessary and sufficient condition that a two-dimensional Finsler space F^2 with an (α, β)-metric (2.2) be a Landsberg space. It is known that in the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I_{|i} y^i = 0$ [9]. Owing to [2], [8], the main scalar I of a two-dimensional Finsler space F^2 with an (α, β)-metric (2.2) is obtained easily as follows:

$$\varepsilon I^2 = \frac{9\gamma^2 Z^2}{4\alpha\Omega^3}, \tag{6.3}$$

where $Z = -4\alpha^4 b^2 + 8\alpha^2\beta^2 - 5\alpha\beta^3 + \beta^4$.

Furthermore, by means of [7] we have

$$\begin{aligned}
&\alpha_{|i} = -\frac{L_\beta}{L_\alpha}\beta_{|i}, \\
&\beta_{|i} y^i = r_{00} - 2b_r B^r, \\
&b^2_{|i} y^i = 2(r_0 + s_0), \\
&\gamma^2_{|i} y^i = 2(r_0 + s_0)\alpha^2 - 2\left(\frac{L_\beta}{L_\alpha} b^2\alpha + \beta\right)(r_{00} - 2b_r B^r).
\end{aligned} \tag{6.4}$$

Now, the covariant differentiation of (6.3) leads to

(6.5)
$$4\alpha^2\Omega^4\varepsilon I^2_{|i} = 9Z\big[\alpha\Omega Z\gamma^2_{|i} + \alpha\gamma^2\{2\Omega(4\beta^3 - 15\alpha\beta^2 + 16\alpha^2\beta) + 9Z(2\alpha\beta - \beta^2)\}\beta_{|i} - \gamma^2\{2\alpha\Omega(5\beta^3 - 16\alpha\beta^2 + 16\alpha^3 b^2) + \Omega Z + 9\alpha Z(2\alpha^2 b^2 - \beta^2)\}\alpha_{|i} - 2\alpha^4\gamma^2(4\alpha\Omega + 3Z)b^2_{|i}\big].$$

Transvecting (6.5) by y^i, we have

$$4\alpha^2\Omega^4\varepsilon I^2_{|i}y^i = 9Z(P\alpha_{|i}y^i + Q\beta_{|i}y^i + R\gamma^2_{|i}y^i + Sb^2_{|i}y^i), \tag{6.6}$$

where

$$\begin{aligned} P = {} & 16\alpha^9 b^6 - 64\alpha^7\beta^2 b^4 + 62\alpha^6\beta^3 b^4 + 28\alpha^5\beta^4 b^2 - 68\alpha^4\beta^5 b^2 \\ & + 27\alpha^3\beta^6 b^2 + \alpha^2\beta^7(6 - b^2) - 7\alpha\beta^8 + \beta^9, \\ Q = {} & -8\alpha^8\beta b^4 - 24\alpha^7\beta^2 b^4 + \alpha^6\beta^3 b^2(16b^2 + 56) - 16\alpha^5\beta^4 b^2 \\ & - \alpha^4\beta^5(7b^2 + 48) + \alpha^3\beta^6(40 - b^2) - 9\alpha^2\beta^7 + \alpha\beta^8, \\ R = {} & -8\alpha^8 b^4 + 28\alpha^6\beta^2 b^2 - 14\alpha^5\beta^3 b^2 + 2\alpha^4\beta^4(b^2 - 12) + 23\alpha^3\beta^5 \\ & - 8\alpha^2\beta^6 + \alpha\beta^7, \\ S = {} & 8\alpha^{10} b^4 - 32\alpha^8\beta^2 b^2 + 22\alpha^7\beta^3 b^2 + \alpha^6\beta^4(24 - 6b^2) \\ & - 22\alpha^5\beta^5 + 6\alpha^4\beta^6. \end{aligned}$$

Consequently, the two-dimensional Finsler space F^2 with (2.2) is a Landsberg space, if and only if

$$9Z(P\alpha_i y^i + Q\beta_{|i}y^i + R\gamma^2_{|i}y^i + Sb^2_{|i}y^i) = 0,$$

which implies

$$P\alpha_{|i}y^i + Q\beta_{|i}y^i + R\gamma^2_{|i}y^i + Sb^2_{|i}y^i = 0, \tag{6.7}$$

because $Z = 0$ implies $\beta = 0$, that is, it is a contradiction.

By means of (6.4), the equation (6.7) is rewritten as follows:

$$\{(2\alpha - \beta)P + \beta Q + 2(2\alpha^2 b^2 - \alpha\beta b^2 - \beta^2)R\}(r_{00} - 2b_r B^r) + 2\beta(\alpha^2 R + S)(r_0 + s_0) = 0.$$

Substituting (6.2), P, Q, R and S into the equation above, we obtain

(6.8)
$$\begin{aligned}&\{-8\alpha^8\beta^3b^4+52\alpha^7\beta^4b^2-\alpha^6\beta^5b^2(10b^2+40)-\alpha^5\beta^6b^2(4b^2+12)\\&+32\alpha^4\beta^7b^2+\alpha^3\beta^8(6-10b^2)-\alpha^2\beta^9(b^2+13)+8\alpha\beta^{10}-\beta^{11}\}r_{00}\\&+\{-32\alpha^{10}\beta^2b^4+224\alpha^9\beta^3b^4-\alpha^8\beta^4b^2(144b^2+160)\\&+\alpha^7\beta^5b^2(4b^2+32)+\alpha^6\beta^6b^2(8b^2+152)+\alpha^5\beta^7(24-104b^2)\\&+\alpha^4\beta^8(16b^2-64)+\alpha^3\beta^9(2b^2+58)-20\alpha^2\beta^{10}+2\alpha\beta^{11}\}s_0\\&+\{-16\alpha^{10}\beta^2b^4+32\alpha^9\beta^3b^4+\alpha^8\beta^4b^2(24-16b^2)\\&-52\alpha^7\beta^5b^2+32\alpha^6\beta^6b^2-\alpha^5\beta^7(4b^2+6)+14\alpha^4\beta^8-10\alpha^3\beta^9\\&+2\alpha^2\beta^{10}\}(r_0+s_0)=0.\end{aligned}$$

Separating (6.8) in the rational and irrational terms of (y^i), we have

$$A_1r_{00}+A_2s_0+A_3(r_0+s_0)+\alpha\{B_1r_{00}+B_2s_0+B_3(r_0+s_0)\}=0,$$

where

$$\begin{aligned}A_1=&-8\alpha^8\beta^3b^4-\alpha^6\beta^5b^2(10b^2+40)+32\alpha^4\beta^7b^2-\alpha^2\beta^9(b^2+13)\\&-\beta^{11},\\A_2=&-32\alpha^{10}\beta^2b^4-\alpha^8\beta^4b^2(144b^2+160)+\alpha^6\beta^6b^2(8b^2+152)\\&+\alpha^4\beta^8(16b^2-64)-20\alpha^2\beta^{10},\\A_3=&-16\alpha^{10}\beta^2b^4+\alpha^8\beta^4b^2(24-16b^2)+32\alpha^6\beta^6b^2+14\alpha^4\beta^8\\&+2\alpha^2\beta^{10},\\B_1=&\,52\alpha^6\beta^4b^2-\alpha^4\beta^6b^2(4b^2+12)+\alpha^2\beta^8(6-10b^2)+8\beta^{10},\\B_2=&\,224\alpha^8\beta^3b^4+\alpha^6\beta^5b^2(4b^2+32)+\alpha^4\beta^7(24-104b^2)\\&+\alpha^2\beta^9(2b^2+58)+2\beta^{11},\\B_3=&\,32\alpha^8\beta^3b^4-52\alpha^6\beta^5b^2-\alpha^4\beta^7(4b^2+6)-10\alpha^2\beta^9,\end{aligned}$$

which yield two equations as follows:

(6.9) $$A_1r_{00}+A_2s_0+A_3(r_0+s_0)=0,$$

(6.10) $$B_1r_{00}+B_2s_0+B_3(r_0+s_0)=0.$$

From (6.9) and (6.10) we obtain respectively

(6.11) $$-\beta^{11}r_{00}\equiv 0\pmod{\alpha^2},$$

(6.12) $$4\beta^{10}r_{00}+\beta^{11}s_0\equiv 0\pmod{\alpha^2}.$$

(6.11) is reduced to

$$(6.11') \qquad \beta^{11} r_{00} \equiv 0 \pmod{\alpha^2}.$$

Then (6.11′) is written as

$$\beta^{11} r_{00} = \alpha^2 U_{11},$$

where U_{11} is $hp(11)$. From $b^2 \neq 0$ it follows that $\alpha^2 \not\equiv 0 \pmod{\beta}$ and there must exist a function $f(x)$ such that $U_{11} = \beta^{11} f(x)$. Hence we have

$$(6.11'') \qquad r_{00} = \alpha^2 f(x); \qquad r_{ij} = a_{ij} f(x).$$

Then (6.12) is reduced to

$$(6.12') \qquad \beta^{11} s_0 \equiv 0 \pmod{\alpha^2}.$$

(6.12′) shows that there exists $hp(10)$ U_{10} satisfying $\beta^{11} s_0 = \alpha^2 U_{10}$, which implies $U_{10} = 0$, because $\alpha^2 U_{10}$ can not contain β^{11} as a factor. Thus we have

$$(6.12'') \qquad s_0 = 0; \qquad s_i = 0.$$

It is obvious taht (6.11″) gives

$$(6.13) \qquad r_0 = \beta f(x); \qquad r_j = b_j f(x).$$

Therefore (6.11) and (6.12) are reduced to (6.11″), (6.12″) and (6.13). Further (6.9) and (6.10) are reduced respectively to

$$(6.14) \qquad \begin{aligned} &f(x)\{24\alpha^8\beta^2 b^4 + 2\alpha^6\beta^4 b^2(13b^2+8) \\ &- 64\alpha^4\beta^6 b^2 + 14\alpha^2\beta^8(b^2-1) - \beta^{10}\} = 0, \end{aligned}$$

$$(6.15) \quad f(x)\{4\alpha^6\beta^3 b^2(8b^2+13) - 4\alpha^4\beta^5 b^2(b^2+16) - 14\alpha^2\beta^7 b^2 - 2\beta^9\} = 0.$$

Let us assume $f(x) \neq 0$. Then (6.14) and (6.15) imply

$$-\beta^{10} = \alpha^2 V_8 \quad \text{and} \quad -2\beta^9 = \alpha^2 W_7,$$

where V_8 and W_7 are $hp(8)$ and $hp(7)$ respectively. Analogously to the above, these imply $V_8 = 0$ and $W_7 = 0$. Thus we arrive at a contradiction. Hence $f(x) = 0$ must hold and we have $r_{00} = 0$; $r_{ij} = 0$ and $s_0 = 0$; $s_i = 0$.

If $b^2 = 0$, then (6.9) and (6.10) are reduced to

$$\begin{aligned}(13\alpha^2\beta^9 + \beta^{11})r_{00} + 4(16\alpha^4\beta^8 + 5\alpha^2\beta^{10})s_0 \\ - 2(7\alpha^4\beta^8 + \alpha^2\beta^{10})(r_0 + s_0) = 0,\end{aligned} \tag{6.16}$$

$$\begin{aligned}(3\alpha^2\beta^8 + 4\beta^{10})r_{00} + (12\alpha^4\beta^7 + 29\alpha^2\beta^9 + \beta^{11})s_0 \\ - (3\alpha^4\beta^7 + 5\alpha^2\beta^9)(r_0 + s_0) = 0.\end{aligned} \tag{6.17}$$

Making use of Lemma 2.1, (6.16) and (6.17) are reduced to

$$(13\delta + \beta)r_{00} + 4(16\delta^2 + 5\delta\beta)s_0 - 2(7\delta^2 + \delta\beta)(r_0 + s_0) = 0, \tag{6.18}$$

$$(3\delta + 4\beta)r_{00} + (12\delta^2 + 29\delta\beta + \beta^2)s_0 - (3\delta^2 + 5\delta\beta)(r_0 + s_0) = 0. \tag{6.19}$$

Since $r_0 + s_0 = b^2_{;i}y^i/2$ vanishes because of $b^2 = 0$, the above equations are written as follows:

$$(13\delta + \beta)r_{00} + 4(16\delta^2 + 5\delta\beta)s_0 = 0, \tag{6.18'}$$

$$(3\delta + 4\beta)r_{00} + (12\delta^2 + 29\delta\beta + \beta^2)s_0 = 0. \tag{6.19'}$$

From (6.18′) adn (6.19′) we have

$$\beta r_{00} \equiv 0 \pmod{\delta}, \tag{6.20}$$

$$4\beta r_{00} + \beta^2 s_0 \equiv 0 \pmod{\delta}. \tag{6.21}$$

From (6.20) there exists $hp(2)$ X_2 such that

$$\beta r_{00} = \delta X_2.$$

Since $\beta \not\equiv 0 \pmod{\delta}$, there exists $hp(1)$ λ satisfying

$$r_{00} = \lambda\delta; \qquad r_{ij} = \frac{1}{2}(\lambda_i d_j + \lambda_j d_i). \tag{6.20'}$$

Substituting (6.20′) into (6.21), we have as follows: there exists $hp(2)$ W_2 such that

$$\beta(4\lambda\delta + \beta s_0) = \delta W_2. \tag{6.21′}$$

From $\beta \not\equiv 0 \pmod{\delta}$ we have $W_2 = \mu\beta$ and $4\lambda\delta + \beta s_0 = \mu\delta$, where μ is $hp(1)$, that is, $\beta s_0 = \delta(\mu - 4\lambda)$. Therefore there exists a function $g(x)$ such that

$$s_0 = g(x)\delta \quad \text{and} \quad \mu - 4\lambda = g(x)\beta. \tag{6.22}$$

Substituting (6.20′) and $s_0 = g(x)\delta$ into (6.18′) and (6.19′), we get respectively

$$(13\delta + \beta)\lambda + 4g(x)\delta(16\delta + 5\beta) = 0, \tag{6.23}$$

$$(3\delta + 4\beta)\lambda + g(x)(12\delta^2 + 29\delta\beta + \beta^2) = 0. \tag{6.24}$$

The term $\lambda\beta$ of (6.23) and the term $4\lambda\beta + g(x)\beta^2$ of (6.24) seemingly do not contain δ, and hence we must have $hp(1)$ X_1 and $hp(1)$ Y_1 satisfying

$$\lambda\beta = \delta X_1 \quad \text{and} \quad 4\lambda\beta + g(x)\beta^2 = \delta Y_1,$$

respectively. Eliminating λ from the equations above, we get

$$g(x)\beta^2 = \delta W_1, \tag{6.25}$$

where $W_1 = Y_1 - 4X_1$ is $hp(1)$, and hence $W_1 = 0$, because δW_1 can not contain β^2 as a factor. Hence we obtain $g(x) = 0$. Substituting $g(x) = 0$ into (6.23), we have

$$(13\delta + \beta)\lambda = 0.$$

If $\lambda \neq 0$, then we have $13\delta + \beta = 0$. It is a contradiction, because (β, δ) are independent. Hence $\lambda = 0$. From (6.20′) and $s_0 = g(x)\delta$ of (6.22) we have $r_{00} = 0$ and $s_0 = 0$ directly.

Summarizing up, we obtain $r_{00} = 0$ and $s_0 = 0$ in both cases of $b^2 \neq 0$ and $b^2 = 0$, that is,

$$b_{i;j} + b_{j;i} = 0, \quad b^r b_{r;i} = 0. \tag{6.26}$$

As is shown in [7], the equation (6.26) is equivalent to $b_{i;j} = 0$. Consequently, we have

THEOREM 6.1. *Let F^2 be a two-dimensional Finsler space with an (α, β)-metric (2.2). If F^2 is a Landsberg space, then it is a Berwald space.*

7. Geodesic equation of dimension two

In the present section, by referring an isothermal coordinate system, we find the differential equations of geodesics of a two-dimensional Finsler space satisfying an (α, β)-metric (2.2).

Substituting (3.1) and $w = 2/(\beta - \alpha)^3$ into (2.11), we obtain the differential equations of geodesics as follows:

$$\tag{7.1} \begin{aligned} &\{\beta^2(\beta-\alpha) + 2aE(b_1\dot{y} - b_2\dot{x})^2\}\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\} \\ &- E^3\{\beta(\beta-\alpha)(\beta-2\alpha)(b_{1y} - b_{2x}) + 2a^2(b_1\dot{y} - b_2\dot{x})b_{0;0}\} = 0, \end{aligned}$$

where $b_{0;0}$ is given by (2.12).

If we take x of (x, y) as the parameter of curve C, that is, $\dot{x} = 1$, $\dot{y} = y'$, $\ddot{x} = 0$, $\ddot{y} = y''$ and we put $Q^2 = 1 + (y')^2$, then (7.1) is reduced to

$$\tag{7.2} \begin{aligned} &\big[(b_1 + b_2y')^3 + a\{(2b_1^2 - b_2^2)(y')^2 + 2b_1b_2y' + (2b_2^2 - b_1^2)\}Q\big]\{ay'' \\ &+ Q^2(a_xy' - a_y)\} - Q^3\big[\{(b_1 + b_2y')^3 - 3a(b_1 + b_2y')^2Q \\ &+ 2a^2(b_1 + b_2y')Q^2\}(b_{1y} - b_{2x}) + 2a^2(b_1y' - b_2)b^*_{0;0}\big] = 0, \end{aligned}$$

where

$$\begin{aligned} b^*_{0;0} = {} & (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y' \\ & + \frac{1}{a}\{(1 + (y')^2)(a_xb_1 + a_yb_2) - 2(b_1 + b_2y')(a_x + a_yy')\}. \end{aligned}$$

Then (7.2) is rewritten in the form

$$\tag{7.3} \begin{aligned} &\big[(b_1 + b_2y')^3\{ay'' + Q^2(a_xy' - a_y)\} + 3a(b_1 + b_2y')^2(b_{1y} - b_{2x})Q^4\big] \\ &+ Q\big[a\{(2b_1^2 - b_2^2)(y')^2 + 2b_1b_2y' + (2b_2^2 - b_1^2)\}\{ay'' + Q^2(a_xy' - a_y)\} \\ &- Q^2\{(b_1 + b_2y')^3(b_{1y} - b_{2x}) + 2a^2(b_1 + b_2y')(b_{1y} - b_{2x})Q^2 \\ &+ 2a^2(b_1y' - b_2)b^*_{0;0}\}\big] = 0. \end{aligned}$$

Since Q is irrational in (y'), (7.3) is divided into two equations as follows:

$$\tag{7.4} (b_1 + b_2y')\{ay'' + Q^2(a_xy' - a_y)\} + 3a(b_{1y} - b_{2x})Q^4 = 0,$$

$$\tag{7.5} \begin{aligned} &a\{(2b_1^2 - b_2^2)(y')^2 + 2b_1b_2y' + (2b_2^2 - b_1^2)\}\{ay'' + Q^2(a_xy' - a_y)\} \\ &- Q^2\{(b_1 + b_2y')^3(b_{1y} - b_{2x}) + 2a^2(b_1 + b_2y')(b_{1y} - b_{2x})Q^2 \\ &+ 2a^2(b_1y' - b_2)b^*_{0;0}\} = 0. \end{aligned}$$

Furthermore, (7.4) and (7.5) are rewritten in the form

$$(7.6)\quad ay'' + \{1+(y)^2\}(a_x y' - a_y) = -\frac{3a(b_{1y} - b_{2x})\{1+2(y)^2+(y')^4\}}{b_1 + b_2(y')},$$

(7.7)

$$\begin{aligned} & ay'' + \{1+(y)^2\}(a_x y' - a_y) \\ & = \frac{Q^2\{(b_1+b_2y')^3(b_{1y}-b_{2x}) + 2a^2(b_1+b_2y')(b_{1y}-b_{2x})Q^2 + 2a^2(b_1y'-b_2)b^*_{0;0}\}}{a\{(2b_2^2-b_1^2)+2b_1b_2y'+(2b_1^2-b_2^2)(y')^2\}}. \end{aligned}$$

Thus we have

THEOREM 7.1. *Let F^2 be a two-dimensional Finsler space with an (α, β)-metric (2.2), where α is assumed to be positive definite. If we refer to an isothermal coordinate system (x, y) such that $\alpha = aE$ and $Q = \sqrt{1+(y')^2}$, then the differential equations of a geodesic $y = y(x)$ of F^2 are given by (7.6) and (7.7).*

Next, we deal with the case where the associated Riemannian space is Euclidean one with an orthonormal coordinate system. Then $a = 1$, $a_x = 0$ and $a_y = 0$. Therefore (7.6) and (7.7) are reduced to

$$(7.6')\qquad y'' = -\frac{3(b_{1y}-b_{2x})\{1+2(y)^2+(y')^4\}}{b_1+b_2(y')},$$

$$(7.7')\qquad y'' = \frac{D_0 + D_1y' + D_2(y')^2 + D_3(y')^3 + D_4(y')^4 + D_5(y')^5}{(2b_2^2-b_1^2)+2b_1b_2y'+(2b_1^2-b_2^2)(y')^2},$$

where

$$\begin{aligned} D_0 &= b_1(b_1^2+2)(b_{1y}-b_{2x}) - 2b_2b_{1x}, \\ D_1 &= b_2(3b_1^2+2)(b_{1y}-b_{2x}) + 2\{b_1b_{1x} - b_2(b_{1y}+b_{2x})\}, \\ D_2 &= b_1(3b_2^2+b_1^2+4)(b_{1y}-b_{2x}) + 2\{b_1(b_{1y}+b_{2x}) - b_2(b_{2y}+b_{1x})\}, \\ D_3 &= b_2(3b_1^2+b_2^2+4)(b_{1y}-b_{2x}) + 2\{b_1(b_{1x}+b_{2y}) - b_2(b_{1y}+b_{2x})\}, \\ D_4 &= b_1(3b_2^2+2)(b_{1y}-b_{2x}) + 2\{b_1(b_{1y}+b_{2x}) - b_2b_{2y}\}, \\ D_5 &= b_2(b_1^2+2)(b_{1y}-b_{2x}) + 2b_1b_{2y}. \end{aligned}$$

Thus we have the following

COROLLARY 7.2. *Let F^2 be a two-dimensional Finsler space with an (α,β)-metric (2.2) whose associated Riemannian space $R^2 = (M^2,\alpha)$ is Euclidean such that $a = 1$ and $a_x = a_y = 0$. In an orthonormal coordinate system (x,y) of R^2, the differential equations of geodesics of F^2 are given by (7.6′) and (7.7′).*

In order to find more concrete forms of (7.6′) and (7.7′), we deal with the case where the associated Riemannian space is Euclidean one with an orthonormal coordinate system. If we take b_1 and b_2 such that $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then $b_{1y} - b_{2x} = 0$. Thus (7.6′) and (7.7′) are reduced to

$$(7.6'') \qquad y'' = 0; \qquad y = cx + d, \quad \text{where } c,\ d \text{ are constants,}$$

$$(7.7'') \qquad y'' = \frac{E_0 + E_1 y' + E_2(y')^2 + E_3(y')^3 + E_4(y')^4 + E_5(y')^5}{(2b_2^2 - b_1^2) + 2b_1 b_2 y' + (2b_1^2 - b_2^2)(y')^2},$$

where

$$\begin{aligned} E_0 &= -2b_y b_{xx}, \quad E_1 = 2(b_x b_{xx} - 2b_y b_{xy}),\\ E_2 &= 2\{2b_x b_{xy} - b_y(b_{yy} + b_{xx})\},\\ E_3 &= 2\{b_x(b_{xx} + b_{yy}) - 2b_y b_{xy}\},\\ E_4 &= 2(2b_x b_{xy} - b_y b_{yy}),\\ E_5 &= 2b_x b_{yy}. \end{aligned}$$

Thus we have

COROLLARY 7.3. *Let F^2 be a two-dimensional Finsler space with an (α,β)-metric (2.2). If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_{1y} - b_{2x} = 0$, where $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then the differential equations of geodesics $y = y(x)$ of F^2 are given by (7.6″), that is, a straight line, and (7.7″).*

ACKNOWLEDGEMENTS. The authors would like to express their sincere gratitude to Professor Dr. MASAO HASHIGUCHI for his invaluable suggestions and encouragement.

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