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REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH PARALLEL SHAPE OPERATOR II

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ABSTRACT. In this paper we consider the notion of ξ-invariant or $\mathfrak{D}^{\perp}$-invariant real hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ and prove that there do not exist such kinds of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel second fundamental tensor on a distribution $\mathfrak{F}$ defined by $\mathfrak{F} = \xi \cup \mathfrak{D}^{\perp}$, where $\mathfrak{D}^{\perp} = \text{Span}\,\{\xi_1, \xi_2, \xi_3\}$.

0.1. Introduction

In the study of real hypersurfaces in complex space forms or in quaternionic space forms it can be easilily checked that there do not exist any real hypersurfaces with parallel second fundamental tensor by the equation of Codazzi.

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the above situation is not so simple when we consider a real hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$.

In this paper we study the analogous question in the complex Grassmann manifold $G_2(\mathbb{C}^{m+2})$ of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold.

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Now let M be a real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with a unit normal vector field N of M in $G_2(\mathbb{C}^{m+2})$. So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometrical conditions for real hypersurfaces that ξ or $\mathfrak{D}^\perp = \text{Span }\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator, where $\xi = -JN$, $\xi_i = -J_iN$ for $i = 1, 2, 3$, and J_i denotes an element in a quaternionic Kähler structure $\mathfrak{J}$.

The main result of this paper is to prove the non-existence of all real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel second fundamental tensor on $[\xi]\cup\mathfrak{D}^\perp$ when $[\xi] = \text{Span }\{\xi\}$ or $\mathfrak{D}^\perp = \text{Span }\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator of M. From this view point Berndt and the present author [4] have proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(1) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(2) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the proof of Theorem A we have proved that the one-dimensional distribution $[\xi]$ is contained in either the 3-dimensional distribution $\mathfrak{D}^\perp$ or in the orthogonal complement $\mathfrak{D}$ such that $T_xM = \mathfrak{D}\oplus\mathfrak{D}^\perp$. The case (2) in Theorem A is just the case that the one dimensional distribution $[\xi]$ is contained in $\mathfrak{D}^\perp$.

We have mentioned that there do not exist any real hypersurfaces with parallel second fundamental form in complex space forms and in quaternionic space forms. But when we consider its situation in $G_2(\mathbb{C}^{n+2})$, it is not so simple to prove it. From this point of a view the present author [11] has proved the following.

Theorem B. *There do not exist any real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel second fundamental form, that is, $\nabla A = 0$.*

On the other hand, Kimura and Maeda [8] have proved that a real hypersurface M in a complex projective space $\mathbb{C}P^n$ satisfying $\nabla_\xi A = 0$ is locally congruent to a real hypersurface of type A_1, A_2, that is, a tube over a totally geodesic complex submanifold $\mathbb{C}P^k$ with radius $0 < r < \frac{\pi}{2}$. In a quaternionic projective space $\mathbb{H}P^m$ Pérez [9] has considered the notion of $\nabla_{\xi_i} A = 0$, $i = 1, 2, 3$, for real hypersurfaces in $\mathbb{H}P^m$ and classified that M is locally congruent to of A_1, A_2-type, that is, a tube over $\mathbb{H}P^k$ with radius $0 < r < \frac{\pi}{4}$. Moreover, in a paper [10] due to Pérez and the present author we have considered the notion of $\nabla_{\xi_i} R = 0$, $i = 1,2,3$, where R denotes the curvature tensor of a real hypersurface M in $\mathbb{H}P^m$, and proved that M is locally congruent to a tube of radius $\frac{\pi}{4}$ over $\mathbb{H}P^k$.

But if we consider such notions in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, then its situations are quite different from the above ones.

Now in this paper let us consider a distribution $\mathfrak{F}$ combined by $[\xi]$ and $\mathfrak{D}^\perp$ in such a way that $\mathfrak{F} = [\xi] \cup \mathfrak{D}^\perp$ for a real hypersurface M in $G_2(\mathbb{C}^{m+1})$. Then naturally as in above classifications we can restrict the condition $\nabla A = 0$ to $\mathfrak{F}$, which means that the second fundamental tensor of M in $G_2(\mathbb{C}^{m+2})$ is parallel on $\mathfrak{F}$. Of course, this condition is much more weaker than $\nabla A = 0$ as in Theorem B. Then by virtue of Theorem A we assert the following remarkable facts :

THEOREM 1. *There do not exist any real hypersurface M in G_2 $(\mathbb{C}^{m+2})$ with the parallel second fundamental tensor on $\mathfrak{F}$ when ξ is invariant by the shape operator of M.*

When the structure vector field $[\xi]$ of a real hypersurface M in G_2 $(\mathbb{C}^{m+2})$ is invariant by the shape operator, that is, $A\xi = \alpha\xi$, a real hypersurface M is said to be a *Hopf* hypersurface. When the distribution $\mathfrak{D}^\perp$ of a hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, we say M a $\mathfrak{D}^\perp$-*invariant* hypersurface.

Theorem 1 means that in the class of Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ there do not exist any real hypersurface with parallel second fundamental tensor on $\mathfrak{F}$. Now we also assert the following.

THEOREM 2. *There do not exist any real hypersurface M in G_2 $(\mathbb{C}^{m+2})$ with the parallel second fundamental tensor on $\mathfrak{F}$ when $\mathfrak{D}^\perp$ is invariant by the shape operator of M.*

The condition $A[\xi] \subset [\xi]$ in Theorem 1 appears to be rather natural, and in fact there is a well-established theory for such hypersurfaces. Any tube around a complex submanifold in $\mathbb{C}P^m$ satisfies this geometrical condition. Cecil and Ryan proved in [6] that these tubes are essentially characterized by this feature. Here the word essentially refers to some additional condition on the focal map. So, roughly speaking, the theory of real hypersurfaces in $\mathbb{C}P^m$ with $A[\xi] \subset [\xi]$ is the theory of tubes around complex submanifolds in $\mathbb{C}P^m$.

The analogous question in quaternionic projective space $\mathbb{H}P^m$ leads to a surprise. The corresponding geometrical feature in Theorem 2 is that the three-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ on M, which is obtained by applying the almost Hermitian structures in the quaternionic Kähler structure $\mathfrak{J}$ of $\mathbb{H}P^m$ to a unit normal N, is invariant under A. In fact, every tube around a quaternionic submanifold of $\mathbb{H}P^m$ has this kind of geometrical feature. (Note that by a result of Alekseevskii [1] such a quaternionic submanifold is necessarily totally geodesic.) But the converse is not true. Berndt proved in [3] that also every tube around a totally geodesic $\mathbb{C}P^m$ in $\mathbb{H}P^m$ satisfies $A\mathfrak{D}^\perp \subset \mathfrak{D}^\perp$, and that there are no other ones. So the real hypersurfaces in $\mathbb{H}P^m$ with $A[\xi] \subset [\xi]$ are precisely the tubes around totally geodesic $\mathbb{H}P^k$, $k \in \{0, \dots, m-1\}$, and $\mathbb{C}P^m$.

Any tube around $G_2(\mathbb{C}^{m+1})$ has four distinct constant principal curvatures and might also be regarded as a tube around the focal set of $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, which is a totally geodesic $\mathbb{C}P^m$. Any tube around $\mathbb{H}P^n$ has five distinct constant principal curvatures, and the other focal set of the tube is a complex hypersurface in $G_2(\mathbb{C}^{m+2})$ which is a Riemannian homogeneous space isomorphic to $Sp(n+1)/(U(1) \times Sp(1) \times Sp(n-1))$. The two families of tubes together with their focal sets are just the orbits of the isometric actions of the subgroups $SU(m+1)$ and $Sp(n+1)$ of $SU(m+2)$, respectively.

In Section 0.2 we recall Riemannian geometry of two dimensional complex Grassmannian $G_2(\mathbb{C}^{m+2})$ and in Section 1 we will show the equation of Codazzi for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ explicitly. Then in Section 2 by the equation of Codazzi we find some fundamental formulas, which will be useful to prove Theorems 1 and 2, for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\nabla_\xi A = 0$ and $\nabla_{\xi_i} A = 0$, $i = 1, 2, 3$.

In Section 3 by using Theorem A and some formulas in previous sections we prove Theorem 1 which is a non-existence theorem for the class of Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Finally in Sections 4 and 5 we will prove Theorem 2 which is another non-existence theorem for real hypersurfaces satisfying $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$ in $G_2(\mathbb{C}^{m+2})$ with the parallel second fundamental tensor on $\mathfrak{F}$.

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0.2. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3],[4] and [5]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the three-dimensional complex projective space $\mathbb{C}P^3$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $Spin(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $tr(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken module three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$. Also this fact will be used frequently.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$. A non-zero tangent vector X of $G_2(\mathbb{C}^{m+2})$ is said to be singular if X is tangent to more than one flat of $G_2(\mathbb{C}^{m+2})$. In $G_2(\mathbb{C}^{m+2})$ there are two types of singular tangent vectors X which are characterized by the properties $JX \perp \mathfrak{J}X$ and $JX \in \mathfrak{J}X$. We will have to compute explicitly Jacobi vector fields along geodesics whose tangent vectors are all singular. For this we need the eigenvalues and eigenspaces of the Jacobi operator $\bar{R}_X := \bar{R}(.,X)X$. Let X be a unit vector tangent to $G_2(\mathbb{C}^{m+2})$. If $JX \perp \mathfrak{J}X$ then the eigenvalues and eigenspaces of $\bar{R}_X$ are

$$\begin{array}{ll} 0 & \mathbb{R}X \oplus \mathfrak{J}JX \\ 1 & (\mathbb{H}\mathbb{C}X)^\perp \\ 4 & \mathbb{R}JX \oplus \mathfrak{J}X\ , \end{array}$$

where $\mathbb{H}\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX \oplus \mathfrak{J}X \oplus \mathfrak{J}JX$. If $JX \in \mathfrak{J}X$, there exists an almost Hermitian structure J_1 in $\mathfrak{J}$ such that $JX = J_1X$. Then the eigenvalues and eigenspaces of $\bar{R}_X$ are

$$\begin{array}{ll} 0 & \mathbb{R}X \oplus \{Y \mid Y \perp \mathbb{H}X,\ JY = -J_1Y\}, \\ 2 & \mathbb{C}^\perp X \oplus \{Y \mid Y \perp \mathbb{H}X,\ JY = J_1Y\}, \\ 8 & \mathbb{R}JX\ , \end{array}$$

where $\mathbb{C}X$ and $\mathbb{H}X$ denote the complex and quaternionic span of X, respectively, and $\mathbb{C}^\perp X$ is the orthogonal complement of $\mathbb{C}X$ in $\mathbb{H}X$.

1. The Codazzi equation for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae from the Codazzi equation for a real hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$.

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ,ξ,η,g). Furthermore, let J_1,J_2,J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$\begin{aligned}
&(\nabla_X A)Y-(\nabla_Y A)X\\
&=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\\
&\quad+\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\}\\
&\quad+\sum_{\nu 1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}\\
&\quad+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu\ .
\end{aligned}$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned}
\phi_{\nu+1}\xi_\nu&=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\
\phi\xi_\nu&=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\
\phi_\nu\phi_{\nu+1}X&=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\
\phi_{\nu+1}\phi_\nu X&=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.
\end{aligned}\tag{1.1}$$

Then in this section let us give some basic formulas which will be used in the later.

Now let us put

$$JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas in Section 0.2 we have the following:

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{1.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{1.3}$$

$$\begin{aligned}(\nabla_X\phi_\nu)Y = &- q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX,Y)\xi_\nu.\end{aligned} \tag{1.4}$$

Summing up these formulas, we know that

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= \nabla_X(\phi\xi_\nu) \\ &= (\nabla_X\phi)\xi_\nu + \phi(\nabla_X\xi_\nu) \\ &= q_{\nu+2}(X)\phi_{\nu+1}\xi - q_{\nu+1}(X)\phi_{\nu+2}\xi + \phi_\nu\phi AX \\ &\quad - g(AX,\xi)\xi_\nu + \eta(\xi_\nu)AX.\end{aligned} \tag{1.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{1.6}$$

2. Some fundamental formulas

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $\nabla_\xi A = 0$ and $\nabla_{\xi_i} A = 0$, where we have put $\mathfrak{D}^\perp = \text{Span}\{\xi_1,\xi_2,\xi_3\}$. Then in this section let us give some basic formulas which will be used in the later.

If we put $Y = \xi$ into the equation of Codazzi, then by the assumption of $\nabla_\xi A = 0$, we have

$$(\nabla_X A)\xi = -\phi X + \Psi X, \tag{2.1}$$

where we have put

$$\begin{aligned}\Psi(X) = &\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\phi_\nu X - 2g(\phi_\nu X,\xi)\xi_\nu\} \\ &- \sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu.\end{aligned}$$

From this it follows that

$$\begin{aligned}\nabla_Y A\xi &= (\nabla_Y A)\xi + A\nabla_Y \xi \\ &= (\nabla_Y A)\xi + A\phi AY \\ &= -\phi Y + \psi Y + A\phi AY.\end{aligned}$$

Now differentiating this formula one more time, we have

$$\begin{aligned}&\nabla_X\nabla_Y A\xi - \nabla_{\nabla_X Y}A\xi \\ &= -(\nabla_X\phi)Y + (\nabla_X\Psi)Y \\ &\quad + (\nabla_X A)\phi AY + A(\nabla_X\phi)AY + A\phi(\nabla_X A)Y \\ &= -\eta(Y)AX + g(AX,Y)\xi + (\nabla_X\Psi)Y + (\nabla_X A)\phi AY \\ &\quad + A\{\eta(AY)AX - g(AX,AY)\xi\} + A\phi(\nabla_X A)Y.\end{aligned}$$

Then we have the following

(2.2)

$$\begin{aligned}R(X,Y)A\xi &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})A\xi \\ &= (\nabla_X A)\phi AY - (\nabla_Y A)\phi AX + \eta(AY)A^2X - \eta(AX)A^2Y \\ &\quad - \eta(Y)AX + \eta(X)AY + (\nabla_X\Psi)Y - (\nabla_Y\Psi)X \\ &\quad + A\phi\{(\nabla_X A)Y - (\nabla_Y A)X\}.\end{aligned}$$

Here the derivative of Ψ is given by

(2.3)

$$\begin{aligned}(\nabla_Y\Psi)X = \sum\nolimits_{\nu=1}^3 &\{(\nabla_Y\eta_\nu)X\phi_\nu\xi + \eta_\nu(X)\nabla_Y(\phi_\nu\xi) \\ &- Y(\eta_\nu(\xi))\phi_\nu X - \eta_\nu(\xi)(\nabla_Y\phi_\nu)X \\ &- 2g((\nabla_Y\phi_\nu)X,\xi)\xi_\nu - 2g(\phi_\nu X,\nabla_Y\xi)\xi_\nu \\ &- 2g(\phi_\nu X,\xi)\nabla_Y\xi_\nu\} - \sum\nolimits_{\nu=1}^3\{(\nabla_Y\eta_\nu)\phi X \\ &+ \eta_\nu((\nabla_Y\phi)X)\}\xi_\nu - \sum\nolimits_{\nu=1}^3 \eta_\nu(\phi X)\nabla_Y\xi_\nu.\end{aligned}$$

On the other hand, from (1.2), (1.3) and (1.5) we have the following formulas

$$\begin{aligned}(\nabla_Y\eta_\nu)X &= Y(\eta_\nu(X)) - \eta_\nu(\nabla_Y X) \\ &= q_{\nu+2}(Y)\eta_{\nu+1}(X) - q_{\nu+2}(Y)\eta_{\nu+2}(X) + g(\phi_\nu AY, X), \\ \eta_\nu((\nabla_Y\phi)X) &= \eta(X)\eta_\nu(AY) - g(AY,X)\eta_\nu(\xi).\end{aligned}$$

Substituting these formulas into (2.3) and taking an inner product with ξ, we have

(2.4)

$$
\begin{aligned}
&g((\nabla_X\Psi)Y,\xi)\\
&=\sum\nolimits_\nu\{\eta_\nu(Y)\eta(\phi_\nu\phi AX)-X(\eta_\nu(\xi))\eta(\phi_\nu Y)\}\\
&\quad-3\sum\eta_\nu(\xi)\{-q_{\nu+1}(X)\eta(\phi_{\nu+2})+q_{\nu+2}(X)\eta(\phi_{\nu+1}Y)+\eta_\nu(Y)\eta(AX)\\
&\quad-g(AX,Y)\eta_\nu(\xi)\}-2\sum g(\phi_\nu Y,\phi AX)\eta_\nu(\xi)\\
&\quad-3\sum g(\phi_\nu Y,\xi)\{q_{\nu+2}(X)\eta_{\nu+1}(\xi)-q_{\nu+1}(X)\eta_{\nu+2}(\xi)+\eta(\phi_\nu AX)\}\\
&\quad-\sum\eta_\nu(\xi)\{q_{\nu+2}(X)\eta_{\nu+1}(\phi Y)-q_{\nu+1}(X)\eta_{\nu+2}(\phi Y)\\
&\quad+g(\phi_\nu AX,\phi Y)\eta(Y)\eta_\nu(AX)-g(AX,Y)\eta_\nu(\xi)\}.
\end{aligned}
$$

On the other hand, by (1.1), (1.2) and (1.4) we know

$$
\begin{aligned}
X(\eta_\nu(\xi))&=g(\nabla_X\xi,\xi_\nu)+g(\xi,\nabla_X\xi_\nu)\\
&=\eta_\nu(\phi AX)+q_{\nu+2}(X)\eta(\xi_{\nu+1})-q_{\nu+1}(X)\eta(\xi_{\nu+2})\\
&\quad+\eta(\phi_\nu AX),\\
\eta(\phi_\nu\phi AX)&=-\eta_\nu(AX)+\eta(AX)\eta_\nu(\xi),\\
g(\phi_\nu Y,\phi AX)&=g(\phi_\nu AX,\phi Y)-\eta_\nu(Y)\eta(AX)+\eta(Y)\eta_\nu(AX).
\end{aligned}
$$

Substituting these formulas into (2.4), we have

(2.5)

$$
\begin{aligned}
&g((\nabla_X\Psi)Y,\xi)\\
&=-\sum\nolimits_\nu\eta_\nu(Y)\eta_\nu(AX)-\sum\nolimits_\nu\{2\eta_\nu(\phi AX)+q_{\nu+2}(X)\eta(\xi_{\nu+1})\\
&\quad-q_{\nu+1}(X)\eta(\xi_{\nu+2})\}\eta(\phi_\nu Y)\\
&\quad-4\sum\nolimits_\nu\eta_\nu(\xi)\{-q_{\nu+1}(X)\eta(\phi_{\nu+2}Y)+q_{\nu+2}(X)\eta(\phi_{\nu+1}Y)\\
&\quad-g(AX,Y)\eta_\nu(\xi)\}\\
&\quad-3\sum\nolimits_\nu\eta_\nu(\xi)\eta(Y)\eta_\nu(AX)-3\sum\nolimits_\nu g(\phi_\nu AX,\phi Y)\eta_\nu(\xi).
\end{aligned}
$$

From this, taking skew-symmetric part, we have

$$
\begin{aligned}
(2.6)\qquad &g((\nabla_X\Psi)Y-(\nabla_Y\Psi)X,\xi)\\
&=-\sum\nolimits_\nu\{\eta_\nu(AX)\eta_\nu(Y)-\eta_\nu(AY)\eta_\nu(X)\}\\
&\quad-2\sum\nolimits_\nu\{\eta_\nu(\phi AX)\eta(\phi_\nu Y)-\eta_\nu(\phi AY)\eta(\phi_\nu X)\}
\end{aligned}
$$

$$
\begin{aligned}
&+4\sum\nolimits_{\nu}\eta_\nu(\xi)\{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y)-q_{\nu+1}(Y)\eta(\phi_{\nu+2}X)\}\\
&-4\sum\nolimits_{\nu}\eta_\nu(\xi)\{q_{\nu+2}(X)\eta(\phi_{\nu+1}Y)-q_{\nu+2}(Y)\eta(\phi_{\nu+1}X)\}\\
&-3\sum\nolimits_{\nu}\eta_\nu(\xi)\{\eta(Y)\eta_\nu(AX)-\eta(X)\eta_\nu(AY)\\
&-g(\phi_\nu AX,\phi Y)+g(\phi_\nu AY,\phi X)\}\\
&-\sum\nolimits_{\nu}\eta(\xi_{\nu+1})\{q_{\nu+2}(X)\eta(\phi_\nu Y)-q_{\nu+2}(Y)\eta(\phi_\nu X)\}\\
&+\sum\nolimits_{\nu}\eta(\xi_{\nu+2})\{q_{\nu+1}(X)\eta(\phi_\nu Y)-q_{\nu+1}(Y)\eta(\phi_\nu X)\}.
\end{aligned}
$$

Now let us take an inner product (2.2) with ξ, we have

(2.7)

$$
\begin{aligned}
&g(R(X,Y)A\xi,\xi)\\
&=g((\nabla_X A)\xi,\phi AY)-g((\nabla_Y A)\xi,\phi AX)+\eta(A^2X)\eta(AY)\\
&\quad-\eta(AX)\eta(A^2Y)-\eta(Y)\eta(AX)+\eta(X)\eta(AY)\\
&\quad+g((\nabla_X\Psi)Y-(\nabla_Y\Psi)X,\xi)\\
&\quad+g(A\phi\{(\nabla_X A)Y-(\nabla_Y A)X\},\xi).
\end{aligned}
$$

The first term of the right side of (2.7) gives

$$
\begin{aligned}
&g((\nabla_X A)\xi,\phi AY)\\
&=\ -g(AX,Y)+\eta(X)\eta(AY)+\sum\nolimits_{\nu}\{\eta_\nu(X)g(\phi_\nu\xi,\phi AY)\\
&\quad-\eta_\nu(\xi)g(\phi_\nu X,\phi AY)-3g(\phi_\nu X,\xi)\eta_\nu(\phi AY)\}.
\end{aligned}
$$

So it follows

$$
\begin{aligned}
&g((\nabla_X A)\xi,\phi AY)-g(\nabla_Y A)\xi,\phi AX)\\
&=\eta(X)\eta(AY)-\eta(Y)\eta(AX)\\
&\quad+\sum\nolimits_{\nu}\{\eta_\nu(X)g(\phi_\nu\xi,\phi AY)-\eta_\nu(Y)g(\phi_\nu,\phi AX)\}\\
&\quad-\sum\nolimits_{\nu}\eta_\nu(\xi)\{g(\phi_\nu X,\phi AY)-g(\phi_\nu Y,\phi AX)\}\\
&\quad-3\sum\nolimits_{\nu}\{\eta(\phi_\nu X)\eta_\nu(\phi AY)-\eta(\phi_\nu Y)\eta_\nu(\phi AX)\}.
\end{aligned}
\tag{2.8}
$$

Moreover, the last term of (2.7) gives

(2.9)

$$
\begin{aligned}
&g(A\phi\{(\nabla_X A)Y-(\nabla_Y A)X\},\xi)\\
&=\ -g((\nabla_X A)Y-(\nabla_Y A)X,\phi A\xi)\\
&=\ -\eta(X)\eta(AY)+\eta(Y)\eta(AX)-\sum\nolimits_{\nu}\{\eta_\nu(X)g(\phi_\nu Y,\phi A\xi)
\end{aligned}
$$

$$- \eta_\nu(Y)g(\phi_\nu X, \phi A\xi) - 2g(\phi_\nu X, Y)g(\xi_\nu, \phi A\xi)\}$$
$$- \sum\nolimits_\nu \{\eta_\nu(\phi X)\eta(A\phi_\nu Y) - \eta_\nu(\phi Y)\eta(A\phi_\nu X)\}.$$

Now substituting (2.6), (2.8) and (2.9) into (2.7), we have

(2.10)
$$\begin{aligned}
&g(R(X,Y)A\xi,\xi)\\
&= \sum\nolimits_\nu \{\eta_\nu(X)g(\phi_\nu\xi, \phi AY) - \eta_\nu(Y)g(\phi_\nu, \phi AX)\}\\
&- \sum\nolimits_\nu \eta_\nu(\xi)\{g(\phi_\nu X, \phi AY) - g(\phi_\nu Y, \phi AX)\}\\
&- \sum\nolimits_\nu \{\eta(\phi_\nu X)\eta_\nu(\phi AY) - \eta(\phi_\nu Y)\eta_\nu(\phi AX)\}\\
&+ \eta(A^2X)\eta(AY) - \eta(AX)\eta(A^2Y) - \eta(Y)\eta(AX) + \eta(X)\eta(AY)\\
&- \sum\nolimits_\nu \{\eta_\nu(X)g(\phi_\nu Y, \phi A\xi) - \eta_\nu(Y)g(\phi_\nu X, \phi A\xi)\\
&- 2g(\phi_\nu X, Y)g(\xi_\nu, \phi A\xi)\}\\
&- \sum\nolimits_\nu \{\eta_\nu(\phi X)\eta(A\phi_\nu Y) - \eta_\nu(\phi Y)\eta(A\phi_\nu X)\}\\
&- \sum\nolimits_\nu \{\eta_\nu(AX)\eta_\nu(Y) - \eta_\nu(AY)\eta_\nu(X)\}\\
&+ 4\sum \eta_\nu(\xi)\{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y)\\
&- q_{\nu+1}(Y)\eta(\phi_{\nu+2}X) - q_{\nu+2}(X)\eta(\phi_{\nu+1}Y)\\
&+ q_{\nu+2}(Y)\eta(\phi_{\nu+1}X)\}\\
&- 3\sum \eta_\nu(\xi)\{\eta(Y)\eta_\nu(AX) - \eta(X)\eta_\nu(AY) - g(\phi_\nu AX, \phi Y)\\
&+ g(\phi_\nu AY, \phi X)\} - \sum \eta(\xi_{\nu+1})\{q_{\nu+2}(X)\eta(\phi_\nu Y) - q_{\nu+2}(Y)\eta(\phi_\nu X)\}\\
&+ \sum \eta(\xi_{\nu+2})\{q_{\nu+1}(X)\eta(\phi_\nu Y) - q_{\nu+1}(Y)\eta(\phi_\nu X)\}.
\end{aligned}$$

On the other hand, by the equation of Gauss, the curvature tensor $R(X,Y)Z$ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is given by

$$\begin{aligned}
&R(X,Y)Z\\
&= g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y\\
&\quad - 2g(\phi X, Y)\phi Z\\
&\quad + \sum\nolimits_\nu \{g(\phi_\nu Y, Z)\phi_\nu X - g(\phi_\nu X, Z)\phi_\nu Y - 2g(\phi_\nu X, Y)\phi_\nu Z\}
\end{aligned}$$

$$+\sum_\nu\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X-g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\}$$
$$-\sum_\nu\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X-\eta(X)\eta_\nu(Z)\phi_\nu\phi Y\}$$
$$-\sum_\nu\{\eta(X)g(\phi_\nu\phi Y,Z)-\eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu$$
$$+g(AY,Z)AX-g(AX,Z)AY.$$

From this together with the formula

$$\eta(\phi_\nu\phi X)g(\phi_\nu\phi Y,A\xi)=\{-\eta_\nu(X)+\eta(X)\eta_\nu(\xi)\}g(\phi_\nu\phi Y,A\xi)$$

we have

(2.11)
$$\begin{aligned}&g(R(X,Y)A\xi,\xi)\\
&=\eta(X)\eta(AY)-\eta(Y)\eta(AX)\\
&\quad+\sum_\nu\{\eta(\phi_\nu X)g(\phi_\nu Y,A\xi)-\eta(\phi_\nu Y)g(\phi_\nu X,A\xi)\\
&\quad-2\eta(\phi_\nu A\xi)g(\phi_\nu X,Y)\}\\
&\quad-\sum_\nu\{\eta_\nu(X)g(\phi_\nu\phi Y,A\xi)-\eta_\nu(Y)g(\phi_\nu\phi X,A\xi)\}\\
&\quad+\sum_\nu\{\eta(Y)\eta_\nu(X)\eta_\nu(A\xi)-\eta(X)\eta_\nu(Y)\eta_\nu(A\xi)\}\\
&\quad+\eta(A^2Y)\eta(AX)-\eta(A^2X)\eta(AY).\end{aligned}$$

From this, if we use the following formulas such that

$$\begin{aligned}g(\phi_\nu\phi Y,A\xi)&=-g(\phi_\nu Y,\phi A\xi)+\eta(Y)\eta_\nu(A\xi)-\eta_\nu(Y)\eta(A\xi)\\
g(\phi_\nu\xi,\phi AY)&=g(\phi\xi_\nu,\phi AY)=\eta_\nu(AY)-\eta(AY)\eta(\xi_\nu),\\
g(\phi_\nu X,\phi AY)&=-g(\phi\phi_\nu X,AY)=-g(\phi_\nu\phi X,AY)+\eta_\nu(X)\eta(AY)\\
&\quad-\eta(X)\eta_\nu(AY)\end{aligned}$$

and compare with (2.10), then we have finally as follows:

(2.12)
$$\begin{aligned}0&=2\sum_\nu\eta_\nu(\xi)\{g(\phi_\nu AY,\phi X)-g(\phi_\nu AX,\phi Y)\}\\
&\quad-2\sum_\nu\eta_\nu(X)\eta(AY)\eta(\xi_\nu)+2\sum_\nu\eta_\nu(Y)\eta(AX)\eta(\xi_\nu)\\
&\quad+2\eta(A^2X)\eta(AY)-2\eta(AX)\eta(A^2Y)\\
&\quad-2\sum_\nu\{\eta_\nu(X)g(\phi_\nu Y,\phi A\xi)-\eta_\nu(Y)g(\phi_\nu X,\phi A\xi)\\
&\quad-2g(\phi_\nu X,Y)g(\xi_\nu,\phi A\xi)\}\end{aligned}$$

$$
\begin{aligned}
&-2\sum\nolimits_{\nu}\{\eta_\nu(\phi X)\eta_\nu(A\phi_\nu Y)-\eta_\nu(\phi Y)\eta(A\phi_\nu X)\}\\
&-2\sum\nolimits_{\nu}\{\eta_\nu(AX)\eta_\nu(Y)-\eta_\nu(AY)\eta_\nu(X)\}\\
&+4\sum\nolimits_{\nu}\eta_\nu(\xi)\{q_{\nu+1}(X)\eta(\phi_{\nu+2}Y)-q_{\nu+1}(Y)\eta(\phi_{\nu+2}X)\}\\
&-4\sum\nolimits_{\nu}\eta_\nu(\xi)\{q_{\nu+2}(X)\eta(\phi_{\nu+1}Y)-q_{\nu+2}(Y)\eta(\phi_{\nu+1}X)\}\\
&-4\sum\nolimits_{\nu}\eta_\nu(\xi)\{\eta(Y)\eta_\nu(AX)-\eta(X)\eta_\nu(AY)\}\\
&-\sum\nolimits_{\nu}\{\eta_\nu(\phi X)\eta_\nu(\phi AY)-\eta_\nu(\phi Y)\eta_\nu(\phi AX)\}\\
&-\sum\nolimits_{\nu}\eta(\xi_{\nu+1})\{q_{\nu+2}(X)\eta(\phi_\nu Y)-q_{\nu+2}(Y)\eta(\phi_\nu X)\}\\
&+\sum\nolimits_{\nu}\eta(\xi_{\nu+2})\{q_{\nu+1}(X)\eta(\phi_\nu Y)-q_{\nu+1}(Y)\eta(\phi_\nu X)\}.
\end{aligned}
$$

3. Hopf real hypersurfaces in $G_2(\mathbb{C}^{n+2})$

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $\nabla_\xi A=0$ and $\nabla_{\xi_i}A=0$, where we have put $\mathfrak{D}^\perp=\text{Span}\{\xi_1,\xi_2,\xi_3\}$. Then from the equation of Codazzi we have

$$
\begin{aligned}
(\nabla_X A)\xi_i &= \eta(X)\phi\xi_i-\eta(\xi_i)\phi X-2g(\phi X,\xi_i)\xi\\
&\quad+\sum\nolimits_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi_i-\eta_\nu(\xi_i)\phi_\nu X-2g(\phi_\nu X,\xi_i)\xi_\nu\}\\
&\quad+\sum\nolimits_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi\xi_i-\eta_\nu(\phi\xi_i)\phi_\nu\phi X\}\\
&\quad+\sum\nolimits_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi\xi_i)-\eta(\xi_i)\eta_\nu(\phi X)\}\xi_\nu.
\end{aligned}
$$

From this, putting $X=\xi$ and using the assumption of $\nabla_\xi A=0$, we have

(3.1)

$$
\begin{aligned}
0=&(\nabla_\xi A)\xi_i\\
=&\phi\xi_i+\sum\nolimits_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\xi_i-\eta_\nu(\xi_i)\phi_\nu\xi-2g(\phi_\nu\xi,\xi_i)\xi_\nu\}\\
&+\sum\nolimits_{\nu=1}^{3}\eta_\nu(\phi\xi_i)\xi_\nu.
\end{aligned}
$$

Thus for $i=1$ in (3.1) we have

$$0=-2g(\xi,\xi_3)\xi_2+2g(\xi,\xi_2)\xi_3. \tag{3.2}$$

This implies ξ is orthogonal to ξ_2 and ξ_3. Moreover, by putting $i=2$ in (3.1), we can assert ξ is orthogonal to ξ_1 and ξ_3. Thus we have

LEMMA 3.1. *Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$ satisfying $\nabla_\xi A = 0$ and $\nabla_{\xi_i} A = 0$ for $i = 1,2,3$. Then $\xi \in \mathfrak{D}$.*

Thus by Lemma 3.1 we have known that $\xi \in \mathfrak{D}$. Then the formula (2.12) reduces to

$$\begin{aligned}
0 = {} & 2\eta(A^2X)\eta(AY) - 2\eta(AX)\eta(A^2Y) \\
& - 2\sum_\nu \{\eta_\nu(X)g(\phi_\nu Y, \phi A\xi) - \eta_\nu(Y)g(\phi_\nu X, \phi A\xi) \\
& - 2g(\phi_\nu X, Y)g(\xi_\nu, \phi A\xi)\} \\
& - 2\sum_\nu \{\eta_\nu(\phi X)\eta(A\phi_\nu Y) - \eta_\nu(\phi Y)\eta(A\phi_\nu X)\} \\
& - 2\sum_\nu \{\eta_\nu(AX)\eta_\nu(Y) - \eta_\nu(AY)\eta_\nu(X)\} \\
& - \sum_\nu \{\eta_\nu(\phi X)\eta_\nu(\phi AY) - \eta_\nu(\phi Y)\eta_\nu(\phi AX)\}.
\end{aligned} \tag{3.3}$$

Let us put $A\xi = \alpha\xi + \beta U$, where $\alpha = \eta(A\xi)$. Then putting $X = \xi$ in (3.3), we have

$$\begin{aligned}
0 = {} & 2\eta(A^2\xi)\eta(AY) - 2\eta(A\xi)\eta(A^2Y) \\
& + 2\{\sum_\nu \eta_\nu(Y)g(\phi_\nu\xi, \phi A\xi) + 2\sum_\nu g(\phi_\nu\xi, Y)g(\xi_\nu, \phi A\xi)\} \\
& + 2\sum_\nu \eta_\nu(\phi Y)\eta(A\phi_\nu\xi) - 2\sum_\nu \eta_\nu(A\xi)\eta_\nu(Y) \\
& + \sum_\nu \eta_\nu(\phi Y)\eta_\nu(\phi A\xi).
\end{aligned}$$

From this, if we substitute the formulas

$$\eta(A\phi_\nu\xi) = -\beta g(\phi U, \xi_\nu),$$

$$\eta_\nu(\phi A\xi) = \beta g(\phi U, \xi_\nu),$$

and

$$\eta_\nu(A\xi) = \beta g(U, \xi_\nu),$$

then

$$2\eta(A\xi)A^2\xi = 2\eta(A^2\xi)A\xi + 5\beta\sum_\nu \eta_\nu(\phi U)\phi\xi_\nu. \tag{3.4}$$

Hereafter in this section let us consider *ξ-invariant* hypersurface in $G_2(\mathbb{C}^{n+2})$, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi, \xi)$. In such a case a real hypersurface M is said to be a *Hopf* hypersurface in $G_2(\mathbb{C}^{m+2})$.

Now we have assumed that M is a Hopf hypersurface. Then by definition its structure vector ξ is principal. So from this fact together with Lemma 3.1 we know that $\eta_\nu(A\xi)=0$, $\nu=1,2,3$.

Putting $X=\xi_1$ in (3.3) and using Lemma 3.1, we have

(3.5)
$$\begin{aligned}0=&-2\{g(\phi_1 Y,\phi A\xi)+\eta_2(Y)g(\xi_3,\phi A\xi)-\eta_3(Y)\eta_2(\phi A\xi)\\&+2\eta_3(Y)\eta_2(\phi A\xi)-2\eta_2(Y)\eta_3(\phi A\xi)\}\\&+2\{\eta_2(\phi Y)\eta(A\xi_3)+\eta_3(\phi Y)\eta(A\xi_2)\}\\&-2\{\eta_1(A\xi_1)\eta_1(Y)+\eta_2(A\xi_1)\eta_2(Y)-\eta_3(A\xi_1)\eta_3(Y)-\eta_1(AY)\}\\&+\{\eta_1(\phi Y)\eta_1(\phi A\xi_1)+\eta_2(\phi Y)\eta_2(\phi A\xi_1)+\eta_3(\phi Y)\eta_3(\phi A\xi_1)\}.\end{aligned}$$

Now we are going to prove the following

LEMMA 3.2. *Let M be a Hopf hypersurface in $G_2(C^{m+1})$ satisfying $\nabla_{\xi_i}A=0$ and $\nabla_\xi A=0$, $i=1,2,3$. Then $g(A\mathfrak{D},\mathfrak{D}^\perp)=0$.*

Proof. By Lemma 3.1, we know $\xi\in\mathfrak{D}$. From this together with (3.5) and ξ is principal we have

(3.6)
$$\begin{aligned}0=&-2\{\eta_1(A\xi_1)\eta_1(Y)+\eta_2(A\xi_1)\eta_2(Y)-\eta_3(A\xi_1)\eta_3(Y)-\eta_1(AY)\}\\&+\{\eta_1(\phi Y)\eta_1(\phi A\xi_1)+\eta_2(\phi Y)\eta_2(\phi A\xi_1)+\eta_3(\phi Y)\eta_3(\phi A\xi_1)\}.\end{aligned}$$

From this, putting $Y=\xi_3$, we have

$$4\eta_3(A\xi_1)=0.$$

Similarly, we can assert $\eta_i(A\xi_j)=0$ for any distinct i and j. So in order to verify the above assertion let us put

$$A\xi_i=\alpha_i\xi_i+\beta_i X_i$$

for some $X_i\in\mathcal{D}$. Then

$$\begin{aligned}\eta_1(\phi A\xi_1)=\beta_1 g(\xi_1,\phi X_1),\ \eta_2(\phi A\xi_1)=\beta_1 g(\xi_2,\phi X_1),\ \eta_3(\phi A\xi_1)\\=\beta_1 g(\xi_3,\phi X_1).\end{aligned}$$

Let us construct an open set $\mathfrak{U}=\{p\in M|\beta_1(p)\neq 0\}$. Then on this open $\mathfrak{U}$ if we substitute the above formulas into (3.6), we have for some $X_1\in\mathfrak{D}$

$$\begin{aligned}0=&-2\{\eta_1(A\xi_1)\eta_1(Y)-\eta_1(AY)\}\\&+\beta_1\{\eta_1(\phi Y)\eta_1(\phi X_1)+\eta_2(\phi Y)\eta_2(\phi X_1)+\eta_3(\phi Y)\eta_3(\phi X_1)\}.\end{aligned}$$

From this, putting $Y = X_1$, we have

$$2\eta_1(AX_1) + \beta_1\{\eta_1(\phi X_1)^2 + \eta_2(\phi X_1)^2 + \eta_3(\phi X_1)^2\} = 0.$$

Since $\eta_1(AX_1) = \beta_1 \neq 0$, the above equation implies $\sum_i \eta_i(\phi X_i)^2 = -2$, which makes a contradiction. That is, there does not exist such an open set. So we should have $\beta_1 = 0$. Similarly, if there exists any open set $\mathfrak{V} = \{p \in M | \beta_2(p) \neq 0\}$ or $\mathfrak{W} = \{p \in M | \beta_3(p) \neq 0\}$, we can also make a contradiction. From this we conclude $\beta_i = 0$, i.e, $A\xi_i = \alpha_i \xi_i$, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. □

Then by Lemmas 3.1 and 3.2 we are able to introduce a Proposition given in the paper of Berndt and the present author ([4]) as follows:

PROPOSITION 3.1. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\ ,\ \beta = 2\cot(2r)\ ,\ \gamma = 0\ ,\ \lambda = \cot(r)\ ,\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1\ ,\ m(\beta) = 3 = m(\gamma)\ ,\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi\ ,\ T_\beta = \mathfrak{J}J\xi\ ,\ T_\gamma = \mathfrak{J}\xi\ ,\ T_\lambda\ ,\ T_\mu\ ,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp\ ,\ \mathfrak{J}T_\lambda = T_\lambda\ ,\ \mathfrak{J}T_\mu = T_\mu\ ,\ JT_\lambda = T_\mu\ .$$

Now let us consider a unit eigenvector $Y \in T_\lambda$. Then by Proposition 3.1 we write the following formulas:

$$\begin{aligned} AY &= \cot rY, \quad A\phi Y = -\tan r\phi Y, \quad \phi AY = \cot r\phi Y,\\ A\phi AY &= \cot rA\phi Y = -\phi Y. \end{aligned}$$

So from these formulas and the equation of Codazzi it follows that for any $Y \in T_\lambda$

$$\begin{aligned} 0 &= (\nabla_\xi A)Y \\ &= (\nabla_Y A)\xi + \eta(\xi)\phi Y \\ &= (\alpha I - A)\phi AY + \phi Y \\ &= \alpha\phi AY - A\phi AY + \phi Y \\ &= \{-2\tan 2r \cdot \cot r + 2\}\phi Y \end{aligned}$$

where in the second equality we have used the fact that

$$\mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu.$$

Then it gives $\tan^2 r + 1 = 0$, which makes a contradiction. Finally we conclude that there do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{n+2})$ with parallel second fundamental tensor along the distribution $\mathfrak{F} = \xi \oplus \mathfrak{D}^\perp$. This completes the proof of our Theorem 1.

4. $\mathfrak{D}^\perp$-invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now as a classification problem concerned with curvature adapted real hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ we assert the following

LEMMA 4.1. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $\nabla_{\mathfrak{F}} A = 0$, $\mathfrak{F} = \xi \oplus \mathfrak{D}^\perp$. Then $\phi U \in \mathfrak{D}^\perp$ or ξ is principal.*

By virtue of Lemma 4.1, hereafter unless otherwise sated, in this section we always refer that $\phi U \in \mathfrak{D}^\perp$ when the structure vector ξ is not principal.

Proof. From the hypothesis we know that by Lemma 3.1

$$A\xi_i = \alpha_i \xi_i, \quad \text{and} \quad \xi, \phi\xi_i, A\xi, A^2\xi, U \in \mathfrak{D}.$$

From these formulas, if we put $X = \xi_1$ in (3.3), we have

$$\begin{aligned} 0 = &- 2\{g(\phi_1 Y, \phi A\xi) + \eta_2(Y)g(\xi_3, \phi A\xi) - \eta_3(Y)\eta_2(\phi A\xi) \\ &+ 2\eta_3(Y)\eta_2(\phi A\xi) - 2\eta_2(Y)\eta_3(\phi A\xi)\}. \end{aligned}$$

Now we assume ξ is not principal. Then if we put $A\xi = \alpha\xi + \beta U$ in the above formula, then the function $\beta \neq 0$ implies

$$g(\phi_1 Y, \phi U) + \eta_3(Y)\eta_2(\phi U) - \eta_2(Y)\eta_3(\phi U) = 0.$$

From this, replacing Y by $\phi_1 Y$, $Y \in \mathfrak{D}$, we have $g(Y, \phi U) = 0$. So $\phi U \in \mathfrak{D}^\perp$. □

Now next let us prove the following

THEOREM 4.2. *Let M be a $\mathfrak{D}^\perp$-invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $\nabla_{\xi_i} A = 0$ and $\nabla_\xi A = 0$, then its structure vector ξ is principal.*

Proof. By Lemma 3.1 we know $\xi \in \mathfrak{D}$. Now let us put

$$A\xi = \alpha\xi + \beta U,$$

where $\alpha = \eta(A\xi)$. Then under the same situation as in (3.4) we find the formula

$$2\eta(A\xi)A^2\xi = 2\eta(A^2\xi)A\xi + 5\beta\sum_\nu \eta_\nu(\phi U)\phi\xi_\nu.$$

Now in order to prove this Theorem 4.2 we consider two cases such that $\eta(A\xi) = 0$ and $\eta(A\xi) \neq 0$. For the case where $\eta(A\xi) \neq 0$ in Theorem 4.2 we will prove it in Section 5.

Now in this section we only consider the case $\eta(A\xi) = 0$. Then in this case let us consider the following two subcases:

Case I: $\beta = 0$.

Then in this case $A\xi = 0$ implies our assertion.

Case II: $\beta \neq 0$.

Then in this case we know

$$\xi, \phi\xi_1, \phi\xi_2, \phi\xi_3 \in \mathfrak{D}.$$

Bearing this in mind, (3.4) implies

$$(*) \qquad \begin{aligned} 0 &= 2\beta^2 A\xi + 5\beta\sum_\nu \eta_\nu(\phi U)\phi\xi_\nu \\ &= 2\beta^3 U + 5\beta\sum_\nu \eta_\nu(\phi U)\phi\xi_\nu. \end{aligned}$$

From this, if we apply the operator ϕ, we have

$$0 = \beta(2\beta^2 - 5)\eta_\nu(\phi U).$$

Thus for a case where $\beta^2 \neq \frac{5}{2}$ we know $\phi U \in \mathfrak{D}$. This makes a contradiction to $\phi U \in \mathfrak{D}^\perp$ in Lemma 4.1.

Now hereafter unless otherwise stated, in this Case II let us only consider the case $\beta^2 = \frac{5}{2}$. Then also in this situation we will show a contradiction. Let us prove this fact as the following. From $\nabla_\xi A = 0$ together with $A\xi = \beta U$, where $\beta = \text{const}$, it follows that

$$\begin{aligned}(\nabla_X A)\xi =& \nabla_X(A\xi) - A\nabla_X\xi \\ =& \beta\nabla_X U - A\phi AX \\ =& -\phi X + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi - 2g(\phi_\nu X, \xi)\xi_\nu\} \\ & -\sum_{\nu=1}^{3}\eta_\nu(\phi X)\xi_\nu,\end{aligned}$$

where in the third equality we have used the equation of Codazzi. From this, if we take an inner product with ξ, then we have for any X in T_xM

$$\beta g(\nabla_X U, \xi) = g(A\phi AX, \xi) = \beta g(\phi AX, U).$$

On the other hand, we have

$$\beta g(\nabla_X U, \xi) = -\beta g(U, \nabla_X \xi) = -\beta g(U, \phi AX).$$

From this and $\beta \neq 0$, we know

$$A\phi U = 0. \tag{4.1}$$

On the other hand, from (*) together with the fact that $2\beta^2 = 5$ we know

$$U + \sum_\nu \eta_\nu(\phi U)\phi\xi_\nu = 0. \tag{4.2}$$

Thus the formulas $A\phi U = 0$ and (4.2) imply

$$0 = A\phi U = \sum \eta_\nu(\phi U)A\xi_\nu = \sum \alpha_\nu \eta_\nu(\phi U)\xi_\nu.$$

That is,

$$\alpha_1\eta_1(\phi U) = \alpha_2\eta_2(\phi U) = \alpha_3\eta_3(\phi U) = 0. \tag{4.3}$$

Sub II.1: $\alpha_1, \alpha_2, \alpha_3 \neq 0$.

Then (4.3) implies

$$\eta_i(\phi U) = 0 \text{ for } i = 1, 2, 3.$$

This contradicts the fact $\sum_{i=1}^{3} \eta_i(\phi U)^2 = 1$.

Sub II.2: $\alpha_1 = 0,\ \alpha_2, \alpha_3 \neq 0$.

That is, one of $\alpha_1, \alpha_2, \alpha_3$ is vanishing. Then (4.3) implies

$$\eta_2(\phi U) = \eta_3(\phi U) = 0.$$

So $\eta_1(\phi U) = \pm 1$. That is, we can put $\phi U = \xi_1$. Then by the equation of Codazzi and $A\xi_1 = 0$, we have

(4.4)

$$\begin{aligned} &\eta(X)\phi\xi_1 - \eta(\xi_1)\phi X - 2g(\phi X, \xi_1)\xi \\ &\quad + \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu\xi_1 - \eta_\nu(\xi_1)\phi_\nu X - 2g(\phi_\nu X, \xi_1)\xi_\nu\} \\ &\quad + \sum_{\nu=1}^{3} \{\eta_\nu(\phi X)\phi_\nu\phi\xi_1 - \eta_\nu(\phi\xi_1)\phi_\nu\phi X\} \\ &\quad + \sum_{\nu=1}^{3} \{\eta(X)\eta_\nu(\phi\xi_1) - \eta(\xi_1)\eta_\nu(\phi X)\}\xi_\nu \\ &= (\nabla_X A)\xi_1 = -A(\nabla_X \xi_1) \\ &= -A\{q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX\} \\ &= -\alpha_2 q_3(X)\xi_2 + \alpha_3 q_2(X)\xi_3 - A\phi_1 AX. \end{aligned}$$

On the other hand, from the assumption $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$ i.e., $A\xi_i = \alpha_i\xi_i$ and the equation of Codazzi it follows

$$(\alpha_i - \alpha_j)q_k(X) = 0$$

for any X in $\mathcal{D}$. This means

$$\text{(4.5)} \qquad \begin{cases} \alpha_1 = \alpha_2 & \text{or} \quad q_3|_{\mathfrak{D}} = 0, \\ \alpha_2 = \alpha_3 & \text{or} \quad q_1|_{\mathfrak{D}} = 0, \\ \alpha_3 = \alpha_1 & \text{or} \quad q_2|_{\mathfrak{D}} = 0. \end{cases}$$

Thus by (4.5), if we consider for any $X \in \mathfrak{D}$, we know that $q_2|_{\mathfrak{D}} = q_3|_{\mathfrak{D}} = 0$ in Sub II.2. So (4.4) implies

(4.6)

$$\begin{aligned} -A\phi_1 AX &= (\nabla_X A)\xi_1 \\ &= \eta(X)\phi\xi_1 - 2g(\phi X, \xi_1)\xi - \phi_1 X + \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi_\nu\phi\xi_1. \end{aligned}$$

From this, putting $X = \xi$ in $\mathfrak{D}$, we have

$$-\beta A\phi_1 U = \phi\xi_1 - \phi_1\xi = 0. \tag{4.7}$$

On the other hand, the formula (4.2) implies

$$\begin{aligned} 0 =& \phi_1 U + \sum_\nu \eta_\nu(\phi U)\phi_1\phi\xi_\nu \\ =& \phi_1 U + \eta_1(\xi_1)\phi_1\phi\xi_1 \\ =& \phi_1 U + \phi_1^2\xi, \end{aligned}$$

because we have put $\phi U = \xi_1$. So it follows

$$\phi_1 U = \xi. \tag{4.8}$$

Then from (4.7) and (4.8) we have

$$0 = A\phi_1 U = A\xi = \beta U,$$

which makes a contradiction.

Sub. II.3: $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \neq 0$.

Now we apply (4.4) and (4.5) for any tangent vector $X \in M$ and for $i = 1, 2$. Then we have

$$\begin{aligned} -A\phi_i AX = \eta(X)\phi\xi_i - 2g(\phi X, \xi_i)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi_i \\ - \eta_\nu(\xi_i)\phi_\nu X - 2g(\phi_\nu X, \xi_i)\xi_\nu\} + \sum_\nu \eta_\nu(\phi X)\phi_\nu\phi\xi_i. \end{aligned}$$

From this, putting $X = \xi$ and also using $\phi\xi_\nu = \phi_\nu\xi$, we have

$$A\phi_1 U = A\phi_2 U = 0.$$

On the other hand, (4.2) and (4.3) imply

$$\begin{aligned} \phi_1 U - \eta_1(\phi U)\xi + \eta_2(\phi U)\phi_3\xi = 0, \\ \phi_2 U - \eta_1(\phi U)\phi_3\xi - \eta_2(\phi U)\xi = 0. \end{aligned} \tag{4.9}$$

Multiplying $\eta_1(\phi U)$ to the first of (4.9) and $\eta_2(\phi U)$ to the second and summing up, then in such a case by (4.2) we have

$$\xi = \eta_1(\phi U)\phi_1 U + \eta_2(\phi U)\phi_2 U.$$

This implies

$$\beta U = A\xi = \eta_1(\phi U)A\phi_1 U + \eta_2(\phi U)A\phi_2 U = 0.$$

So this case also can not occur.

Sub. II.4: $A\xi_i = 0, i = 1,2,3$.

Then we are able to apply the formula (4.4) for $i = 1,2,3$ as follows:

(4.10)

$$\begin{aligned} -A\phi_i AX = & -A(\nabla_X \xi_i) = (\nabla_X A)\xi_i \\ & = \eta(X)\phi\xi_i - 2g(\phi X, \xi_i)\xi + \sum_{\nu=1}^{3} \{\eta_\nu(X)\phi_\nu\xi_i \\ & \quad - \eta_\nu(\xi_i)\phi_\nu X - 2g(\phi_\nu X, \xi_i)\xi_\nu\} + \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi_\nu\phi\xi_i, \end{aligned}$$

where we have used $\xi \in \mathfrak{D}$ and $\phi\xi_\nu = \phi_\nu\xi$ for any $i = 1,2,3$. From this, putting $X = \xi$ and also using $\phi\xi_\nu = \phi_\nu\xi$, we have respectively for any $i = 1,2,3$

$$A\phi_1 U = 0, \ A\phi_2 U = 0, \text{and } A\phi_3 U = 0. \tag{4.11}$$

On the other hand, (4.2) implies

$$\begin{aligned} &\phi_1 U - \eta_1(\phi U)\xi + \eta_2(\phi U)\phi\xi_3 - \eta_3(\phi U)\phi\xi_2 = 0, \\ &\phi_2 U - \eta_1(\phi U)\phi_3\xi - \eta_2(\phi U)\xi + \eta_3(\phi U)\phi\xi_1 = 0, \\ &\phi_3 U + \eta_1(\phi U)\phi_2\xi - \eta_2(\phi U)\phi_1\xi - \eta_3(\phi U)\xi = 0. \end{aligned} \tag{4.12}$$

From this, multiplying $\eta_1(\phi U), \eta_2(\phi U)$ and $\eta_3(\phi U)$ respectively to the first, the second and the third and summing up all of these formulas and using (4.2), we have

$$\xi = \sum_\nu \eta_\nu(\phi U)\phi_\nu U.$$

Then from this together with (4.11) we have

$$\beta U = A\xi = \sum_\nu \eta_\nu(\phi U)A\phi_\nu U = 0.$$

So it follows $\beta = 0$. This contradicts $2\beta^2 = 5$. □

5. $\mathfrak{D}^{\perp}$-invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\eta(A\xi)\neq 0$

Now in this section let us continue the proof of Theorem 4.2 satisfying the assumption stated above. In particular, in this section we only treat the case that $\eta(A\xi)\neq 0$. Moreover, in this section we will show that $\alpha = \eta(A\xi)$ is constant. Now the formula (3.4) implies

$$A^2\xi = \lambda A\xi + \mu\sum_{\nu}\eta_{\nu}(\phi U)\phi\xi_{\nu}, \tag{5.1}$$

where we have put $\lambda = \frac{\alpha^2+\beta^2}{\alpha}$ and $\mu = \frac{5\beta}{2\alpha}$. From this, together with the assumptions of $\alpha, \beta\neq 0$, we have

$$AU = \beta\xi + \frac{\beta^2}{\alpha}U + \frac{5}{2\alpha}\sum_{\nu}\eta_{\nu}(\phi U)\phi\xi_{\nu}. \tag{5.2}$$

LEMMA 5.1. *Let M be a real hypersurface in a complex two-plane Grassmannian $G_2(C^{n+1})$ satisfying $\nabla_{\mathfrak{F}}A = 0$ and $\mathfrak{D}^{\perp}$ is invariant by the shape operator of M. Then the function $\alpha = \eta(A\xi)$ is constant.*

Proof. Let $A\xi = \alpha\xi + \beta U$. Then differentiating this implies

$$(\nabla_X A)\xi + A\nabla_X\xi = (X\alpha)\xi + \alpha\nabla_X\xi + (X\beta)U + \beta\nabla_X U. \tag{5.3}$$

From this, using the equation of Codazzi and $\nabla_{\xi}A = 0$, we have

(5.4)
$$\begin{aligned}
&-A\phi AX + (X\alpha)\xi + \alpha\phi AX + (X\beta)U + \beta\nabla_X U\\
&=(\nabla_X A)\xi\\
&=-\phi X + \sum\{\eta_{\nu}(X)\phi_{\nu}\xi - 2g(\phi_{\nu}X,\xi)\xi_{\nu}\} - \sum\eta_{\nu}(\phi X)\xi_{\nu}.
\end{aligned}$$

From this, taking an inner product with ξ, we have

$$X\alpha + 2\beta g(A\phi U, X) = 0.$$

On the other hand, by Lemma 4.1 and the assumption that $\mathfrak{D}^{\perp}$ is invariant we know that $A\phi U\in\mathfrak{D}^{\perp}$. This implies

$$\xi\alpha = 0 \text{ and } X\alpha = 0$$

for any $X \in \mathfrak{D}$.

Now it remains only to check $\xi_i \alpha = 0$ for any $\xi_i \in \mathfrak{D}^\perp$. Putting $X = \xi_i$ in (5.4), we have

$$\begin{aligned} 0 &= (\nabla_{\xi_i} A)\xi \\ &= -A\phi A\xi_i + (\xi_i\alpha)\xi + \alpha\phi A\xi_i + (\xi_i\beta)U + \beta\nabla_{\xi_i} U. \end{aligned}$$

From this, taking an inner product with ξ, it follows

$$\begin{aligned} \xi_i\alpha &= g(A\phi A\xi_i, \xi) - \beta g(\nabla_{\xi_i} U, \xi) \\ &= -\alpha_i\beta g(\xi_i, \phi U) + \beta g(U, \phi A\xi_i) \\ &= -2\alpha_i\beta g(\xi_i, \phi U). \end{aligned} \tag{5.5}$$

On the other hand, if we use the equation of Codazzi in (5.3), we have

$$\begin{aligned} &(X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) \\ &+ (X\beta)g(U, Y) - (Y\beta)g(U, X) + \beta g(\nabla_X U, Y) \\ &- \beta g(\nabla_Y U, X) - 2g(A\phi AX, Y) \\ =& g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ =& -2g(\phi X, Y) + \sum\{\eta_\nu(X)\eta(\phi_\nu Y) - \eta_\nu(Y)\eta(\phi_\nu X)\} \\ &+ \sum\{\eta_\nu(\phi X)\eta(\phi_\nu\phi Y) - \eta_\nu(\phi Y)\eta(\phi_\nu\phi X)\}. \end{aligned}$$

Then putting $X = \xi \in \mathcal{D}$ and $Y = \xi_i$ gives

$$\begin{aligned} \xi_i\alpha &= \alpha g(\phi A\xi, \xi_i) + \beta g(\nabla_\xi U, \xi_i) - \beta g(\nabla_{\xi_i} U, \xi) \\ &\quad - 2g(A\phi A\xi, \xi_i) \\ &= 2\alpha\beta\eta_i(\phi U) - 3\alpha_i\beta\eta_i(\phi U). \end{aligned} \tag{5.6}$$

Let $\mathfrak{D}$ be the maximal quaternionic subbundle of the tangent bundle TM of M and $\mathfrak{D}^\perp$ the orthogonal complement of $\mathfrak{D}$ in TM. Now we assume that $A\mathfrak{D} \subset \mathfrak{D}$. Then $\mathfrak{D}^\perp$ is also invariant under A, and by a suitable choice of J_1, J_2, J_3 the vector fields ξ_ν are principal curvature vectors everywhere, say $A\xi_\nu = \beta_\nu\xi_\nu$ for $\nu = 1, 2, 3$. The Codazzi equation then implies (see Berndt and the present author [4] and [5])

$$\begin{aligned}
&2\eta(X)\eta_\nu(\phi Y) - 2\eta(Y)\eta_\nu(\phi X) - 2g(\phi X, Y)\eta(\xi_\nu) \\
&\quad + 2\eta_{\nu+1}(X)\eta_{\nu+2}(Y) - 2\eta_{\nu+1}(Y)\eta_{\nu+2}(X) - 2g(\phi_\nu X, Y) \\
&\quad + 2\eta_{\nu+1}(\phi X)\eta_{\nu+2}(\phi Y) - 2\eta_{\nu+1}(\phi Y)\eta_{\nu+2}(\phi X) \\
&= g((\nabla_X A)Y - (\nabla_Y A)X, \xi_\nu) \\
&= g((\nabla_X A)\xi_\nu, Y) - g((\nabla_Y A)\xi_\nu, X) \\
&= (X\beta_\nu)\eta_\nu(Y) - (Y\beta_\nu)\eta_\nu(X) \\
&\quad + (\beta_\nu - \beta_{\nu+1})(q_{\nu+2}(X)\eta_{\nu+1}(Y) - q_{\nu+2}(Y)\eta_{\nu+1}(X)) \\
&\quad - (\beta_\nu - \beta_{\nu+2})(q_{\nu+1}(X)\eta_{\nu+2}(Y) - q_{\nu+1}(Y)\eta_{\nu+2}(X)) \\
&\quad + \beta_\nu g((A\phi_\nu + \phi_\nu A)X, Y) - 2g(A\phi_\nu AX, Y)\ .
\end{aligned}$$

Putting $X = \xi_\nu$ in this equation yields

$$\begin{aligned}
Y\alpha_j = (\xi_j\alpha_j)\eta_j(Y) + (\alpha_j - \alpha_{j+1})q_{j+2}(\xi_j)\eta_{j+1}(Y) \\
- (\alpha_j - \alpha_{j+2})q_{j+1}(\xi_j)\eta_{j+2}(Y),
\end{aligned} \tag{5.7}$$

where we have used $\xi \in \mathfrak{D}$.

Now from (5.5) and (5.6) it follows

$$\beta(2\alpha - \alpha_j)\eta_j(\phi U) = 0 \text{ for } j = 1, 2, 3. \tag{5.8}$$

Since we have proved that $X\alpha = 0$ for any $X \in \mathfrak{D}$, so $\xi\alpha = 0$. Now it remains only to show that $\xi_j\alpha = 0$ for $j = 1, 2, 3$. Then from (5.8) we are able to consider the following Cases:

Case I: All of $\eta_i(\phi U) \neq 0$, $i = 1, 2, 3$.

Then $2\alpha = \alpha_1 = \alpha_2 = \alpha_3$. So (5.7) gives $Y\alpha = (\xi_j\alpha_j)\eta_j(Y)$. From this it follows

$$\xi_k\alpha_j = 2\xi_k\alpha = 0, \quad k = 1, 2, 3.$$

Case II: Two of $\eta_i(\phi U), i = 1, 2, 3$ are nonvanishing.

For convenience sake let us say $\eta_1(\phi U), \eta_2(\phi U) \neq 0, \eta_3(\phi U) = 0$. Then (5.8) gives $2\alpha = \alpha_1 = \alpha_2$. From this, (5.7) implies

$$\begin{aligned}
Y\alpha_1 &= (\xi_1\alpha_1)\eta_1(Y) - (\alpha_1 - \alpha_3)q_2(\xi_1)\eta_3(Y), \\
Y\alpha_2 &= (\xi_2\alpha_2)\eta_2(Y) + (\alpha_2 - \alpha_3)q_1(\xi_2)\eta_3(Y).
\end{aligned}$$

So these formulas imply

$$2\xi_2\alpha = \xi_2\alpha_1 = 0 \text{ and } 2\xi_1\alpha = \xi_1\alpha_2 = 0.$$

Moreover, (5.6) implies $\xi_3\alpha = 0$.

Case III: One of $\eta_i(\phi U), i = 1, 2, 3$ is nonvanishing.

Let us say $\eta_1(\phi U) \neq 0$, $\eta_2(\phi U) = \eta_3(\phi U) = 0$. Then by (5.6) and (5.8) we know $2\alpha = \alpha_1$ and $\xi_2\alpha = \xi_3\alpha = 0$. Moreover, we can put $\phi U = \xi_1$. So it remains only to prove $\xi_1\alpha = 0$. For this let us differentiate along the direction ξ_1

$$A\phi U = A\xi_1 = \alpha_1\xi_1 = 2\alpha\phi U,$$

then the assumption $\nabla_{\xi_1} A = 0$ implies

$$A\phi\nabla_{\xi_1} U = 2(\xi_1\alpha)\phi U + 2\alpha\phi\nabla_{\xi_1} U,$$

where we have used $U \in \mathfrak{D}, \xi_1 = \phi U \in \mathfrak{D}^{\perp}$. Then from this, taking an inner product with ϕU implies

$$\begin{aligned} 2\xi_1\alpha &= g(A\phi\nabla_{\xi_1} U, \phi U) \\ &= 2\alpha g(\phi\nabla_{\xi_1} U, \phi U) \\ &= 0. \end{aligned}$$

Now summing up the above Cases I,II and III, we conclude that

$$\xi\alpha = 0, \quad X\alpha = 0, \ X \in \mathfrak{D}, \quad \text{and} \quad \xi_j\alpha = 0, \quad j = 1, 2, 3.$$

Thus the function α is constant on any real hypersurface M in $G_2(C^{m+2})$. Hence we have completed the proof of Lemma 5.1 □

Now we are going to prove that the structure vector ξ is principal. That is, we will complete the proof of Theorem 4.2 in Section 4. So in order to get this purpose we should make a contradiction, under the case that the function α and β are both nonvanishing. So if we make a contradiction, then we can assert $\alpha = 0$ or $\beta = 0$. When $\beta = 0$, our assertion holds. When we consider the case $\alpha = 0$, our result already has been asserted in Section 4.

For this let us differentiate $\alpha = \eta(A\xi)$. Then for any tangent X in M we have

$$g((\nabla_X A)\xi, \xi) + 2g(A\phi AX, \xi) = 0.$$

From this, together with the assumption $\nabla_\xi A = 0$, the equation of Codazzi gives

$$A\phi U = 0, \ \phi U \in \mathcal{D}^{\perp}. \tag{5.9}$$

Then (4.1) and (4.2) imply

$$\phi U = \sum_\nu \eta_\nu(\phi U)\xi_\nu. \tag{5.10}$$

From this together with (5.9) it follows

$$\sum_\nu \eta_\nu(\phi U) A\xi_\nu = \sum_\nu \alpha_\nu \eta_\nu(\phi U)\xi_\nu = 0. \tag{5.11}$$

Then we consider the following four Cases:

Case I: $\alpha_1, \alpha_2, \alpha_3 \neq 0$.

Then $\eta_i(\phi U) = 0$ for any $i = 1, 2, 3$. So $\phi U = 0$. This makes a contradiction.

Case II: One of $\alpha_1, \alpha_2, \alpha_3$ is vanishing.

Let us say $\alpha_1 = 0, \alpha_2, \alpha_3 \neq 0$. Then $\eta_2(\phi U) = \eta_3(\phi U) = 0$. So $\eta_1(\phi U) = \pm 1$. We may put $\phi U = \xi_1$. Then by using the similar method we also have the same formula as in (4.4), (4.5) and (4.6). From this, putting $X = \xi \in \mathfrak{D}$ in (4.6), we have

$$\begin{aligned} 0 &= -A\phi_1 A\xi \\ &= -A\phi_1(\alpha\xi + \beta U) \\ &= -\alpha A\phi_1\xi - \beta A\phi_1 U. \end{aligned} \tag{5.12}$$

On the other hand, from (5.10) we may write

$$\begin{aligned} 0 &= \phi_1 U + \sum \eta_\nu(\phi U)\phi_1\phi\xi_\nu \\ &= \phi_1 U + \phi_1^2\xi. \end{aligned}$$

Since $\xi \in \mathfrak{D}$, we know $\phi_1 U = \xi$. This gives $-U = \phi_1\xi$. So (5.12) and $\alpha \neq 0$ implies

$$\begin{aligned} 0 &= \alpha AU - \beta A\xi \\ &= \alpha AU - \beta(\alpha\xi + \beta U). \end{aligned}$$

So we have $AU = \beta\xi + \frac{\beta^2}{\alpha}U$. Then from this together with (5.2) it follows

$$\begin{aligned} 0 &= \sum \eta_\nu(\phi U)\phi\xi_\nu \\ &= \eta_1(\phi U)\phi\xi_1 \\ &= -U, \end{aligned}$$

which makes a contradiction.

Case III: $\alpha_1 = \alpha_2 = 0$, $\alpha_3 \neq 0$.

Then from $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$ we know $A\xi_1 = 0, A\xi_2 = 0$. Moreover, we know $\eta_3(\phi U) = 0$. Then for any $X \in \mathfrak{D}$ (4.6) gives

$$-A\phi_1 AX = \eta(X)\phi\xi_1 - 2g(\phi X, \xi_1)\xi - \phi_1 X + \sum \eta_\nu(\phi X)\phi_\nu\phi\xi_1,$$
$$-A\phi_2 AX = \eta(X)\phi\xi_2 - 2g(\phi X, \xi_2)\xi - \phi_2 X + \sum \eta_\nu(\phi X)\phi_\nu\phi\xi_2.$$

So it follows that

$$0 = A\phi_i A\xi = \alpha A\phi_i\xi + \beta A\phi_i U, \ i = 1, 2. \tag{5.13}$$

On the other hand, (5.10) gives $U + \sum_{\nu=1}^{2} \eta_\nu(\phi U)\phi\xi_\nu = 0$. Then using the same method as in (4.9), we have

$$\xi = \eta_1(\phi U)\phi_1 U + \eta_2(\phi U)\phi_2 U,$$

from this it follows that

$$\begin{aligned} A\xi &= \alpha\xi + \beta U \\ &= \eta_1(\phi U)A\phi_1 U + \eta_2(\phi U)A\phi_2 U \\ &= -\frac{\alpha}{\beta}\eta_1(\phi U)A\phi_1\xi - \frac{\alpha}{\beta}\eta_2(\phi U)A\phi_2\xi \\ &= \frac{\alpha}{\beta}AU \end{aligned}$$

where we have used (5.12) in the third equality. This implies

$$AU = \beta\xi + \frac{\beta^2}{\alpha}U.$$

So also from (5.2) we know that

$$\sum \eta_\nu(\phi U)\phi\xi_\nu = 0$$

which implies $U = 0$. This makes a contradiction. So this case also can not be occurred.

Case IV: $\alpha_1 = \alpha_2 = \alpha_3 = 0$

Then $A\xi_1 = A\xi_2 = A\xi_3 = 0$. From this differentiating and using (4.6) it follows that

$$\begin{aligned} -A\phi_i AX &= \eta(X)\phi\xi_i - 2g(\phi X, \xi_i)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu\xi_i - \eta_\nu(\xi_i)\phi_\nu X - 2g(\phi_\nu X, \xi_i)\xi_\nu\} \\ &\quad + \sum \eta_\nu(\phi X)\phi_\nu\phi\xi_i. \end{aligned}$$

From this, putting $X = \xi$, we have

$$\alpha A\phi_i\xi + \beta A\phi_i U = 0, \ i = 1, 2, 3.$$

Moreover, (5.10) holds if and only if $U = -\sum \eta_i(\phi U)\phi_i\xi$. So by using the same method as in (4.12) we have $\xi = \sum \eta_\nu(\phi U)\phi_\nu U$. Then the formulas in above imply

$$\begin{aligned} \alpha\xi + \beta U &= A\xi \\ &= \sum \eta_\nu(\phi U)A\phi_\nu U \\ &= -\frac{\alpha}{\beta}\sum_\nu \eta_\nu(\phi U)A\phi_\nu\xi \\ &= \frac{\alpha}{\beta}AU. \end{aligned}$$

Thus as in Case III if we compare this to (5.2), we also make a contradiction. So this case also can not occur.

So summing up all the cases mentioned above, we know that there do not exist any open set $\mathfrak{U} = \{p \in M | \beta \neq 0\}$ provided the function $\alpha \neq 0$. That is, we can assert that its structure vector ξ is principal. From this we complete the proof of our Theorem 4.2.

From Theorem 4.2 and Theorem A in the Introduction it follows that M is also locally congruent to an open part of a tube around a totally real totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{n+2})$. Then by using the same method given in Section 3 we can also make a contradiction. So we complete the proof of our Theorem 2.

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