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ON DECOMPOSABILITY OF FINITE GROUPS

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Abstract. Let G be a finite group and N be a normal subgroup of G. We denote by $ncc(N)$ the number of conjugacy classes of N in G and N is called n-decomposable, if $ncc(N) = n$. Set $K_G = \{ncc(N) \mid N \lhd G\}$. Let X be a non-empty subset of positive integers. A group G is called X-decomposable, if $K_G = X$.

In this paper we characterise the $\{1,3,4\}$-decomposable finite non-perfect groups. We prove that such a group is isomorphic to *SmallGroup* (36, 9), the 9^{th} group of order 36 in the small group library of GAP, a metabelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$, in which n is odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime or a metabelian group of order $2^n(2^{\frac{n}{3}} - 1)$, where $3|n$ and $2^{\frac{n}{3}} - 1$ is a Mersenne prime. Moreover, we calculate the set K_G, for some finite group G.

1. Introduction and preliminaries

Let G be a finite group and let N_G be the set of proper normal subgroups of G. An element K of N_G is said to be n-decomposable if K is a union of n distinct conjugacy classes of G. In this case we denote n by $ncc(K)$. Suppose $K_G = \{ncc(N) \mid N \lhd G\}$ and X is a non-empty subset of positive integers. A group G is called X-decomposable, if $K_G = X$. For simplicity, if $X = \{1, n\}$ and G is X-decomposable, then we say that G is n-decomposable.

In [14], Wujie Shi defined the notion of complete normal subgroup of a finite group, which we called it 2-decomposable. He proved that if G is a group and N a complete normal subgroup of G. Then N is a minimal normal subgroup of G and it is an elementary abelian p-group. Moreover, $N \subseteq Z(O_p(G))$, where $O_p(G)$ is a maximal p-normal subgroup of G, and $\mid N \mid (\mid N \mid -1) \mid\mid G \mid$ and in particular, $\mid G \mid$ is even.

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Also, Shi proved some deep results about finite groups of order $p^a q^b$ containing a 2-decomposable normal subgroup. Next, Wang Jing, a student of Wujie Shi, continued his work and defined the notion of sub-complete normal subgroup of a group G [18], which we called it 3-decomposable. She proved that if N is a sub-complete normal subgroup of a finite group G, then N is a group in which every element has prime power order. Moreover, if N is a minimal normal subgroup of G, then $N \subseteq Z(O_p(G))$, where p is a prime factor of $\mid G \mid$. If N is not a minimal normal subgroup of G, then N contains a complete normal subgroup N_1, N_1 is an elementary abelian group with order p^a and we have: (a) $N = N_1 Q$ has order $p^a q$ and every element of N has prime power order, $\mid Q \mid = q$, $q \neq p, q$ is a prime and $G = MN_1$, $M \cap N_1 = 1$, where $M = N_G(Q)$, (b) N is an abelian p-group with exponent $\leq p^2$ or a special group; if N is not elementary abelian, then $N_1 \leq \Phi(G)$.

In [12] and [13], Shahryari and Shahabi, independent from Shi and Jing, investigated the structure of finite groups which contains a 2- or 3- decomposable subgroup. Riese and Shahabi continue in [7], investigating of the structure of finite groups with a 4-decomposable subgroup. Using these works in some joint papers [1], [2], [3] and [4], the author characterized the finite non-perfect X-decomposable finite groups, for $X = \{1, n\}$, $n \leq 6$ and $X = \{1, 2, 3\}$. He also obtained the structure of solvable n-decomposable non-perfect finite groups.

Throughout this paper $A = \{1, 3, 4\}$. We continue the mentioned problem and investigate the structure of A-decomposable finite groups. In fact, we prove that:

THEOREM. *Let G be a finite non-perfect group. If G is A-decomposable then G is isomorphic to $SmallGroup(36, 9)$, a metabelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$, in which n is odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime or a metabelian group of order $2^n(2^{\frac{n}{3}} - 1)$, where $3|n$ and $2^{\frac{n}{3}} - 1$ is a Mersenne prime.*

Throughout this paper, as usual, G' denotes the derived subgroup of G, Z_n is the cyclic group of order n, $\Phi(G)$ denotes the Frattini subgroup of G and $Z(G)$ is the center of G. G is called non-perfect, if $G' \neq G$. Also, $D(n)$ denotes the set of positive divisors of n and $SmallGroup(n, i)$ is the i^{th} group of order n in the small group library of GAP, [11]. All groups considered are assumed to be finite. Our notation is standard and taken mainly from [5], [6] and [8].

2. Examples

In this section we calculate the set K_G for some finite group G and present some open questions.

EXAMPLE 2.1. Suppose that G is a non-abelian group of order pq, in which p and q are primes and $p > q$. It is well known that $q|p-1$ and G has exactly one normal subgroup. Suppose that H is the normal subgroup of G. Then H is $(1+\frac{p-1}{q})$-decomposable.

By the previous example, if p, q are primes, $q|p-1$ and $X = \{1, 1+\frac{p-1}{q}\}$ then there exists a non-abelian X-decomposable finite group. Therefore, the problem of existing $\{1,n\}$-decomposable finite groups can be reduced to a number theoretic problem:

QUESTION 2.2. Is it true that every odd positive integer has a representation of the form $n = 1+\frac{p-1}{q}$, where p, q are primes and $q|p-1$?

Suppose $k(G)$ denotes the number of conjugacy classes of the group G. If H is a simple group with $n = k(G)$ conjugacy class and $G = H \times H$ then G is $\{1,n\}$-decomposable. Therefore, the problem of existing $\{1,n\}$-decomposable finite groups can be reduced to a problem about finite simple groups, as follows:

QUESTION 2.3. Suppose $n \geq 5$ is a given positive integer. Is there a finite simple group G with $n = k(G)$?

EXAMPLE 2.4. Let G be a non-abelian group of order p^3, p is prime. It is well-known fact that this group has p^2+p-1 conjugacy classes. Since every conjugacy class of G has length p, G is $\{1,p,2p-1\}$-decomposable.

EXAMPLE 2.5. Let D_{2n} be the dihedral group of order $2n$, $n \geq 3$. This group can be presented by

$$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle.$$

We first assume that n is odd and $X = \{\frac{d+1}{2} \mid d|n\}$. In this case every proper normal subgroup of D_{2n} is contained in $\langle a\rangle$ and so D_{2n} is X-decomposable. Next we assume that n is even and $Y = \{\frac{d+1}{2} \mid d|n\ ;\ 2 \not| d\} \cup \{\frac{d+2}{2} \mid d|n\ ; 2|d\}$. In this case, we can see that D_{2n} has exactly two other normal subgroups $H = \langle a^2, b\rangle$ and $K = \langle a^2, ab\rangle$. To complete the example, we must compute $ncc(H)$ and $ncc(K)$. Obviously, $ncc(H) = ncc(K)$. If $4|n$ then $ncc(H) = \frac{n}{4}+2$ and if $4 \not| n$ then $ncc(H) = \frac{n+2}{4}+1$. Set $A = Y \cup \{\frac{n}{4}+2\}$ and $B = Y \cup \{\frac{n+2}{4}+1\}$. Our calculations show that if $4|n$ then D_{2n} is A-decomposable and if $4 \not| n$ then dihedral group D_{2n} is B-decomposable.

EXAMPLE 2.6. Let Q_{4n} be the generalized quaternion group of order $4n$, $n \geq 2$. This group can be presented by

$$Q_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle.$$

Set $X = \{\frac{d+1}{2} \mid d|n$ & d is odd$\} \bigcup \{\frac{d+2}{2} \mid d|2n$ & d is even$\}$ and $Y = X \cup \{\frac{n+4}{2}\}$. It is a well-known fact that Q_{4n} has $n+3$ conjugacy classes, as follows:

$$\{1\} \; ; \; \{a^n\} \; ; \; \{a^r, a^{-r}\}(1 \leq r \leq n-1);$$
$$\{a^{2j}b \mid 0 \leq j \leq n-1\} \; ; \; \{a^{2j+1}b \mid 0 \leq j \leq n-1\}.$$

We consider two separate cases that n is odd or even. If n is odd then every normal subgroup of Q_{4n} is contained in the cyclic subgroup $\langle a \rangle$. Thus, in this case Q_{4n} is X-decomposable. If n is even, we have two other normal subgroups $\langle a^2, b \rangle$ and $\langle a^2, ab \rangle$ which are both $\frac{n+4}{2}$-decomposable. Therefore, Q_{4n} is Y-decomposable.

Now it is natural to generally ask about the set $K_G = \{ncc(A) \mid A \lhd G\}$. We end this section with the following question:

QUESTION 2.7. Suppose X is a finite subset of positive integers containing 1. Is there a finite group G which is X-decomposable?

3. On A-decomposable finite groups

The aim of this section is to prove the main result of the paper. First of all, we consider the abelian case.

LEMMA 3.1. *Let G be an abelian finite group. Set $X = D(n) - \{n\}$, in which $n = |G|$. Then G is X-decomposable.*

Proof. The proof is straightforward. □

COROLLARY. *There is no abelian A-decomposable finite group.*

Set $I = \{8, 12, 18, 20, 24, 28, 30, 42, 48, 54, 78, 96, 100, 294\}$. In the end of this paper, we write a GAP program to show that there is no finite group G of order $n, n \in I$, such that G is $A-decomposable$. Using this program, we have:

LEMMA 3.2. *There is no A-decomposable finite groups of order $n, n \in I$. Moreover, if G is a A-decomposable group of order 36 then $G \cong SmallGroup(36, 9)$.*

Proof. It follows from a GAP program in the end of the paper. □

From the Corollary of Lemma 3.1, there is no abelian A-decomposable finite group. So we can restrict our investigation on the structure of non-abelian A-decomposable finite groups. From now on G denotes a non-abelian finite group.

LEMMA 3.3. *Suppose G is an A-decomposable finite group and K and H are 3- and 4-decomposable subgroups of G, respectively. Then $K \subset H$ and G has a unique 3-decomposable subgroup. Moreover, G is centerless or $|Z(G)| = 3$ and G has a unique 4-decomposable subgroup, which is a 3-group containing $Z(G)$.*

Proof. Suppose L is another 3-decomposable subgroup of G and $T = LK$. Then $G = K \times L$, a contradiction. Therefore, the 3-decomposable subgroup of G is unique. Also, if $K \not\subset H$ then $K \cap H = 1$ and so $G \cong H \times K$, which is impossible. We now assume that $Z(G) \neq 1$. Since G is A-decomposable, $|Z(G)| = 3$ and $Z(G) \subset H$.

We next prove that H is a 3-group. By [7, Theorem 1], H is a p-group and $H'' = 1$, $H \cong A_5$, the alternating group of degree 5, and $\frac{G}{C_G(H)} \cong S_5$ or it is a solvable group of order $3^a p^b$, where $p \neq 3$ is prime and a, b are positive integers. Since $Z(G) \subset H$, H is not isomorphic to A_5. Suppose $|H| = 3^a p^b$. Then by [7, Theorem 2], H is a Frobenius group with kernel $M \supset Z(G)$, where M is a Sylow 3-subgroup of H, which is a union of three conjugacy classes and $\frac{H}{M}$ is cyclic of order p. But a Frobenius group is centerless, a contradiction. Therefore H is 3-group.

Finally, suppose that M is another 4-decomposable subgroup of G. By our argument M is a 3-group. Consider $T = MH$. Since G is A-decomposable, $T = G$, i.e., G is 3-group with a 4-decomposable subgroup H of order, say 3^r, $r > 1$. Suppose $|G| = 3^n$, $n > r$. Then $3^r - 3 | 3^n$, which is our final contradiction. □

PROPOSITION 3.4. *Let G be a non-perfect A-decomposable finite group. G' is 3-decomposable if and only if $G' \cong SmallGroup(36, 9)$*

Proof. Suppose G' is 3-decomposable. First of all, we claim that G is not a p-group. To do this, we assume that G is a non-perfect A-decomposable p-group of order p^n. By Lemma 3.3, $p = 3$ and G has a unique 4- decomposable subgroup H containing $Z(G)$ of order, say 3^r. So $3^r - 3 | 3^n$, a contradiction. Next, we show that G' is elementary abelian. Assume that $G' = 1 \cup Cl_G(g) \cup Cl_G(h)$, where $Cl_G(x)$ denotes the conjugacy class of G containing x. If $g^{-1} \in Cl_G(h)$ then by [13, Proposition 1], G' is elementary abelian, as desired. Suppose $g^{-1} \in$

$Cl_G(g)$. If $(o(g), o(h)) = 1$ then by [13, Lemma 5], $1 \neq G'' < G'$, which is impossible. Also, if $(o(g), o(h)) \neq 1$ then by [13, Proposition 2], G' is a metabelian p-group. It is easy to see that G' is abelian and $\Phi(G') = 1$. Thus G' is elementary abelian.

We now assume that H is a 4-decomposable subgroup of G. By Lemma 3.3, $G' \subset H$ and so $H = G' \cup Cl_G(k)$. By [7, Theorem 1], H is a p-group or a solvable group of order $p^a q^b$, where p and q are distinct primes and a, b are positive integers. In what follows, we consider two separate cases that whether or not G has a 4-decomposable p-subgroup.

Case 1. G does not have a 4-decomposable p-subgroup. In this case, we can assume that H has order $p^a q^b$. Let N be a minimal normal subgroup of H. Then N is a 3-decomposable subgroup of G and so $N = G'$. On the other hand, G' is a Sylow subgroup of H. Suppose $|G'| = p^n$. Thus $|H| = p^n q$ and H contains a G-conjugacy class of order $p^n(q-1)$. Since H is a maximal subgroup of G, $|G| = p^n qr$, where r is prime. Assume that $r \notin \{p, q\}$ then G is a solvable group of order $p^n qr$ and contains a 4-decomposable subgroup of order $p^n r$. This implies that $q = 2, r = 3$ or $q = 3, r = 2$. Thus $|G| = 6p^n, p \neq 2, 3$. Suppose $1, a$ and b are the lengths of the G-conjugacy classes of G'. Consider the equation $p^n = 1 + a + b$ and possible pairs (a, b). It is easy to see that $p \nmid a$ or $p \nmid b$. Using a simple calculation one can see that $|G| \in \{30, 42, 78, 294\}$. But, this contradicts by Lemma 3.2. Next we assume that $r = q$. Using similar argument as in above, we can see that $q = 2$ and $|G| \in \{1220, 28, 36, 100\}$. Apply Lemma 3.2, we have $G \cong SmallGroup(36, 9)$. Finally, if $r = p$ then $|G'| = p^{n+1}$, which is impossible.

Case 2. G has a 4-decomposable p-subgroup H. Suppose $|H| = p^n$, where p is prime and $n > 1$. Since H is maximal, $|G| = p^n q$, where q is a prime. Since G is not a p-group, $q \neq p$. By assumption G' has order p^{n-1} and $|Cl_G(k)| = p^{n-1}(p-1)$. Thus $p = 2$ or $p = 1 + q$. Suppose $p = 2$ and $y \in H$ is an element of order q . Then $|G| = 2^n q$. Consider the subgroup $T = G'\langle y \rangle$. Clearly T is a 4-decomposable subgroup of G and so $q = 3$. This shows that $|G| = 2^n \cdot 3$. Write $2^{n-1} = 1 + a + b$, where a, b are class lengths of G. We can assume that $2 \nmid a$ and $2 | b$. Thus $a = 1$ or 3. If $a = 1$ then by Lemma 3.3, $|Z(G)| = 3$ and H is a 3-group, which is impossible. If $a = 3$ then $|G| = 48$ or 96 and by Lemma 3.2, we lead to a contradiction. Hence $p = 1 + q$ and $|G| = 2 \cdot 3^n$. Using a similar argument as in above, we can see that $|G| = 18$. This is our final contradiction. □

PROPOSITION 3.5. *Let G be a non-perfect A-decomposable finite group and G' is 4-decomposable subgroup of G. Then G is isomorphic to a metabelian group of order $2^n(2^{\frac{n-1}{2}}-1)$, in which n is odd positive integer and $2^{\frac{n-1}{2}}-1$ is a Mersenne prime or a metabelian group of order $2^n(2^{\frac{n}{3}}-1)$, where $3|n$ and $2^{\frac{n}{3}}-1$ is a Mersenne prime.*

Proof. Suppose G' is 4-decomposable and H is a 3-decomposable subgroup of G. By Lemma 3.3, $H \subset G'$. By [7, Theorem 1], G' is a p-group or a solvable group of order p^aq^b, where p and q are distinct primes and a, b are positive integers. We conclude that G is solvable and so $G'' = 1$ or H. We first assume that $G'' = H$. If G' is not a p-group, then $|G'| = p^nq^m$, where p, q are distinct primes and m, n are positive integers. Since G' is non-abelian, it is a Frobenius group with kernel $M \supseteq H$, where M is a Sylow q-subgroup of G' and $\frac{G'}{M}$ is cyclic of order p. Obviously, $M = H$ is of order q^m, $|G'| = pq^m$ and $|G| = spq^m$, for a prime s. Hence we get $p = 2$ or $p = 1 + s$. If $p = 2$ then G has a subgroup of index 2, say T. Since T has a G-conjugacy class of order $(s-1)q^m$, $s = 2, 3$. Thus $|G| = 4q^m$ or $6q^m$. Using a similar method as in Proposition 3.4, we can see that $|G| \in \{12, 18, 20, 28, 30, 36, 42, 54\}$, which by Lemma 3.2, leads to a contradiction. So assume that $p = 3$ and $s = 2$ then $|G| = 6q^m$ and $|G| \in \{18, 24, 30, 42, 48, 54, 78, 96, 294\}$, which is impossible.

Therefore, G' is abelian and since it contains only one normal subgroup of G, G' is p-group. Suppose $|G'| = p^n, |H| = p^t$ and $|G| = p^nq$, in which p and q are distinct primes and n, t are positive integers with $t < n$. Thus G' has a G-conjugacy class of length $p^t(p^{n-t}-1)$. This implies that $p = 2$ and $t = n-1$ or $q = p^{n-t}-1$. Suppose $p = 2$ and $t = n-1$. Then $|G| = 2^nq$ and $|H| = 2^{n-1}$. Choose an element y of order q and define the subgroup T to generate by y and H. Since H has a G-conjugacy class of length $2^{n-1}(q-1)$, $q = 2, 3$. Now we can use again, a similar method as in Proposition 3.4, to prove that $|G| \in \{24, 48, 96\}$, contradict by Lemma 3.2. Therefore, $q = 2^{n-t}-1$. Suppose that $2^t = 1 + a + b$, where a and b are class lengths of G. Without lose of generality, we can assume that $a|q$. By assumption $a \neq 1$. Thus $a = q$ and $b = 2^t - 2^{n-t}$. This shows that $t = \frac{n+1}{2}$ or $\frac{2n}{3}$, which completes the proof. □

We now ready to state the main result of the paper.

THEOREM. *Let G be a finite non-perfect A-decomposable finite group. Then G is isomorphic to SmallGroup$(36, 9)$, a metabelian group of order $2^n(2^{\frac{n-1}{2}}-1)$, in which n is odd positive integer and $2^{\frac{n-1}{2}}-1$ is* *a Mersenne prime or a metabelian group of order* $2^n(2^{\frac{n}{3}} - 1)$, *where* $3|n$ *and* $2^{\frac{n}{3}} - 1$ *is a Mersenne prime.*

Proof. It follows from Lemma 3.1, Proposition 3.4 and Proposition 3.5. □

A GAP Program

```
AppendTo("x.txt", "Beginning the Program", "\n");
E:=[8,12,18,20,24,28,30,42,48,54,78,96,100,294];
for m in E do
  n:=NrSmallGroups(m);
  F:=Set([1,3,4]);
        for i in [1,2..n] do
          G1:=[];
          G:=[];
          g:=SmallGroup(m,i);
          h:=NormalSubgroups(g);
          d1:=Size(h);d:=d1-1;
            for j in [1,2..d] do
              s:=FusionConjugacyClasses(h[j],g);
              s1:=Set(s);
              Add(G,s1);
            od;
            for k in G do
              a:=Size(k);
              Add(G1,a);
            od;
          G2:=Set(G1);
   if G2=F then AppendTo("x.txt","S(",m,",",i, ")",""); fi;
        od;
od;
```

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