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Existence and Uniqueness of Fuzzy Solutions for the nonlinear Fuzzy Integro-Differential Equation on E_N^n

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Abstract

In this paper we study the existence and uniqueness of fuzzy solutions for the nonlinear fuzzy integro-differential equations on E_N^n by using the concept of fuzzy number of dimension n whose values are normal, convex, upper semicontinuous and compactly supported surface in R^n. E_N^n be the set of all fuzzy numbers in R^n with edges having bases parallel to axis $X_1, X_2, \cdots, X_n$.

Key Words : fuzzy number of dimension n, fuzzy solution, nonlinear fuzzy integro-differential equation

I. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva[2] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported surface in R^n. Seikkala[9] proved the existence and uniqueness of fuzzy solution for the initial value problem on E^1. Subrahmanyam and Sudarsanam[10] studied fuzzy volterra-integral equation. Recently, Park et. al.[8] are proved the existence and uniqueness of fuzzy solution for the nonlinear fuzzy differential equation on E_N^n with nonlocal initial condition, Kwun et.al.[6] are studied controllability for the nonlinear fuzzy control system on E_N^n, where E_N^n be the set of all fuzzy numbers in R^n with edges having bases parallel to axis $X_1, X_2, \cdots, X_n$. For example E_N^2 be the set of all fuzzy pyramidal numbers in R^2 with edges having rectangular bases parallel to the axis X_1 and X_2.

In this paper we consider the existence and uniqueness of fuzzy solutions for the following nonlinear integro-differential equations:

$$\begin{cases} \frac{dx(t)}{dt} = a(t)x(t) + f(t, x(t), \\ \int_0^t k(t,s,x(s))ds)) + u(t), \ t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $a: [0, T] \times \to E_N^n$ is fuzzy coefficient, initial value $x_0 \in E_N^n$ and $f: [0, T] \times E_N^n \times E_N^n \to E_N^n$ and $k: [0, T] \times [0, T] \times E_N^n \to E_N^n$ are nonlinear regular fuzzy functions.

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II. Properties of n-dimensional fuzzy numbers and metric

In this section, we give some definitions, properties and notations of the fuzzy number of dimension n.

Definition 2.1.([6, 8]) We consider a fuzzy graph $G \subset R^n$ that is functional fuzzy relation in R^n such that its membership function $m_G(x_1, x_2, \cdots, x_n) \in [0, 1]$, $(x_1, x_2, \cdots, x_n) \in R^n$ has the following properties :

(1) For all $x_i \in R (i = 1, 2, \cdots, n)$, $m_G(x_1, x_2, \cdots, x_n) \in [0, 1]$ is a convex membership function.

(2) For all $\alpha \in [0, 1]$, $\{(x_1, x_2, \cdots, x_n) \in R^n | m_G(x_1, x_2, \cdots, x_n) \geq \alpha$ is convex set.

(3) There exists $(x_1, x_2, \cdots, x_n) \in R^n$, $m_G(x_1, x_2, \cdots, x_n) = 1$

If the above conditions are satisfied, the fuzzy subset G is called a fuzzy number of dimension n.

We denote by fuzzy number in E_N^n, $A = \{(a_1, a_2, \cdots, a_n)$ where a_i is projection of A to axis $X_i (i = 1, 2, \cdots, n)$, respectively. And $a_i (i = 1, 2, \cdots, n)$ is fuzzy number in R

Definition 2.2. The α-level set of fuzzy number in E_N^n is defined by

$$[A]^\alpha = \{(x_1, x_2, \cdots, x_n) \in R^n | (x_1, x_2, \cdots, x_n) \in \prod_{i=1}^n [a_i]^\alpha, \ 0 < \alpha \leq 1\} \tag{2.2}$$

where

$$[a_i]^\alpha = \{x_i \in R | m_{a_i}(x_i) \geq \alpha,\ 0 < \alpha \leq 1\} \tag{2.3}$$

and $\prod$ is the Cartesian product of sets.

Definition 2.3. Let $A, B \in E^n_N$, for all $\alpha \in (0,1]$, $A = B \Leftrightarrow [A]^\alpha = [B]^\alpha$.

Definition 2.4. Let $A, B \in E^n_N$, for all $\alpha \in (0,1]$,

$$[A *_n B]^\alpha = \prod_{i=1}^n [a_i * b_i]^\alpha \tag{2.4}$$

where $*_n$ is operation in E^n_N and $*$ is operation in E_N. And $(A *_n B)^\alpha_i = A^\alpha_i * B^\alpha_i$.

Definition 2.5. The derivative $\frac{dx(t)}{dt} \in E^n_N$ of fuzzy process $x(t) \in E^n_N$ is defined by

$$\left[\frac{dx(t)}{dt}\right]^\alpha = \prod_{i=1}^n \left[\frac{d}{dt} x^\alpha_{il}(t),\ \frac{d}{dt} x^\alpha_{ir}(t)\right],\ 0 < \alpha \leq 1. \tag{2.5}$$

Definition 2.6. The fuzzy integral $\int_a^b x(t)dt$ is defined by

$$\left[\int_a^b x(t)dt\right]^\alpha = \prod_{i=1}^n \left[\int_a^b x^\alpha_{il}(t)dt,\ \int_a^b x^\alpha_{ir}(t)dt\right],\ 0 < \alpha \leq 1. \tag{2.6}$$

where $x(t) \in E^n_N, a, b \in R$.

Let $\prod_{i=1}^n [a_i]^\alpha$, $0 < \alpha \leq 1$, be a given family of nonempty areas. If

$$\prod_{i=1}^n [a_i]^\beta \subset \prod_{i=1}^n [a_i]^\alpha,\ 0 < \alpha < \beta \leq 1 \tag{2.7}$$

and

$$\prod_{i=1}^n \lim_{k\to\infty} [a_i]^{\alpha_k} = \prod_{i=1}^n [a_i]^\alpha \tag{2.8}$$

whenever (α_k) is a nondecreasing sequence converging to $\alpha \in (0,1]$, then the family $\prod_{i=1}^n [a_i]^\alpha$, $0 < \alpha \leq 1$, represents the α-level sets of fuzzy number $A \in E^n_N$.

Conversely, if $\prod_{i=1}^n [a_i]^\alpha$, $0 < \alpha \leq 1$, are the α-level sets of fuzzy number R^n, then the condition (2.7) and (2.8) hold true.

We define the metric d_∞ on E^n_N and the supremum metric H on $C([0, T]: E^n_N)$.

Definition 2.7. Let $A, B \in E^n_N$,

$$d_\infty = \sup\{d_H([A]^\alpha, [B]^\alpha) \mid \alpha \in (0,1]\}$$
$$= \sup\left\{ \left(\sum_{i=1}^n (d_H([a_i]^\alpha, [b_i]^\alpha))^2\right)^{\frac{1}{2}} \mid \alpha \in (0,1]\right\}$$

where d_H is Hausdorff distance and $a_i,\ b_i \in E_N$.

Definition 2.8. The supremum metric H on $C([0, T]: E^n_N)$ is defined by

$$H(x, y) = \sup\{d_\infty(x(t), y(t)) \mid t \in [0, T]\}$$

where $x, y \in C([0, T]: E^n_N)$.

Definition 2.9. Nonlinear regular fuzzy function $f: [0, T] \times E^n_N \times E^n_N \to E^n_N$ is satisfied, $x, y \in E^n_N$,

$$f(t, [x]^\alpha, [y]^\alpha) = f\left(t, \prod_{m=1}^n [x_m]^\alpha, \prod_{m=1}^n [y_m]^\alpha\right)$$
$$= \prod_{m=1}^n f_m(t, [x_m]^\alpha, [y_m]^\alpha)$$
$$= \prod_{m=1}^n f^\alpha_m(t, x, y)$$
$$= f^\alpha(t, x, y).$$

III. Existence and Uniqueness of Fuzzy Solutions

In this section, we show the existence and uniqueness of fuzzy solution for the following nonlinear fuzzy integro-differential equations :

$$\begin{cases} \frac{dx(t)}{dt} = a(t)x(t) + f(t, x(t), \\ \qquad \int_0^t k(t, s, x(s))\ ds),\ t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{3.1}$$

where $a: [0, T] \times \to E^n_N$ is fuzzy coefficient, initial value $x_0 \in E^n_N$ and nonlinear regular fuzzy function $f: [0, T] \times E^n_N \times E^n_N \to E^n_N$ and $k: [0, T] \times [0, T] \times E^n_N \to E^n_N$ are satisfies a global Lipshitz condition.

Definition 3.1. The fuzzy process $x : [0, T] \to E^n_N$ is fuzzy solution of the equation (3.1) without inhomogeneous term if and only if

$$\frac{d}{dt} x^\alpha_{ml} = \min\{a^\alpha_{mi}(t) \cdot x^\alpha_{mj}(t) | m = 1, 2, \cdots, n, i, j = l, r\},$$
$$\frac{d}{dt} x^\alpha_{mr} = \max\{a^\alpha_{mi}(t) \cdot x^\alpha_{mj}(t) | m = 1, 2, \cdots, n, i, j = l, r\},$$
$$x^\alpha_{ml}(0) = x^\alpha_{0ml},\quad m = 1, 2, \cdots, n,$$
$$x^\alpha_{mr}(0) = x^\alpha_{0mr},\quad m = 1, 2, \cdots, n.$$

Theorem 3.1. For every $x_0 \in E^n_N$,

$$\begin{cases} \frac{dx(t)}{dt} = a(t)x(t), \\ x(0) = x_0. \end{cases} \tag{3.2}$$

has a unique fuzzy solution $x \in C([0, T]: E^n_N)$.

Proof. Let $x_0, a(t) \in E_N^n$. From the definition of fuzzy solution,

$$\frac{d}{dt} x^a_{ml}(t) = a^a_{ml}(t) \cdot x^a_{ml}(t),$$

$$\frac{d}{dt} x^a_{mr}(t) = a^a_{mr}(t) \cdot x^a_{mr}(t), \quad m=1,2,\cdots n,$$

and

$$x^a_{ml}(t) = x^a_{0ml}(t) \cdot \exp\left(\int_0^t a^a_{ml}(s)\,ds \right),$$

$$x^a_{mr}(t) = x^a_{0mr}(t) \cdot \exp\left(\int_0^t a^a_{mr}(s)\,ds \right),$$

$$m=1,2,\cdots n$$

Let

Therefore

$$[x(t)]^a = \prod_{m=1}^n x^a_m(t)$$

$$= \prod_{m=1}^n [x^a_{ml}(t),\ x^a_{mr}(t)]$$

$$= \prod_{m=1}^n [x^a_{0ml} \cdot S^a_{ml}(t),\ x^a_{0mk} \cdot S^a_{mr}(t)]$$

where

$$S^a_{mi}(t) = \exp\left(\int_0^t a^a_{mi}(s)\,ds \right), \quad i=l, r, \quad m=1,2,\cdots n.$$

and $S^a_{mi}(t)$ $(i=l, r,\ m=1,2,\cdots n)$ is continuous. That is, there exists a constant $C \geq 0$ such that $|S^a_{mi}(t)| \leq C$ for all $t \in [0, T]$. Hence

$$[x(t)]^a = [S(t)x_0]^a$$

By the resolution identity,

$$x(t) = S(t)x_0$$

From the definiton of fuzzy derivative,

$$\left[\frac{d}{dt} x(t)\right] = \frac{d}{dt} x^a(t) = \prod_{m=1}^n \left[\frac{d}{dt} x^a_m(t)\right]$$

$$= \prod_{m=1}^n \left[\frac{d}{dt} x^a_{ml}(t),\ \frac{d}{dt} x^a_{mr}(t)\right]$$

$$= \prod_{m=1}^n [x^a_{0ml} \cdot a^a_{ml}(t) \cdot S^a_{ml}(t),$$

$$x^a_{0mr} \cdot a^a_{mr}(t) \cdot S^a_{mr}(t)]$$

$$= \prod_{m=1}^n [a^a_{ml}(t) \cdot x^a_{ml}(t),\ a^a_{mr}(t) \cdot x^a_{mr}(t)]$$

$$= \prod_{m=1}^n a^a_m(t) \cdot x^a_m(t)$$

$$= [a(t) \cdot x(t)]^a.$$

Thus, by the resolution identity

$$\frac{d}{dt} x(t) = a(t) \cdot x(t)$$

Hence $x(t) = S(t)x_0$ is fuzzy solution.

The equation (3.1) is related to the following fuzzy integral equations :

$$(3.3)\begin{cases} x(t) = S(t)x_0 \\ \quad + \int_0^t S(t-s)\left\{f(s, x(s), \int_0^s k(s,t,x(\tau)))\,d\tau\right\} ds \\ \quad + \int_0^t S(t-s)\,u(s)\,ds, \\ x(0) = x_0. \end{cases}$$

where $S(t) \in E_N^n$ and

$$[S(t)]^a = \prod_{m=1}^n [S_m(t)]^a = \prod_{m=1}^n [S^a_{ml}(t),\ S^a_{mr}(t)]$$

where

$$S^a_{mi}(t) = \exp\left\{ \int_0^t a^a_{mi}(s)\,ds \right\}, \quad i=l, r$$

is continuous. That is, there exists a constant $C > 0$ such that $|S^a_{mi}(t)| \leq C$ for all $t \in [0, T]$.

Assume that the following hypotheses :

(H1)Nonlinear regular function f: $[0, T] \times E_N^n \times E_N^n \to E_N^n$ and k: $[0, T] \times [0, T] \times E_N^n \to E_N^n$ are satisfy a global Lipschitz condition, there exist $K > 0$ and $p > 0$ such that

$$d_H(f^a_i(t, x_1, y_1),\ f^a_i(t, x_2, y_2))$$

$$\leq K(d_H([x_1]^a,\ [x_2]^a) + d_H([y_1]^a,\ [y_2]^a)),$$

$$d_H(k^a_i(t, s, x_1),\ k^a_i(t, s, x_2))$$

$$\leq p\ d_H([x_1]^a,\ [x_2]^a).$$

where $x_i, y_i \in E_N$ $(i=1,\ 2)$.

Theorem 3.2. Let $T > 0$. f and k are satisfied (H1). And $CKT(1 + \frac{p}{2}T) < 1$ then for every $x_0 \in E_N^n$, equation (3.3) has a unique fuzzy solution $x \in C([0,\ T]: E_N^n)$.

Proof. For each $x(t) \in E_N^n$; $t \in [0, T]$. Define

$$(\Phi x)(t) = S(t)x_0 + \int_0^t S(t-s)\,u(s)\,ds$$

$$+ \int_0^t S(t-s) f\left(s, x(s), \int_0^s k(s,t,x(\tau))\,d\tau\right) ds$$

Then Φ is the continuous function from $C([0,\ T]: E_N^n)$ to itself. Then there exists $\Phi_m (m=1,2,\cdots,n)$ is continuous function from $C([0,\ T]: E_N^n)$ to itself. Let $x(t), y(t) \in C([0,\ T]: E_N^n)$ then there exists $x_m(t), y_m(t) \in C([0,\ T]: E_N)$ $(m=1,2,\cdots,n)$. Thus

$$d_H([\Phi_m x_m(t)]^\alpha, [\Phi_m y_m(t)]^\alpha)$$

$$= d_H\Big(\Big[S_m(t)x_{0m} + \int_0^t S_m(t-s) f_m(s, x_m(s), \int_0^s k_m(s,\tau,x_m(\tau))d\tau)\,ds + \int_0^t S_m(t-s)\,u_m(s)\,ds\Big]^\alpha,$$

$$\Big[S_m(t)x_{0m} + \int_0^t S_m(t-s) f_m(s, y_m(s), \int_0^s k_m(s,\tau,y_m(\tau))\,d\tau)\,ds + \int_0^t S_m(t-s)\,u_m(s)\,ds\Big]^\alpha\Big)$$

$$\le d_H\Big(\Big[\int_0^t S_m(t-s) f_m(s,x_m(s), \int_0^s k_m(s,\tau,x_m(\tau))\,d\tau)ds\Big]^\alpha, \Big[\int_0^t S_m(t-s) f_m(s,y_m(s), \int_0^s k_m(s,\tau,y_m(\tau))\,d\tau)\,ds\Big]^\alpha\Big)$$

$$\le CK\int_0^t \big(d_H([x_m(s)]^\alpha,[y_m(s)]^\alpha) + d_H([x_m(\tau)]^\alpha,[y_m(\tau)]^\alpha)\big)ds$$

$$\le CK\int_0^t (1+ps)\,d_H([x_m(s)]^\alpha,[y_m(s)]^\alpha)\,ds$$

$$\le CK\Big(t+\frac{pt^2}{2}\Big)d_H([x_m(t)]^\alpha,[y_m(t)]^\alpha).$$

Therefore

$$d_\infty(\Phi x, \Phi y)$$

$$= \sup_{\alpha\in[0,1]} d_H([\Phi x)]^\alpha, [\Phi y]^\alpha)$$

$$= \sup_{\alpha\in[0,1]} d_H\Big(\Big[\prod_{m=1}^n \Phi_m x_m(t)\Big]^\alpha, \Big[\prod_{m=1}^n \Phi_m y_m(t)\Big]^\alpha\Big)$$

$$= \sup_{\alpha\in[0,1]}\Big\{\sum_{m=1}^n \big(d_H([\Phi_m x_m(t)]^\alpha, [\Phi_m y_m(t)]^\alpha)\big)^2\Big\}^{\frac{1}{2}}$$

$$\le \sup_{\alpha\in[0,1]}\Big\{\sum_{m=1}^n \Big(CK\Big(t+\frac{p}{2}t^2\Big) \times d_H([x_m(t)]^\alpha,[y_m(t)]^\alpha)\Big)^2\Big\}^{\frac{1}{2}}$$

$$= CK\Big(t+\frac{p}{2}t^2\Big) \times \sup_{\alpha\in[0,1]}\Big[\sum_{m=1}^n d_H([x_m(t)]^\alpha, [y_m(t)]^\alpha)^2\Big]^{\frac{1}{2}}$$

$$= CK\Big(t+\frac{p}{2}t^2\Big)d_\infty(x(t),y(t)).$$

Hence

$$H(\Phi x(t), \Phi y(t))$$

$$= \sup_{t\in[0,1]} CK\Big(t+\frac{p}{2}t^2\Big)d_\infty(x(t),y(t))$$

$$\le CKT\Big(1+\frac{p}{2}T\Big)H(x(t), y(t)).$$

We take sufficiently small T, $CKT(1+\frac{p}{2}T) < 1$. Hence Φ is a contraction mapping. By the Banach fixed point theorem, equation (3.3) has a unique fixed point $x \in C([0,T]: E_N^n)$.

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