Out-of-plane Free Vibration Analysis of Curved Timoshenko Beams by the Pseudospectral Method

Jinhee Lee[1,#]

[1] Department of Mechano-Informatics, Hongik University, Choongnam, South Korea

ABSTRACT

The pseudospectral method is applied to the analysis of out-of-plane free vibration of circularly curved Timoshenko beams. The analysis is based on the Chebyshev polynomials and the basis functions are chosen to satisfy the boundary conditions. Natural frequencies are calculated for curved beams of circular cross sections under hinged-hinged, clamped-clamped and hinged-clamped end conditions. The present method gives good accuracy with only a limited number of collocation points.

Key Words : Pseudospectral Method, Eigenvalue Analysis, Curved Timoshenko Beam, Out-of-plane Mode

Nomenclature

A = cross sectional area of the beam
B_k, C_k, D_k = basis functions
b_k, c_k, d_k = pseudospectral coefficients
E = Young's modulus
G = shear modulus
I = second moment of area
I_p = polar moment of area
J = torsional constant
M, Q, S = stress resultants
R = radius of curvature of the curved beam
s_l = slenderness ratio
T_n = Chebyshev polynomial of the first kind
W, w = displacement in the longitudinal direction
κ = shear coefficient
ρ = density of the beam
Θ = total angle of the curved beam
Φ, ϕ = torsional rotation
Ψ, ψ = bending rotation
ω = natural frequency in [rad/sec]

☞ Manuscript received: July 1, 2003 ;
Accepted: January 17, 2004
Corresponding Author : Jinhee Lee
Email : jinhlee@hongik.ac.kr
Tel. +82-41-860-2589 ; Fax +82-41-863-0559

1. Introduction

Free vibration analysis of curved beams based on the Timoshenko theory has been carried out using various methods such as the transfer matrix method (Bickford and Strom[1], Irie et al.[2, 3], Yildirim[4]), the dynamic stiffness method (Issa et al.[5], Howson et al.[6], Tseng et al.[7], Howson and Jemah[8]), the differential quadrature method (Kang et al.[9]) and the finite element method (Davis et al.[10], Prathap and Babu[11], Heppler[12], Lee and Sin[13], Yang and Sin[14]).

The performances of the computers have been improved drastically during the last decade and algorithms that are easier to implement are preferred these days to those that run faster. The pseudospectral method can be considered as a spectral method that performs a collocation process, which makes its formulation straightforward and efficient for writing a code for computation. Also it can be made as spatially accurate as desired through exponential rate of convergence with mesh refinement. The pseudospectral method was applied to the eigenvalue problems of straight Timoshenko beams and axisymmetric Mindlin plates (Lee[15]) and rectangular Mindlin plates (Lee[16]). In this study an out-of-plane free vibration analysis of curved Timoshenko beams using the pseudospectral method is presented.

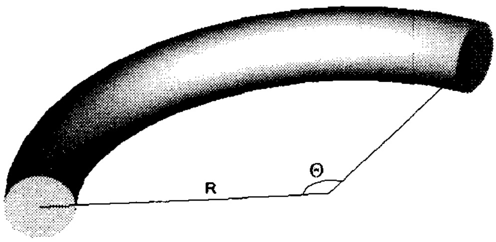

Fig. 1 Geometric configuration of a circularly curved beam

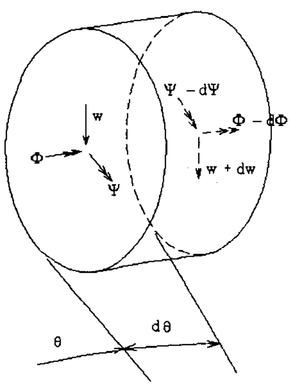

Fig. 2 Generalized displacements

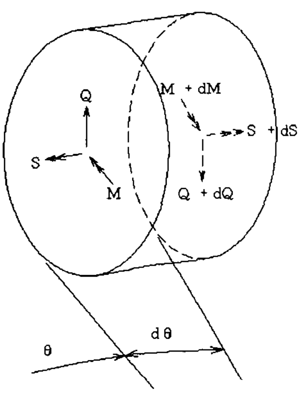

Fig. 3 Stress resultants acting on an infinitesimal element

2. Pseudospectral Formulations

The geometric configuration of a circularly curved beam is shown in Fig. 1. The slenderness ratio s_l of the curved beam is defined by

$$s_l = \sqrt{AR^2/I}\ . \tag{1}$$

The dependent variables and the stress resultants applied to the infinitesimal element of the beam are described in Fig. 2 and Fig. 3, respectively.

The equations of motion for the out-of-plane modes of the curved beam vibration are given as follows:

$$\begin{aligned} &\frac{1}{R}\frac{\partial Q}{\partial \theta} = \rho A \frac{\partial^2 W}{\partial t^2}, \\ &\frac{1}{R}\frac{\partial M}{\partial \theta} + \frac{S}{R} - Q = \rho I \frac{\partial^2 \Psi}{\partial t^2}, \\ &\frac{1}{R}\frac{\partial S}{\partial \theta} - \frac{M}{R} = \rho I_p \frac{\partial^2 \Phi}{\partial t^2}. \end{aligned} \tag{2}$$

The stress resultants M , Q and S are defined by

$$\begin{aligned} &M = \frac{EI}{R}\left(\frac{\partial \Psi}{\partial \theta} + \Phi\right), \\ &Q = \kappa AG\left(\frac{1}{R}\frac{\partial W}{\partial \theta} + \Psi\right), \\ &S = \frac{GJ}{R}\left(\frac{\partial \Phi}{\partial \theta} - \Psi\right). \end{aligned} \tag{3}$$

Assuming the simple harmonic motions in time

$$\begin{aligned} &W(\theta,t) = w(\theta)\cos\omega t, \\ &\Phi(\theta,t) = \phi(\theta)\cos\omega t, \\ &\Psi(\theta,t) = \psi(\theta)\cos\omega t \end{aligned} \tag{4}$$

the substitution of Eq. (3) into Eq. (2) yields

$$\begin{aligned} &\frac{\kappa AG}{R^2}\frac{d^2 w}{d\theta^2} + \frac{\kappa AG}{R}\frac{d\psi}{d\theta} = -\omega^2 \rho A w, \\ &-\frac{\kappa AG}{R}\frac{dw}{d\theta} + \frac{EI+GJ}{R^2}\frac{d\phi}{d\theta} + \frac{EI}{R^2}\frac{d^2\psi}{d\theta^2} \\ &-\left(\frac{GJ}{R^2} + \kappa AG\right)\psi = -\omega^2 \rho I \psi, \\ &\frac{GJ}{R^2}\frac{d^2\phi}{d\theta^2} - \frac{EI}{R^2}\phi - \frac{EI+GJ}{R^2}\frac{d\psi}{d\theta} = -\omega^2 \rho I_p \phi. \end{aligned} \tag{5}$$

When the range of the independent variables is given by $(0 \le \theta \le \Theta)$ it is convenient to use the normalized variable

$$\xi = \frac{2\theta - \Theta}{\Theta} \in [-1,1], \tag{6}$$

and Eq. (5) can be rewritten as

$$\frac{4\kappa AG}{R^2\Theta^2}w'' + \frac{2\kappa AG}{R\Theta}\psi' = -\omega^2\rho A w,$$
$$-\frac{2\kappa AG}{R\Theta}w' + \frac{2(EI+GJ)}{R^2\Theta}\phi' + \frac{4EI}{R^2\Theta^2}\psi''$$
$$-\left(\frac{GJ}{R^2} + \kappa AG\right)\psi = -\omega^2\rho I\psi, \tag{7}$$
$$\frac{4GJ}{R^2\Theta^2}\phi'' - \frac{EI}{R^2}\phi - \frac{2(EI+GJ)}{R^2\Theta}\psi'$$
$$= -\omega^2\rho I_p\phi$$

where $'$ stands for the differentiation with respect to ξ. The series expansions of the exact solutions $w(\xi)$, $\phi(\xi)$ and $\psi(\xi)$ have infinite numbers of terms. In this study, however, they are approximated by the K-th partial sums as follows:

$$w(\xi) \approx \tilde{w}(\xi) = \sum_{k=1}^{K} b_k B_k(\xi),$$
$$\phi(\xi) \approx \tilde{\phi}(\xi) = \sum_{k=1}^{K} c_k C_k(\xi), \tag{8}$$
$$\psi(\xi) \approx \tilde{\psi}(\xi) = \sum_{k=1}^{K} d_k D_k(\xi).$$

The end conditions considered in this study are clamped-clamped, hinged-hinged, and clamped-hinged boundary conditions. The boundary conditions are expressed by

$$\begin{cases} \text{hinged}: w=0,\ \phi=0,\ M=0, \\ \text{clamped}: w=0,\ \phi=0,\ \psi=0. \end{cases} \tag{9}$$

The basis functions

$$B_{2n-1}(\xi) = C_{2n-1}(\xi) = T_{2n}(\xi) - T_0(\xi),$$
$$B_{2n}(\xi) = C_{2n}(\xi) = T_{2n+1}(\xi) - T_1(\xi), \tag{10}$$
$$(n = 1, 2, \cdots)$$

identically satisfy the boundary conditions $u=0$ and $\phi=0$ at $\xi=\pm1$. The basis function $D_k(\xi)$ is required to satisfy either $\psi=0$ or $M=0$ at the ends. It is worthwhile to note that in the event $\phi=0$ is already specified at $\xi=\pm1$ the boundary condition $M=0$ is reduced to $\psi'=0$, and $D_k(\xi)$ is assumed to be

$$D_{2n-1}(\xi) = T_{2n}(\xi) - T_0(\xi) + a_1\xi^2 + a_2\xi,$$
$$D_{2n}(\xi) = T_{2n+1}(\xi) - T_1(\xi) + a_3\xi^2 + a_4\xi, \tag{11}$$
$$(n = 1, 2, \cdots).$$

The procedure to compute the constants a_1, a_2, a_3 and a_4 that satisfy the boundary conditions is given in the Appendix.

By substituting Eq. (8) into Eq. (7) and by setting the residuals equal to zero at the collocation points

$$\xi_i = -\cos\frac{(2i-1)}{2K}, \qquad (i = 1, \cdot\cdot, K) \tag{12}$$

the pseudospectral equations for the out-of-plane mode is given by

$$\sum_{k=1}^{K}\left\{b_k\frac{4\kappa AG}{R^2\Theta^2}B_k''(\xi_i) + d_k\frac{2\kappa AG}{R\Theta}D_k'(\xi_i)\right\}$$
$$= -\omega^2\sum_{k=1}^{K}\rho A b_k B_k(\xi_i),$$
$$\sum_{k=1}^{K}\left[-b_k\frac{2\kappa AG}{R\Theta}B_k'(\xi_i) + c_k\frac{2(EI+GJ)}{R^2\Theta}C_k'(\xi_i)\right.$$
$$\left. + d_k\left\{\frac{4EI}{R^2\Theta^2}D_k''(\xi_i) - \left(\frac{GJ}{R^2} + \kappa AG\right)D_k(\xi_i)\right\}\right]$$
$$= -\omega^2\sum_{k=1}^{K}\rho I d_k D_k(\xi_i), \tag{13}$$
$$\sum_{k=1}^{K}\left[c_k\left\{\frac{4GJ}{R^2\Theta^2}C_k''(\xi_i) - \frac{EI}{R^2}C_k(\xi_i)\right\}\right.$$
$$\left. - d_k\frac{2(EI+GJ)}{R^2\Theta}D_k'(\xi_i)\right]$$
$$= -\omega^2\sum_{k=1}^{K}\rho I_p c_k C_k(\xi_i),$$
$$(i = 1, \cdots, K).$$

The total number of pseudospectral coefficients $b_1, \cdot\cdot, b_K, c_1, \cdot\cdot, c_K, d_1, \cdot\cdot, d_K$ is $3K$, which matches the total number of equations in Eq. (13), and the equation is solved for the eigenvalues.

Table 1 Convergence test of nondimensionalized frequency parameter λ_i (circular cross section, clamped-clamped boundary condition, $s_l = 100$)

	K=15	K=20	K=25	K=30	2
1	4.4731	4.4731	4.4731	4.4731	4.473
2	12.892	12.892	12.892	12.892	12.89
3	26.081	26.081	26.081	26.081	26.08
4	43.684	43.684	43.684	43.684	43.68
5	65.561	65.561	65.561	65.561	
6	91.654	91.582	91.582	91.582	
7	116.83	116.83	116.83	116.83	
8	121.77	121.61	121.61	121.61	
9	162.84	155.50	155.50	155.50	
10	199.01	193.23	193.10	193.10	

Table 2 Nondimensionalized frequency parameter λ_i (circular cross section, clamped-clamped boundary condition, $K = 30$)

s_l		$\Theta = 60^\circ$	$\Theta = 120^\circ$	$\Theta = 180^\circ$
	1	12.783	3.9032	1.7139
	2	19.730	9.6368	4.5178
10	3	26.891	11.513	8.6029
	4	37.635	17.012	9.3073
	5	43.609	19.704	13.469
	1	16.885	4.3094	1.7908
	2	39.700	11.796	5.0324
20	3	40.934	22.510	10.232
	4	70.581	23.303	16.917
	5	75.611	35.482	18.738
	1	19.062	4.4515	1.8147
	2	51.731	12.737	5.2138
50	3	98.146	25.534	10.884
	4	99.483	42.300	18.537
	5	155.34	58.386	28.037
	1	19.454	4.4731	1.8182
	2	54.148	12.892	5.2415
100	3	105.86	26.081	10.989
	4	173.16	43.684	18.813
	5	199.01	65.561	28.633
	1	19.556	4.4785	1.8191
	2	54.811	12.931	5.2485
200	3	108.12	26.223	11.016
	4	178.81	44.055	18.885
	5	266.47	66.354	28.788

Table 3 Nondimensionalized frequency parameter λ_i (circular cross section, hinged-hinged boundary condition, $K = 30$)

s_l		$\Theta = 50^\circ$	$\Theta = 110^\circ$	$\Theta = 180^\circ$
	1	9.3708	1.3157	2.4332
	2	23.205	7.7508	6.4665
10	3	30.221	12.182	9.2605
	4	44.924	16.481	11.450
	5	52.790	21.283	14.083
	1	10.749	1.3604	2.5590
	2	41.306	8.7294	7.1728
20	3	46.721	20.292	13.429
	4	80.829	24.640	18.737
	5	90.254	34.856	21.032
	1	11.288	1.3739	2.5985
	2	48.308	9.0942	7.4261
50	3	105.96	22.087	14.248
	4	117.05	39.913	22.954
	5	179.71	61.805	33.458
	1	11.373	1.3758	2.6043
	2	49.712	9.1510	7.4650
100	3	112.47	22.394	14.380
	4	197.93	40.880	23.285
	5	234.17	64.472	34.144
	1	11.395	1.3763	2.6057
	2	50.087	9.1654	7.4748
200	3	114.35	22.474	14.414
	4	203.66	41.136	23.371
	5	317.38	65.097	34.324

3. Numerical Examples

A preliminary test is run to check the convergence of the pseudospectral method applied to the out-of-plane free vibration analysis of curved Timoshenko beams. The eigenvalues of circularly curved beam of circular cross section with clamped-clamped boundary condition for the slenderness ratio $s_l = 100$ are computed for different collocation number K , and the computed results are listed in Table 1. This shows the rapid convergence nature of the pseudospectral method such that the convergence of the lowest 5 eigenvalues to 5 digits is achieved for $K = 15$, and the lowest 9 eigenvalues for less than $K = 20$. Poisson's ratio ν is 0.3 and the shear correction factor κ for circular cross section of beam is 0.89 throughout the paper. The numbers given in Tables 1 to 4 are the nondimensionalized frequency parameters λ_i defined as

$$\lambda_i = \sqrt{\rho A R^4 \omega_i^2 / EI} \ . \tag{14}$$

The eigenvalues computed by the transfer matrix method (Irie *et al.*[2]) are also given for comparison in Table 1, which are in excellent agreement with those of present study.

Eigenvalues are computed with $K = 30$ for various slenderness ratios s_l and curved beam angles Θ under clamped-clamped, hinged-hinged and clamped-hinged boundary conditions, and lowest 5 eigenvalues for each boundary condition are listed in Tables 2 to 4. It is shown that the eigenvalues of the curved beams in Tables 2 to 4 tend to grow larger as the slenderness ratio increases.

Table 4 Nondimensionalized frequency parameter λ_i (circular cross section, clamped-hinged boundary condition, $K = 30$)

s_l		$\Theta = 60^\circ$	$\Theta = 120^\circ$	$\Theta = 180^\circ$
	1	9.6368	2.4265	0.9011
	2	19.704	8.1148	3.5037
10	3	25.124	11.511	7.5772
	4	37.624	15.648	9.2677
	5	42.520	19.702	12.502
	1	11.796	2.5842	0.9244
	2	35.482	9.4587	3.7888
20	3	39.701	19.808	8.6902
	4	65.654	23.278	15.168
	5	75.610	32.638	18.738
	1	12.737	2.6347	0.9312
	2	42.300	9.9891	3.8834
50	3	86.345	21.858	9.1148
	4	99.473	37.799	16.349
	5	142.14	57.472	25.455
	1	12.892	2.6422	0.9322
	2	43.684	10.073	3.8976
100	3	91.582	22.215	9.1814
	4	155.50	38.796	16.546
	5	199.01	59.682	25.906
	1	12.931	2.6441	0.9325
	2	44.055	10.095	3.9012
200	3	93.074	22.307	9.1984
	4	159.58	39.060	16.596
	5	243.24	60.283	26.023

4. Conclusions

The Chebyshev pseudospectral method is applied to the analysis of out-of-plane free vibration of curved Timoshenko beams. Numerical examples are provided for circularly curved beams of circular cross section under clamped-clamped, hinged-hinged and clamped-hinged boundary conditions for various slenderness ratios and curved beam angles. The results under the clamped-clamped boundary condition are compared with the solutions by the transfer matrix method and it is shown that they are in excellent agreement. The title problem demonstrates the rapid convergence and accuracy as well as the conceptual simplicity of the pseudospectral method.

Acknowledgement

This work was supported by 2003 Hongik University Research Fund.

References

1. Bickford, W. B. and Strom, B. T., "Vibration of Plane Curved Beams," Journal of Sound and Vibration, Vol. 39, No. 2, pp. 135-146, 1975.
2. Irie, T., Yamada, G. and Tanaka, K., "Natural Frequencies of Out-of-plane Vibration of Arcs," Journal of Applied Mechanics, Transactions of ASME, Vol. 49, No. 4, pp. 910-913, 1982.
3. Irie, T., Yamada, G. and Tanaka, K., "Natural Frequencies of In-plane Vibration of Arcs," Journal of Applied Mechanics, Transactions of ASME, Vol. 50, No. 2, pp. 449-452, 1983.
4. Yildrim, V., "A Computer Program for the free Vibration Analysis of Elastic Arcs," Computers & Structures, Vol. 62, No. 3, pp. 475-485, 1997.
5. Issa, M. S., Wang, T. M. and Hsiao, B. T., "Extensional Vibrations of Continuous Circular Curved Beams with Rotary Inertia and Shear Deformation I: Free Vibration," Journal of Sound and Vibration, Vol. 114, No. 2, pp. 297-308, 1987.
6. Howson, W. P., Jemah, A. K. and Zhou, J. Q., "Exact Natural Frequencies for Out-of-plane Motion of Plane Structures Composed of Curved Beam Members," Computers & Structures, Vol. 55, No. 6, pp. 989-995, 1995.
7. Tseng, Y.-P., Huang, C. S. and Lin, C.-J., "Dynamic Stiffness Analysis for In-plane Vibrations of Arches with Variable Curvature," Journal of Sound and Vibration, Vol. 207, No. 1, pp. 15-31, 1997.
8. Howson, W. P. and Jemah, A. K., "Exact Out-of-plane Natural Frequencies of Curved Timoshenko Beam," Journal of Engineering Mechanics, Vol. 125, No. 1, pp. 19-25, 1999.
9. Kang, K., Bert, C. W. and Striz, A. G., "Vibration Analysis of Shear Deformable Circular Arches by the Differential Quadrature Method," Journal of Sound and Vibration, Vol. 183, No. 2, pp. 353-360, 1995.
10. Davis, R., Henshell, R. D. and Warburton, G. B., "Constant Curvature Beam Finite Elements for In-plane Vibration," Journal of Sound and Vibration,

Vol. 25, No. 4, pp. 561-576, 1972.
11. Prathap, G. and Babu, C. R., "An Isoparametric Quadratic Thick Curved Beam Element," International Journal for Numerical Methods in Engineering, Vol. 23, No. 9, pp. 1583-1600, 1986.
12. Heppler, G. R., "An Element for Studying the Vibration of Unrestrained Curved Timoshenko Beams," Journal of Sound and Vibration, Vol. 158, No. 3, pp. 387-404, 1992.
13. Lee, P.-G. and Sin, H.-C., "Locking Free Curved Beam Element Based on Curvature," International Journal for Numerical Methods in Engineering, Vol. 37, No. 6, pp. 989-1007, 1994.
14. Yang, S.-Y. and Sin, H.-C., "Curvature Base Beam Element for the Analysis of Timoshenko and Shear-Deformable Beam," Journal of Sound and Vibration, Vol. 187, No. 4, pp. 569-584, 1995.
15. Lee, J., "Eigenvalue Analysis of Axisymmetric Circular Mindlin Plates by Pseudospectral Method," International Journal of KSPE, Vol. 3, No. 3, pp. 44-49, 2002.
16. Lee, J., "Eigenvalue Analysis of Rectangular Mindlin Plates by Chebyshev Pseudospectral Method," KSME International Journal, Vol. 17, No. 3, pp. 370-379, 2003.

Appendix: Constants of Function $D_k(\xi)$

A1. The clamped-clamped boundary condition for the Timoshenko beam is given by

$$w=0, \quad \phi=0, \quad \psi=0 \quad \text{at} \quad \xi=\pm 1. \tag{A1}$$

$w=0$ and $\phi=0$ at $\xi=\pm 1$ are satisfied by the condition given in Eq. (10), and the remaining condition $\psi=0$ at $\xi=\pm 1$ can be satisfied simply by choosing

$$\begin{aligned} D_{2n-1}(\xi) &= T_{2n}(\xi)-T_0(\xi) \\ D_{2n}(\xi) &= T_{2n+1}(\xi)-T_1(\xi) \\ &(n=1,2,\cdots), \end{aligned} \tag{A2}$$

which can be accomplished by $a_1=a_2=a_3=a_4=0$.

A2. The hinged-hinged boundary condition is

$$\begin{cases} w=0, & \phi=0, \quad M=0 \quad \text{at} \quad \xi=-1 \\ w=0, & \phi=0, \quad M=0 \quad \text{at} \quad \xi=1. \end{cases} \tag{A3}$$

$w=0$ and $\phi=0$ at $\xi=\pm 1$ are satisfied by the condition given in Eq. (10), and the remaining condition is

$$M\Big|_{\xi=\pm 1} = \frac{EI}{R}\frac{d\psi}{d\theta}\bigg|_{\xi=\pm 1} = \frac{2EI}{R\Theta}\frac{d\psi}{d\xi}\bigg|_{\xi=\pm 1} = 0. \tag{A4}$$

Using the relationship (8), it is worthwhile to note that

$$\frac{dD_k}{d\xi}\bigg|_{\xi=\pm 1} = 0 \qquad (k=1,\ 2,\ \cdots,K) \tag{A5}$$

is a sufficient condition for the zero-moment condition (A4). Having the differentiation of the odd numbered terms of $D_k(\xi)$ with respect to ξ equal to zero makes

$$\frac{dD_{2n-1}}{d\xi}\bigg|_{\xi=\pm 1} = \left(\frac{dT_{2n}}{d\xi}+2a_1\xi+a_2\right)\bigg|_{\xi=\pm 1} = 0 \quad (n=1,\ 2,\ \cdots) \tag{A6}$$

Eq. (A6) is rewritten as

$$\begin{cases} -4n^2-2a_1+a_2=0 & \text{at} \quad \xi=-1 \\ 4n^2+2a_1+a_2=0 & \text{at} \quad \xi=1, \end{cases} \tag{A7}$$

and we have

$$a_1=-2n^2, \quad a_2=0. \tag{A8}$$

The differentiation of the even numbered terms with respect to ξ makes

$$\frac{dD_{2n}}{d\xi}\bigg|_{\xi=\pm 1} = \left(\frac{dT_{2n+1}}{d\xi}-1+2a_3\xi+a_4\right)\bigg|_{\xi=\pm 1} = 0. \tag{A9}$$

Eq. (A9) is also rewritten as

$$\begin{cases} (2n+1)^2-1-2a_3+a_4=0 & \text{at} \quad \xi=-1 \\ (2n+1)^2-1+2a_3+a_4=0 & \text{at} \quad \xi=1, \end{cases} \tag{A10}$$

from which the constants a_3 and a_4 are found to be

$$a_3=0, \quad a_4=-4n(n+1). \tag{A11}$$

A3. The clamped-hinged boundary condition is given by

$$\begin{cases} w=0,\ \phi=0,\ \psi=0 & \text{at} \quad \xi=-1 \\ w=0,\ \phi=0,\ M=0 & \text{at} \quad \xi=1. \end{cases} \tag{A12}$$

$w=0$ and $\phi=0$ at $\xi=\pm1$ are satisfied by the condition given in Eq. (10), and the remaining condition is satisfied by the introduction

$$\begin{cases} D_k=0 & \text{at} \quad \xi=-1 \\ \dfrac{dD_k}{d\xi}=0 & \text{at} \quad \xi=1. \end{cases} \tag{A13}$$

Using the relationships of Eq. (11), the condition for the odd numbered terms is given by

$$\begin{cases} D_{2n-1}\big|_{\xi=-1} = \left(T_{2n}-T_0+a_1\xi^2+a_2\xi\right)\big|_{\xi=-1} \\ \qquad = a_1-a_2=0 \\ \dfrac{dD_{2n-1}}{d\xi}\bigg|_{\xi=1} = \left(\dfrac{dT_{2n}}{d\xi}+2a_1\xi+a_2\right)\bigg|_{\xi=1} \\ \qquad = 4n^2+2a_1+a_2=0\ , \end{cases} \tag{A14}$$

from which we have

$$a_1=a_2=-\frac{4n^2}{3}. \tag{A15}$$

For the even numbered terms the condition is expressed by

$$\begin{cases} D_{2n}\big|_{\xi=-1} = \left(T_{2n+1}-T_1+a_3\xi^2+a_4\xi\right)\big|_{\xi=-1} \\ \qquad = a_3-a_4=0 \\ \dfrac{dD_{2n}}{d\xi}\bigg|_{\xi=1} = \left(\dfrac{dT_{2n+1}}{d\xi}-1+2a_3\xi+a_4\right)\bigg|_{\xi=1} \\ \qquad = (2n+1)^2-1+2a_3+a_4=0\ , \end{cases} \tag{A16}$$

from which we have

$$a_3=a_4=-\frac{4n(n+1)}{3}. \tag{A17}$$