

## RANK AND PERIMETER PRESERVERS OF BOOLEAN RANK-1 MATRICES

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ABSTRACT. For a Boolean rank-1 matrix  $A = \mathbf{ab}^t$ , we define the *perimeter* of  $A$  as the number of nonzero entries in both  $\mathbf{a}$  and  $\mathbf{b}$ . We characterize the Boolean linear operators that preserve rank and perimeter of Boolean rank-1 matrices.

### 1. Introduction and preliminaries

The *Boolean algebra* consists of the set  $\mathbb{B} = \{0, 1\}$  equipped with two binary operations, addition and multiplication. The operations are defined as usual except that  $1 + 1 = 1$ .

There are many papers on linear operators that preserve the rank of matrices over several semirings. Boolean matrices also have been the subject of research by many authors. Beasley and Pullman [1] obtained characterizations of rank-preserving operators of Boolean matrices. They were unable to find necessary and sufficient conditions for a Boolean operator to preserve the rank of Boolean rank-1 matrices. This remains open until now. We consider this problem by adding conditions on perimeter of the Boolean rank-1 matrices.

Let  $M_{m,n}(\mathbb{B})$  denote the set of all  $m \times n$  matrices with entries in the Boolean algebra  $\mathbb{B}$ . The usual definitions for adding and multiplying matrices apply to Boolean matrices as well. Throughout this paper, we shall adopt the convention that  $m \leq n$  unless otherwise specified.

If an  $m \times n$  Boolean matrix  $A$  is not zero, then its *Boolean rank*,  $b(A)$ , is the least  $k$  for which there exist  $m \times k$  and  $k \times n$  Boolean matrices  $B$  and  $C$  with  $A = BC$ . The Boolean rank of the zero matrix is 0. It is

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Received February 27, 2003.

2000 Mathematics Subject Classification: Primary 15A03, 15A04.

Key words and phrases: Boolean linear operator, perimeter, (U,V)-operator.

This work was supported by Korea Research Foundation Grant KRF-2002-042-C00001.

well known that  $b(A)$  is the least  $k$  such that  $A$  is the sum of  $k$  matrices of Boolean rank 1 ([3], [4]).

Let  $\Delta_{m,n} = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$ , and  $E_{ij}$  be the  $m \times n$  matrix whose  $(i, j)$ th entry is 1 and whose other entries are all 0, and  $\mathbb{E}_{m,n} = \{E_{ij} | (i, j) \in \Delta_{m,n}\}$ . We call  $E_{ij}$  a *cell*.

An  $n \times n$  Boolean matrix  $A$  is said to be *invertible* if for some  $X$ ,  $AX = XA = I_n$ , where  $I_n$  is the  $n \times n$  identity matrix. This matrix  $X$  is necessarily unique when it exists. It is well known that the permutation matrices are the only invertible Boolean matrices (see [2]).

If  $A$  and  $B$  are in  $\mathbb{M}_{m,n}(\mathbb{B})$ , we say  $A$  *dominates*  $B$  (written  $B \leq A$  or  $A \geq B$ ) if  $a_{ij} = 0$  implies  $b_{ij} = 0$  for all  $i, j$ . Equivalently,  $B \leq A$  if and only if  $A + B = A$ .

Also lowercase, boldface letters will represent vectors, all vectors  $\mathbf{u}$  are column vectors ( $\mathbf{u}^t$  is a row vector) for  $\mathbf{u} \in \mathbb{B}^m [= \mathbb{M}_{m,1}(\mathbb{B})]$ .

It is easy to verify that the Boolean rank of  $A$  is 1 if and only if there exist nonzero (Boolean) vectors  $\mathbf{a} \in \mathbb{M}_{m,1}(\mathbb{B})$  and  $\mathbf{b} \in \mathbb{M}_{n,1}(\mathbb{B})$  such that  $A = \mathbf{a}\mathbf{b}^t$ . And these vectors  $\mathbf{a}$  and  $\mathbf{b}$  are uniquely determined by  $A$ . We call  $\mathbf{a}$  the *left factor*, and  $\mathbf{b}$  the *right factor* of  $A$ . Therefore there are exactly  $(2^m - 1)(2^n - 1)$  rank-1  $m \times n$  Boolean matrices.

For any vector  $\mathbf{u} \in \mathbb{M}_{m,1}(\mathbb{B})$ , let  $|\mathbf{u}|$  be the number of nonzero entries in  $\mathbf{u}$ , and when  $A = \mathbf{a}\mathbf{b}^t$  is not zero, define the *perimeter* of  $A$ ,  $P(A)$ , as  $|\mathbf{a}| + |\mathbf{b}|$ . Since the factorization of  $A$  as  $\mathbf{a}\mathbf{b}^t$  is unique, the perimeter of  $A$  is also unique.

PROPOSITION 1.1. *If  $A, B$  and  $A + B$  are rank-1 matrices in  $\mathbb{M}_{m,n}(\mathbb{B})$ , then  $P(A + B) < P(A) + P(B)$ .*

*Proof.* Let  $A = \mathbf{a}\mathbf{x}^t, B = \mathbf{b}\mathbf{y}^t$  and  $A + B = \mathbf{c}\mathbf{z}^t$  be the factorizations of  $A, B$  and  $A + B$ . We have for all  $i, j$

$$(1.1) \quad a_i\mathbf{x} + b_i\mathbf{y} = c_i\mathbf{z}$$

and

$$(1.2) \quad x_j\mathbf{a} + y_j\mathbf{b} = z_j\mathbf{c}.$$

Now, consider three cases :

Case (1)  $B \leq A$ . Then  $A + B = A$ . Thus  $P(A + B) = P(A) < P(A) + P(B)$  since  $P(B) \neq 0$ , as required.

Case (2)  $A \leq B$ . It is similar to the case (1).

Case (3)  $A \not\leq B$  and  $B \not\leq A$ .

Subcase (3.1)  $\mathbf{a} \not\leq \mathbf{b}$  and  $\mathbf{b} \not\leq \mathbf{a}$ . From (1.1) we have for some  $i, j$ ,  $a_i = 1$  and  $b_i = 0$ , and  $a_j = 0$  and  $b_j = 1$  so that  $\mathbf{x} = c_i \mathbf{z}$  and  $\mathbf{y} = c_j \mathbf{z}$ . But  $\mathbf{x} \neq 0$  and  $\mathbf{y} \neq 0$ , so  $\mathbf{x} = \mathbf{y} = \mathbf{z}$ . Thus

$$\begin{aligned} P(A + B) &= P((\mathbf{a} + \mathbf{b})\mathbf{z}^t) = |\mathbf{a} + \mathbf{b}| + |\mathbf{z}| \\ &< (|\mathbf{a}| + |\mathbf{z}|) + (|\mathbf{b}| + |\mathbf{z}|) = P(A) + P(B), \end{aligned}$$

as required.

Subcase (3.2)  $\mathbf{a} \leq \mathbf{b}$ . Then  $\mathbf{x} \not\leq \mathbf{y}$ . Thus  $\mathbf{a} = z_j \mathbf{c} = \mathbf{c}$  for some  $j$  from (1.2). But  $\mathbf{c} \geq \mathbf{b}$  from (1.2). Therefore  $\mathbf{a} = \mathbf{b} = \mathbf{c}$  and we have

$$\begin{aligned} P(A + B) &= P(\mathbf{c}(\mathbf{x} + \mathbf{y})^t) = |\mathbf{c}| + |\mathbf{x} + \mathbf{y}| \\ &< (|\mathbf{c}| + |\mathbf{x}|) + (|\mathbf{c}| + |\mathbf{y}|) = P(A) + P(B), \end{aligned}$$

as required.

Subcase (3.3)  $\mathbf{b} \leq \mathbf{a}$ . It is similar to the subcase (3.2). □

A mapping  $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{m,n}(\mathbb{B})$  is called a *Boolean linear operator* if  $T$  preserves sums and 0. A Boolean linear operator  $T : \mathbb{M}_{m,n}(\mathbb{B}) \rightarrow \mathbb{M}_{m,n}(\mathbb{B})$  is *invertible* if  $T$  is injective and surjective. As with vector spaces over fields, the inverse,  $T^{-1}$ , of a Boolean linear operator  $T$  is also linear.

Beasley and Pullman obtained the following:

LEMMA 1.2.([1]) *If  $T$  is a Boolean linear operator on  $\mathbb{M}_{m,n}(\mathbb{B})$ , then the following statements are equivalent:*

- (a)  $T$  is invertible;
- (b)  $T$  is surjective;
- (c)  $T$  permutes  $\mathbb{E}_{m,n}$ .

In this paper, we obtain characterizations of the linear operators that preserve the rank and the perimeter of every Boolean rank-1 matrix. These are motivated by analogous results for the preserver of Boolean rank. However, we obtain results and proofs in the view of the perimeter analog.

In the following, “linear operator” on  $\mathbb{M}_{m,n}(\mathbb{B})$  means “Boolean linear operator”.

## 2. Linear operators preserving the rank and perimeter of Boolean rank-1 matrices.

In this section, we will characterize the linear operators that preserve the rank and the perimeter of every Boolean rank-1 matrix.

Suppose  $T$  is a linear operator on  $\mathbb{M}_{m,n}(\mathbb{B})$ . Then

(i)  $T$  is a  $(U, V)$ -operator if there exist invertible matrices  $U$  and  $V$  such that  $T(A) = UAV$  for all  $A$  in  $\mathbb{M}_{m,n}(\mathbb{B})$ , or  $m = n$  and  $T(A) = UA^tV$  for all  $A$  in  $\mathbb{M}_{m,n}(\mathbb{B})$ .

(ii)  $T$  preserve Boolean rank 1 if  $b(T(A)) = 1$  whenever  $b(A) = 1$  for all  $A \in \mathbb{M}_{m,n}(\mathbb{B})$ .

(iii)  $T$  preserve perimeter  $k$  of Boolean rank-1 matrices if  $P(T(A)) = k$  whenever  $P(A) = k$  for all  $A \in \mathbb{M}_{m,n}(\mathbb{B})$  with  $b(A) = 1$ .

The following example shows that not all rank-1 preserving operators  $T$  are of the form  $T(X) = UXV$  for some invertible matrices  $U, V$ .

EXAMPLE 2.1. Let  $T : \mathbb{M}_{2,3}(\mathbb{B}) \rightarrow \mathbb{M}_{2,3}(\mathbb{B})$  be defined by

$$T \left( \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \right) = \begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \end{bmatrix} + (c + d + e + f) \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then  $T$  preserves Boolean rank 1, but  $T \left( \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . So we can not find invertible matrices  $U$  and  $V$  such that  $T(X) = UXV$  for  $X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbb{M}_{2,3}(\mathbb{B})$ . This shows that  $T$  is not a  $(U, V)$ -operator.  $\square$

PROPOSITION 2.2. If  $T$  is a  $(U, V)$ -operator on  $\mathbb{M}_{m,n}(\mathbb{B})$ , then  $T$  preserves both rank and perimeter of Boolean rank-1 matrices.

*Proof.* Let  $A$  be a matrix in  $\mathbb{M}_{m,n}(\mathbb{B})$  with  $b(A) = 1$  and  $A = \mathbf{a}\mathbf{u}^t$  be the factorization of  $A$  with  $P(A) = |\mathbf{a}| + |\mathbf{u}|$ . Since  $T(A) = UAV = (U\mathbf{a})(\mathbf{u}^tV) = (U\mathbf{a})(V^t\mathbf{u})^t$  and  $U, V$  are invertible matrices (and hence permutations), we have

$$b(T(A)) = b((U\mathbf{a})(V^t\mathbf{u})^t) = 1,$$

and

$$P(T(A)) = |U\mathbf{a}| + |V^t\mathbf{u}| = |\mathbf{a}| + |\mathbf{u}| = P(A).$$

If  $m = n$  and  $T(A) = UA^tV$ , then we can show that  $b(T(A)) = 1$  and  $P(T(A)) = |\mathbf{a}| + |\mathbf{u}|$  by a similar method as above.

Hence  $(U, V)$ -operators preserve the rank and perimeter of every Boolean rank-1 matrix.  $\square$

Since a Boolean matrix has perimeter 2 if and only if it is a cell, we have the following Lemma:

LEMMA 2.3. Let  $T$  be a linear operator on  $\mathbb{M}_{m,n}(\mathbb{B})$ . If  $T$  preserves Boolean rank 1 and perimeter 2 of Boolean rank-1 matrices, then (1)  $T$  maps a cell into a cell and (2)  $T$  maps a row (or a column) of a Boolean matrix into a row or a column if  $m=n$ .

*Proof.* (1) follows from the property that  $T$  preserves perimeter 2. Suppose (2) is not true. Then there exists two distinct cells  $E_{i,j}, E_{i,h}$  in some  $i$ th row such that  $T(E_{i,j})$  and  $T(E_{i,h})$  lie in two different rows and different columns. Then the Boolean rank of  $E_{i,j} + E_{i,h}$  is 1 but that of  $T(E_{i,j} + E_{i,h}) = T(E_{i,j}) + T(E_{i,h})$  is 2. Therefore  $T$  does not preserve Boolean rank 1, a contradiction.  $\square$

Now, we give an example of a linear operator that preserves Boolean rank 1 and perimeter 2.

EXAMPLE 2.4. Let  $T : \mathbb{M}_{2,2}(\mathbb{B}) \rightarrow \mathbb{M}_{2,2}(\mathbb{B})$  be defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = (a + b + c + d) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then  $T$  is a linear operator and preserves Boolean rank 1. And  $T$  preserves perimeter 2 of each Boolean rank-1 matrix. But  $T$  does not preserve perimeter 3; for,  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 1]$  has Boolean rank 1 and perimeter 3, but  $T(A) = (1 + 1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0]$  has Boolean rank 1 and perimeter 2.

Moreover,  $T$  is not a  $(U, V)$ -operator: if there existed invertible matrices  $U$  and  $V$  such that  $T(X) = UXV$  for all  $X \in \mathbb{M}_{2,2}(\mathbb{B})$ , then for  $j = 1, 2$ , we have  $T(E_{1,j}) = Ue_1e_j^tV = \mathbf{u}_1\mathbf{v}_j^t$  where  $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and  $\mathbf{u}_1$  is the first column of  $U$  and  $\mathbf{v}_j$  is the  $j$ th column of  $V$ .

But  $T(E_{1,1}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0]$  and  $T(E_{1,2}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0]$  and hence  $V = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ , which is not invertible. This contraction implies that  $T$  is not a  $(U, V)$ -operator.  $\square$

Let  $R_i = \{E_{i,j} | 1 \leq j \leq n\}, C_j = \{E_{i,j} | 1 \leq i \leq m\}, \mathbb{R} = \{R_i | 1 \leq i \leq m\}$  and  $\mathbb{C} = \{C_j | 1 \leq j \leq n\}$ . Define  $T^*(R_i) = \{T(E_{i,j}) | 1 \leq j \leq n\}$  for all  $i$  and  $T^*(C_j) = \{T(E_{i,j}) | 1 \leq i \leq m\}$  for all  $j$ .

LEMMA 2.5. Let  $T$  be a linear operator on  $\mathbb{M}_{m,n}(\mathbb{B})$ . Suppose that  $T$  preserves Boolean rank 1 and perimeters 2 and  $p (\geq 3)$  of Boolean rank-1 matrices. Then,

- (1)  $T$  maps two distinct cells in a row (or a column) into two distinct cells in a row or in a column,

- (2) if  $T$  maps a row of a Boolean matrix  $A$  into a row, then  $T$  maps each row of  $A$  into a row of  $T(A)$ . Similarly, if  $T$  maps a column of a Boolean matrix  $A$  into a column, then  $T$  maps each column of  $A$  into a column of  $T(A)$ .

*Proof.* (1) Suppose  $T(E_{i,j}) = T(E_{i,h}) = E_{i',j'}$  for some distinct pairs  $(i, j) \neq (i, h)$ . Then  $T$  maps the  $i$ th row into the  $i'$ th row and both the  $j$ th and  $h$ th column into the  $j'$ th column by Lemma 2.3. Thus for any Boolean rank-1 matrix  $A$  with perimeter  $p (\geq 3)$  which dominates  $E_{i,j}$  and  $E_{i,h}$ , we can show that  $T(A)$  has perimeter at most  $p-1$ , a contradiction. Thus  $T$  maps distinct cells in a row into distinct cells in either a row or a column by Lemma 2.3(2).

(2) If not, then there exist rows  $R_i$  and  $R_j$  such that  $T(R_i) \subseteq R_r$  and  $T(R_j) \subseteq C_s$  for some  $r, s$ . Consider a matrix  $D = E_{i,p} + E_{i,q} + E_{j,p} + E_{j,q}$  with Boolean rank 1. Then

$$T(D) = T(E_{i,p} + E_{i,q}) + T(E_{j,p} + E_{j,q}) = (E_{r,p'} + E_{r,q'}) + (E_{p'',s} + E_{q'',s})$$

for some  $p' \neq q'$  and  $p'' \neq q''$  by (1). Therefore  $b(T(D)) \neq 1$  and  $T$  does not preserve Boolean rank 1, a contradiction. Hence  $T$  maps each row into a row. Similarly,  $T$  maps each column into a column.  $\square$

Now we have interesting examples:

EXAMPLE 2.6. (1) Consider a linear map  $T$  on  $M_{2,3}(\mathbb{B})$  such that

$$T\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{bmatrix}\right) = \begin{bmatrix} a_{1,1} + a_{2,2} & a_{1,2} + a_{2,3} & a_{1,3} + a_{2,1} \\ 0 & 0 & 0 \end{bmatrix}.$$

Then  $T$  maps each row and each column into the first row. Moreover,  $T$  preserves both Boolean rank and perimeters 2, 3 and 4 of Boolean rank-1 matrices. But  $T$  does not preserve perimeter 5 of Boolean rank-1 matrix and this map  $T$  is not a  $(U, V)$ -operator.

(2) Consider a linear map  $L$  on  $M_{m,n}(\mathbb{B})$  with  $m \geq 2$  and  $n \geq 4$  such that

$$L(A) = B = (b_{i,j})$$

where  $A = (a_{i,j})$ ,  $b_{i,j} = 0$  with  $i \geq 2$  and  $b_{1,j} = \sum_{i=1}^m a_{i,r}$  with  $r \equiv i + (j - 1) \pmod{n}$  and  $1 \leq r \leq n$ . Then  $L$  maps each row and each column into the first row. And  $L$  preserves both Boolean rank and perimeters 2, 3 and  $n + 1$  of Boolean rank-1 matrices. But  $L$  does not preserve perimeters  $k$  ( $k \geq 4$  and  $k \neq n + 1$ ) of Boolean rank-1 matrices: For, if  $4 \leq k \leq n$ , then we can choose a  $2 \times (k - 2)$  submatrix with perimeter  $k$  which is mapped to distinct  $k$  position in the first row of  $B$

under  $L$ . Then this  $1 \times k$  submatrix has perimeter  $k + 1$ . Therefore  $L$  does not preserve perimeter  $k$  of Boolean rank-1 matrices.  $\square$

**THEOREM 2.7.** *Let  $T$  be a linear operator on  $M_{m,n}(\mathbb{B})$  with  $m \geq 2$  and  $n \geq 4$ . Then the following are equivalent:*

- (1)  $T$  is a  $(U, V)$ -operator;
- (2)  $T$  preserves both rank and perimeter of Boolean rank-1 matrices;
- (3)  $T$  preserves both rank and perimeters 2 and  $k$  ( $k \geq 4, k \neq n + 1$ ) of Boolean rank-1 matrices.

*Proof.* (1) implies (2) by Proposition 2.2. It is obvious that (2) implies (3). We now show that (3) implies (1). We will first show that  $T(E_{i,j}) = E_{r,l}$  and the corresponding mapping  $f : \Delta_{m,n} \rightarrow \Delta_{m,n}$  defined by  $f(i, j) = (r, l)$  is a bijection.

By Lemma 2.3,  $T(E_{i,j}) = E_{r,l}$  for some  $(r, l) \in \Delta_{m,n}$ . Without loss of generality, we may assume that  $T$  maps the  $i$ th row into the  $r$ th row. Then  $T$  maps each row into a row. Suppose  $T(E_{i,j}) = T(E_{p,q}) = E_{r,l}$  for some pairs  $(i, j) \neq (p, q)$ . If  $i = p$  or  $j = q$ , then we have a contradiction by Lemma 2.5. So let  $i \neq p$  and  $j \neq q$ .

If  $k = n + k' \geq n + 2$ , consider the matrix

$$D = \sum_{j'=1}^n E_{i,j'} + \sum_{q'=1}^n E_{p,q'} + \sum_{g=1}^{k'-2} \sum_{h=1}^n E_{g,h}$$

with Boolean rank 1 and perimeter  $n + k' = k$ . Then  $T$  maps the  $i$ th and  $p$ th row of  $D$  into the  $r$ th row by Lemma 2.5. Then the perimeter of  $T(D)$  is less than  $n + k' = k$ , a contradiction.

If  $4 \leq k \leq n$ , we will show that we can choose a  $2 \times (k - 2)$  submatrix from the  $i$ th and  $p$ th row whose image under  $T$  has  $1 \times k$  submatrix in the  $r$ th row as follows: Since  $T(E_{i,j}) = T(E_{p,q}) = E_{r,l}$ ,  $T$  maps the  $i$ th row and the  $p$ th row into the  $r$ th row. But  $T$  maps distinct cells in each row (or column) to distinct cells by Lemma 2.5. Now, choose  $E_{i,j}$ ,  $E_{p,j}$  but do not choose  $E_{i,q}$ ,  $E_{p,q}$ . Since there is a cell  $E_{p,h}$  ( $h \neq j, q$ ) in the  $p$ th row such that  $T(E_{p,h}) = T(E_{i,q})$  but  $T(E_{i,h}) \neq T(E_{p,j})$ , we choose  $2 \times 2$  submatrix  $E_{i,j} + E_{i,h} + E_{p,j} + E_{p,h}$  whose image under  $T$  is  $1 \times 4$  submatrix in the  $r$ th row. And we can choose a cell  $E_{p,s}$  ( $s \neq q, j, h$ ) such that  $T(E_{i,s}) \neq T(E_{p,j}), T(E_{p,q}), T(E_{p,h})$ . Then we have  $2 \times 3$  submatrix  $E_{i,j} + E_{i,h} + E_{i,s} + E_{p,j} + E_{p,h} + E_{p,s}$  whose image under  $T$  is  $1 \times 5$  submatrix in the  $r$ th row. Similarly, we can choose a  $2 \times (k - 2)$  submatrix whose image under  $T$  is a  $1 \times k$  submatrix in the  $r$ th row. This shows that  $T$  does not preserve the perimeter  $k$  of a Boolean rank-1 matrix, a contradiction.

Therefore we have shown that  $T$  permutes  $\mathbb{E}_{m,n}$  and the corresponding mapping  $f$  is a bijection on  $\Delta_{m,n}$ .

Now, there are two cases : (a)  $T$  maps  $\mathbb{R}$  onto  $\mathbb{R}$  and maps  $\mathbb{C}$  onto  $\mathbb{C}$  or (b)  $T$  maps  $\mathbb{R}$  onto  $\mathbb{C}$  and  $\mathbb{C}$  onto  $\mathbb{R}$ .

Case (a). Let  $\alpha(i) = r$  if and only if  $T^*(R_i) = R_r$ . Then  $\alpha$  permutes  $\{1, 2, \dots, m\}$ . Let  $\beta(j) = h$  if and only if  $T^*(C_j) = C_h$ . Then  $\beta$  permutes  $\{1, 2, \dots, n\}$ . Let  $U$  be the  $m \times m$  permutation matrix corresponding to  $\alpha$ , and  $V$  be the  $n \times n$  permutation matrix corresponding to  $\beta$ . Then for all  $X$ ,  $T(X) = UXV$ .

Case (b). In this case  $m = n$ . Let  $\alpha(i) = r$  if and only if  $T^*(R_i) = C_r$  and  $\beta(j) = h$  if and only if  $T^*(C_j) = R_h$ . Then  $\alpha$  and  $\beta$  permute  $\{1, 2, \dots, m\}$ . By choosing the permutation matrices  $U$  and  $V$  corresponding to  $\alpha$  and  $\beta$  respectively, we have  $T(X) = UX^tV$  for all  $X$ .  $\square$

We say that a linear operator  $T$  *strongly preserves* perimeter  $k$  of Boolean rank-1 matrices if  $P(T(A)) = k$  if and only if  $P(A) = k$ .

Consider the linear operator on  $\mathbb{M}_{2,2}(\mathbb{B})$  defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + b + c + d) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then  $T$  preserves both rank and perimeter 2 of Boolean rank-1 matrices but does not strongly preserve perimeter 2, since  $T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  with  $P\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 4$  but  $P\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = 2$ .

**THEOREM 2.8.** *Let  $T$  be a linear operator on  $\mathbb{M}_{m,n}(\mathbb{B})$ . Then  $T$  preserves both rank and perimeter of Boolean rank-1 matrices if and only if it preserves perimeter 3 and strongly preserves perimeter 2 of Boolean rank-1 matrices.*

*Proof.* Suppose  $T$  preserves perimeter 3 and strongly preserves perimeter 2 of Boolean rank-1 matrices. Then  $T$  maps each row into a row (or a column if  $m = n$ ). Since  $T$  strongly preserves perimeter 2,  $T$  maps each cell onto a cell. That is,  $T$  permutes  $\mathbb{E}_{m,n}$  by Lemma 1.2. Thus  $T$  preserves both rank and perimeter of Boolean rank-1 matrices by the similar method in the proof of Theorem 2.7.

The converse is immediate.  $\square$

**THEOREM 2.9.** *Let  $T$  be a linear operator on  $\mathbb{M}_{m,n}(\mathbb{B})$  that preserves the rank of Boolean rank-1 matrices. Then  $T$  preserves perimeter of Boolean rank-1 matrices if and only if it strongly preserves perimeter 2 of Boolean rank-1 matrices.*



*Proof.* Suppose  $T$  strongly preserves perimeter 2 of Boolean rank-1 matrices. Then  $T$  maps each cell onto a cell and hence permutes  $\mathbb{E}_{m,n}$  by Lemma 1.2. Since  $T$  preserves Boolean rank 1, it maps a row into a row (or a column if  $m = n$ ). Thus  $T$  preserves both rank and perimeter of Boolean rank-1 matrices by the similar method in the proof of Theorem 2.7.

The converse is immediate.  $\square$

Consider the linear operator on  $M_{2,2}(\mathbb{B})$  defined by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & b \\ d & c \end{bmatrix}.$$

Then  $T$  is invertible but does not preserve Boolean rank 1, since

$$T \left( \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

However any  $(U, V)$ -operator is invertible. Its inverse is  $(U^t, V^t)$ -operator.

**COROLLARY 2.10.** *Let  $T$  be a linear operator on  $M_{m,n}(\mathbb{B})$  that preserves both rank and perimeter of Boolean rank-1 matrices. Then  $T$  is invertible.*

Thus we obtain characterizations of linear operators that preserve rank and perimeter of Boolean rank-1 matrices.

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